Stochastic Processes - 1
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Lecture - 55
Time Reversible CTMC and Birth Death Process

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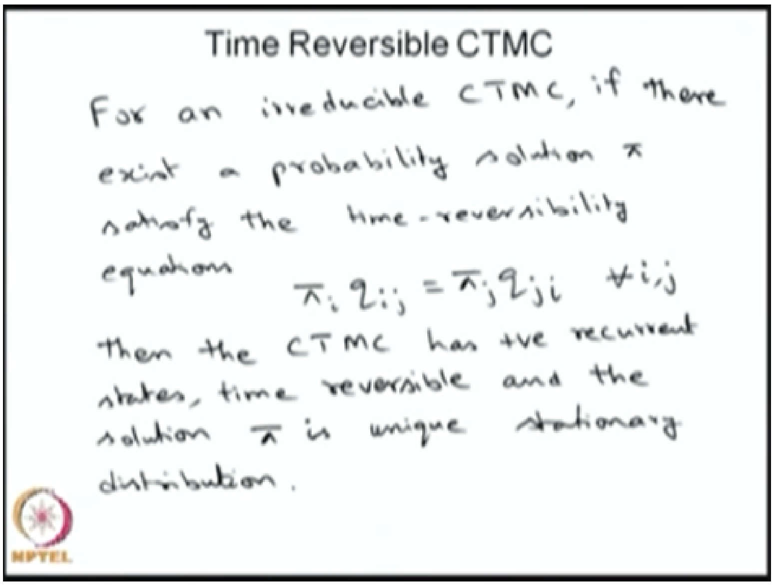

The way I have explained the time reversible concept in the DTMC. The CTMC also has the time reversible concept. So the reversible equation is a pi I times QIJ is equal to pi J times QJI. The Qs are nothing, but the rates and the pi are nothing but the probability values. So if pi I exist this stationary probability or stationary distribution exist then if the stationary distribution exist as well as the time reversibility is satisfied by CTMC.

Then that CTMC is a positive recurrent and we can say that it is a time reversible and the solution pi can be pi is nothing but a stationary distribution. So this result says for irreducible CTMC if there exist a probability solution pi satisfy the time reversibility equation this is s time reversible equation where Qs are rates pi are the probability solution.

If it satisfied by the irreducible CTMC the time reversible equation then that CTMC has a positive recurrent states and that CTMC is a called a time reversible as well as the pi is called the stationary distribution. So initially we have not taken as a stationary distribution some probability solution satisfies the time reversibility equation and it is a irreducible CTMC then that CTMC has a positive recurrent states and pi is nothing.

But the unique stationary distribution. So the usage of this concept is whenever any CTMC is first it is irreducible and satisfies the time reversibility equation of this form. Then you do not want to solve pi Q is equal to 0 and the summation of pi I is equal to 1 to get the stationary distribution instead of that use this time reversibility equation instead of solving pi Q equal to 0 and then use a summation of pi I is equal to 1 to get the one on one.

That means, use the time reversibility equation repeatedly recursively and get all this in terms of one or no either pi 0 or pi 1 whatever it is. Then use a summation of pi 1 is equal to 1 to find that unknown instead of solving pi Q is equal to 0. So whenever it is model is irreducible and time reversibility equations are satisfied then you can conclude all the states are positive recurrent and you can find the pi the stationary distribution in easy way instead of solving pi Q is equal to 0.

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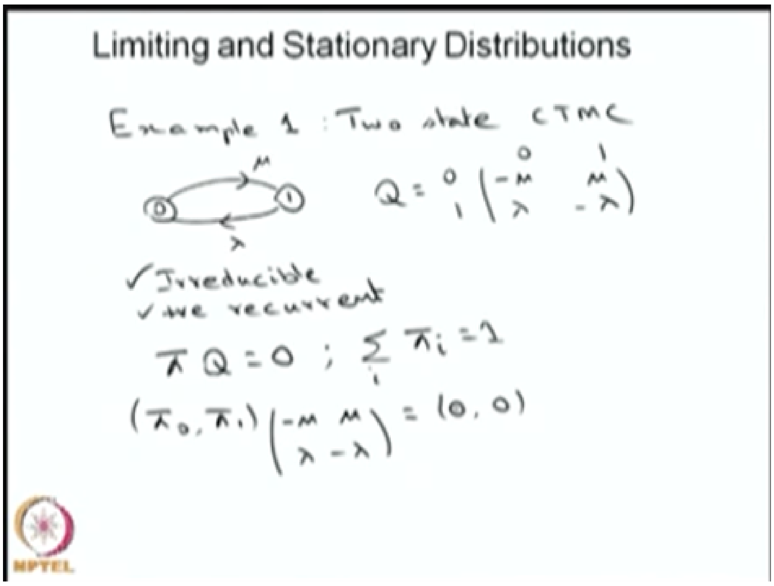

I am going to give one simple example what is a limiting and a stationary distribution. Take the two states as CTMC and we know that Q matrix and you can verify whether this is going to be irreducible and a positive recurrent since it is a finite state model and both the states are communicating each other therefore it is irreducible positive recurrent states. So you can solve pi Q is equal to 0 and the summation of pi I is equal to 1. So pi times Q pi is the vector Q is the matrix and again 0.

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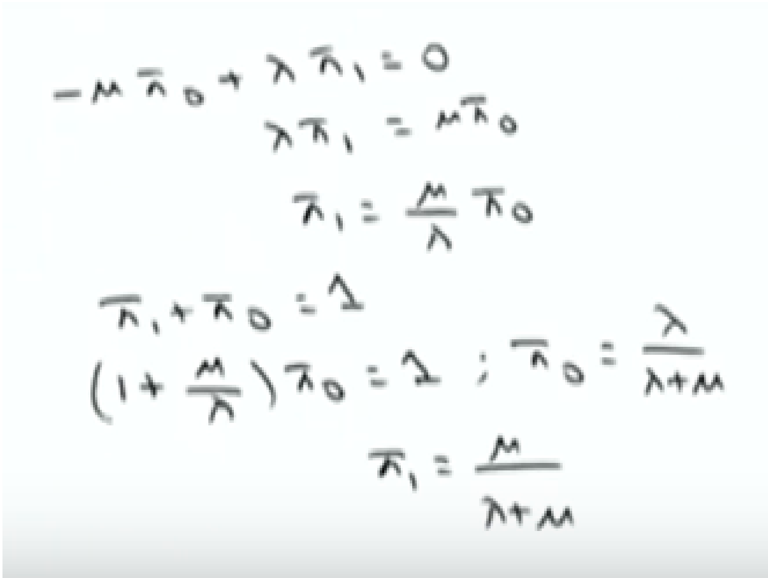

Therefore, if I take the first equation, I will get minus mu times pi 0 lambda times pi 1 is equal to 0 by taking the first equation minus mu pi 0 plus lambda times pi 1 that is equal to 0. From this I can get the pi 1 in terms of pi 0. Since it is a homogenous equation I have to use a non homogenous and normalizing condition summation of pi I is equal to 1. So using that I will get pi 0 is equal lambda divided by lambda plus mu.

Once I know a pi 0 then pi 1 is equal to mu divided by lambda-plus mu. So this is the stationary distribution as well as the limiting distribution because it satisfies both the conditions.

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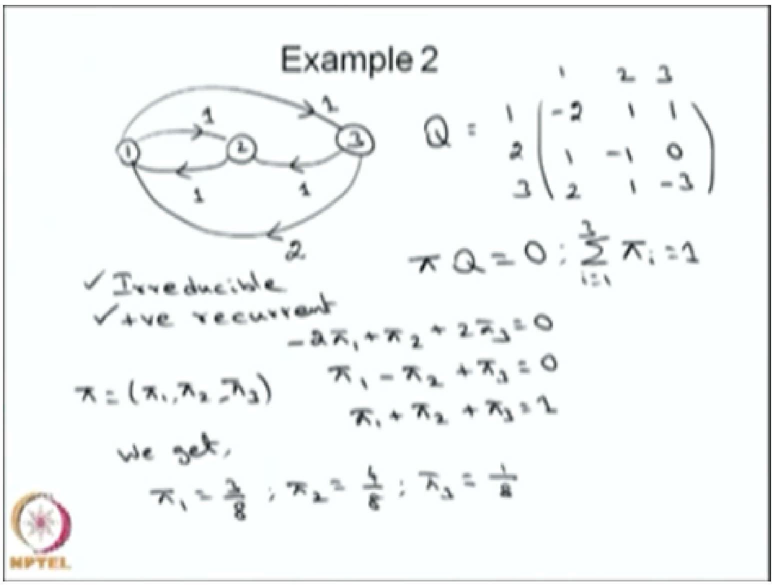

Take the second example. Second example also finite state model all the states are communicating with all other states therefore it is a irreducible. Since it is a finite state model you would not have a null recurrent it is a positive recurrent model. So I can solve pi Q is equal to 0 and the summation of pi I is equal to 1 so there are three equations. So I will take the first two equations and one normalizing equation and solve this three equation I can get the pi 1, pi 2, pi 3.

You can verify that the summation is going to be 1. So this is the limiting distribution as well as the stationary distribution because the model is the irreducible positive recurrent model. So this limiting distribution and the stationary distribution are one of the same. Instead of solving pi Q is equal to 0 you can use the time reversibility, but before that you should verify whether the time reversibility equation are satisfied by this model.

If this model satisfies the time reversibility equation for all I, J then you can conclude it is a time reversible Markov Chain and so on. But example 1 is the Time Reversible Markov Chain where as that example two is not a Time Reversible Markov Chain you can verify it.

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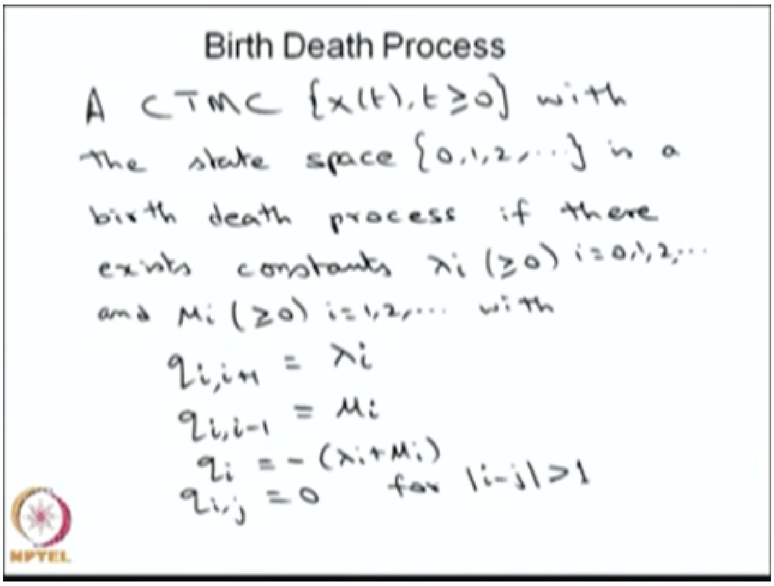

Now I am moving into the special case of Continuos Time Markov Chain that is a birth death process. This is a very important time homogenous Continuos Time Markov Chain because many of the scenario can be mapped with the birth death process either with the finite state or infinite state. Let me first give the definition of birth, death process.

I started with Continuos Time Markov Chain it is a time homogenous Continuos Time Markov Chain with the state space countably infinite it can be finite also that CTMC is going to be call it as a birth death process. If there exist a constants lambda i and mu i such that and this are all are nothing, but the Infinitesimal Generator Matrix elements and this is i to i plus 1 that rate is always lambda i.

And rate in which the system is moving from the state i to i minus 1 that rate is mu i and the diagonal elements are minus of lambda i plus mu i. Whereas all the other rates, the system is moving from the state i to j other than i to i plus 1, i to i minus i and i to i and all other rates are always 0, absolute of i minus i is greater than 1.

That means you will have the Infinitesimal Generator Matrix in which you will only have a diagonal matrix and all other elements are going to be 0. I can write down the condition so that it land up the rates are going to be only lambda i and mu i so on not all other rates are going to be 0.

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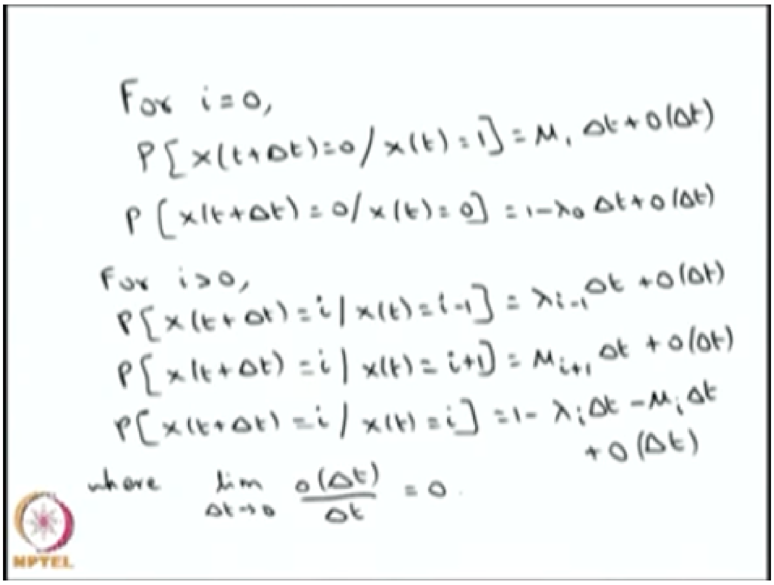

So if I start with i is equal to 0 the system is moving from the state 1 to 0 in the interval of delta t because it is a time homogenous model. So this is nothing but this probability the system is moving from the state 1 to 0 in the interval of delta t that is nothing, but the rate is mu 1 times delta t plus order of delta t.

Similarly, the system is moving from the state 0 to 0 from the time t to t plus delta t or during the interval delta t that is nothing, but 1 minus lambda 0 times delta t plus order of the delta t. This mu i and the lambda 0 and so on this values are always going to be greater than or equal to 0 strictly greater than 0 also. For I is greater than 0 the system is moving from the state i to i that is 1 minus lambda i times delta t minus mu i times delta t plus order of delta t.

Whereas the system is moving from i plus 1 to i one step backward that is mu i plus 1 delta t. The system is moving from the state i minus 1 to i or i is greater than 0 that is a forward one step more that is lambda times i minus 1 delta t plus order of delta t. This is the order of delta t it maybe a function of delta t need not be the same as the delta t times to 0 this quantity are going to be 0 or of delta t divided by delta t is going to be 0.

Therefore, this is the way the system is moving from the one state to either one step forward or either one step backward or move anywhere. So these are only three possibilities with these probabilities. Therefore, we will end up the Q matrix is going to be the system is moving from the state i to i plus I forward one more that rate is lambda i.

The system is moving from the i to i minus 1 one step backward that is mu i or the system being in the same state that rate is minus lambda i plus mu i. Therefore, there is no other move from the system from one state to all other states either one step forward or one step backward.

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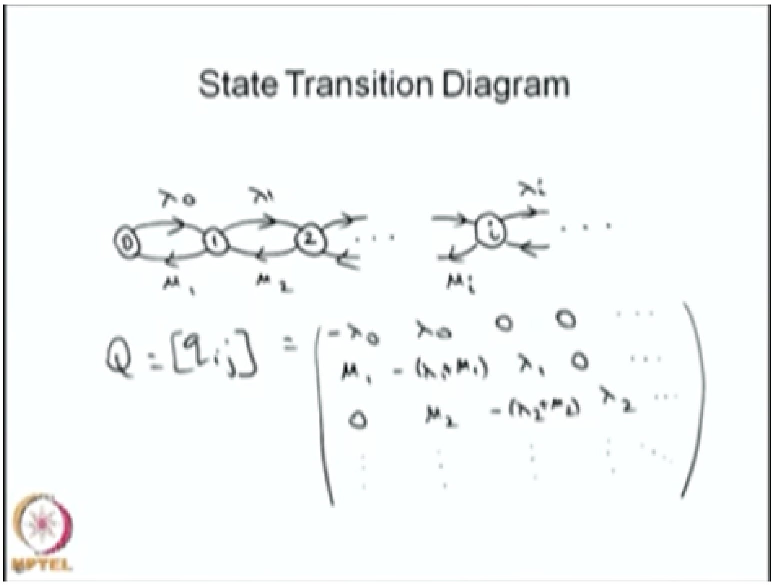

So this can be visualized in the state transition diagram. Since I started with the state space 0 to infinity there is a possibility you can have a label from some negative integers to the positive integers so we can always transform into something therefore default scenario or the simplest one I discussed from 0 to infinity. Therefore, we can visualize whatever will be the label that can be transformed in a one a one fashion.

So this is the rate in which the system is moving from the state 0 to 1 that rate is lambda 0. The system is moving from the state 1 to 2 that rate is lambda 1 or the system is moving from the state 1 to 0 that rate is mu 1. Therefore, the time spent in the state 1 before moving into any other states that is a minimum of the time spending in the state 1 before moving into the state 2 or the system time spending in the state 1 before moving into the state 0.

So both are exponentially distributed with the parameters lambda 1 and the mu 1and the minimum of that time is the spending time or the waiting time in the state 1 that is going to be exponential distribution with the parameter lambda 1 plus mu 1 because both are independent. The time spending in the state 1 before moving into the state 2 and similarly the time spending in the state 1 before moving into state 0.

And both the random variables are independent that is the assumption therefore it is going to be a exponentially distributed the time spending in the state 1 that is exponentially distributed with the parameter lambda 1 plus mu 1. Like that you can discuss for all other states so whenever you have a birth death process the system either move one step forward or one step backward then it is a called a birth death process.

Therefore, here this lambda is called the system is moving from one state to forward one step therefore this lambda is called the birth rates. The system is moving from one state to previous one state and the corresponding rate mu i M1, M2, M3 and so on and these rates are going to be call it as a death rates.

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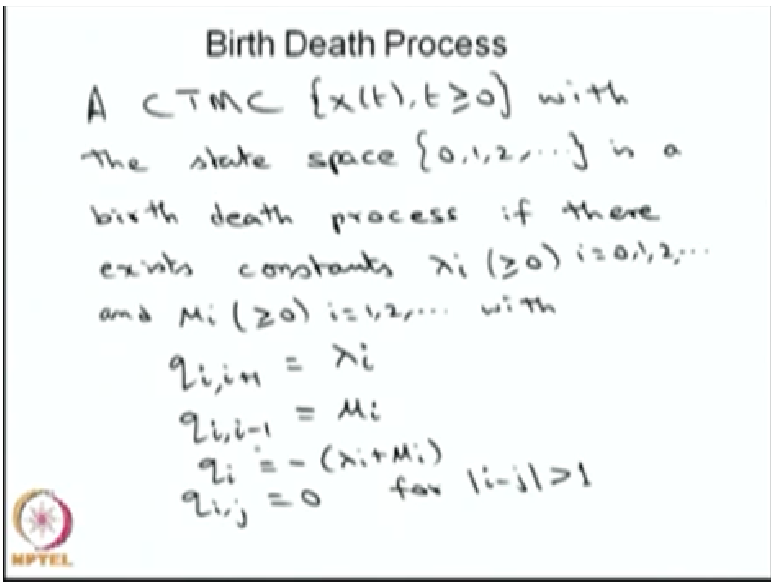

So lambda i are nothing, but the birth rates that means the rate in which the system is moving from the state i to i plus 1 that depends on I. Therefore, that rate is lambda i. The system is moving from the state i to i minus 1 that is related to the death by 1 that is a function of i. Therefore, that death rate is mu i. So lambda i are the birth rate and the mu i are the death rates.

Suppose example the system moving from the state 2 to 1 the death rate will be mu suffix 2. So you can fill up the Q matrix if you see the Q matrix it is a tridiagonal matrix. So here I am giving few examples of a birth death process. The first example consist of the first example is a finite state model. The birth rates are lambda 0, lambda 1 till lambda N minus 1. The death rates are mu 1, mu 2, and mu N. It is a finite state birth death process.

The second example is the infinite state of birth death process. The third example the all the death rates are 0 that is also possible. The fourth example all the birth rates are 0 that is also possible, but one can discuss the state classification also. The first one it is a finite state model all the states are communicating with all other states therefore it is a irreducible positive recurrent birth death process.

The second one is the infinite state all the states are communicating with all other states it is a Irreducible, but one cannot conclude without knowing the values about the lambda 0 and lambda i and mu i one cannot conclude it is a positive recurrent or null recurrent. The mean recurrence time that is going to be a finite one then you can conclude it is positive recurrent otherwise it is null recurrent.

So as such you cannot discuss now the positive recurrent or null recurrent but you can conclude it is a recurrent state. The third example the system keep moving forward therefore all the states are transient states. It is not irreducible it is a reducible model all the states are transient states that mean as T times to infinity the system will be in the some infinite state. So one cannot define the infinite state therefore the limiting distribution would not exist in this situation.

The fourth example it is a finite model, but all the states are not communicating with all other states therefore it is a not a irreducible it is a reducible model. Whenever the system start from some state other than 0 over the time the system is keep moving backward and once it reaches the state 0, it will be forever. Therefore, state 0 is observing barrier.