Stochastic Processes - 1

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Lecture - 53
Introduction and Examples of Continuous Time Markov Chain

This is module 5 Continuous Time Markov Chain. In the first lecture, we have discussed the definition of a Continuous Time Markov Chain then we have explained how we can derive the Chapman-Kolmogorov equation then we have defined Infinitesimal Generators Matrix then I have given the Kolmogorov differential equations in the first lecture.

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In the lecture 2, I am planning to discuss the limiting distribution, stationary distribution and a steady state distribution followed by that I am planning to give a description about the brith, death processes and also some simple examples for the limiting distribution stationary steady state distributions and birth, death processes.

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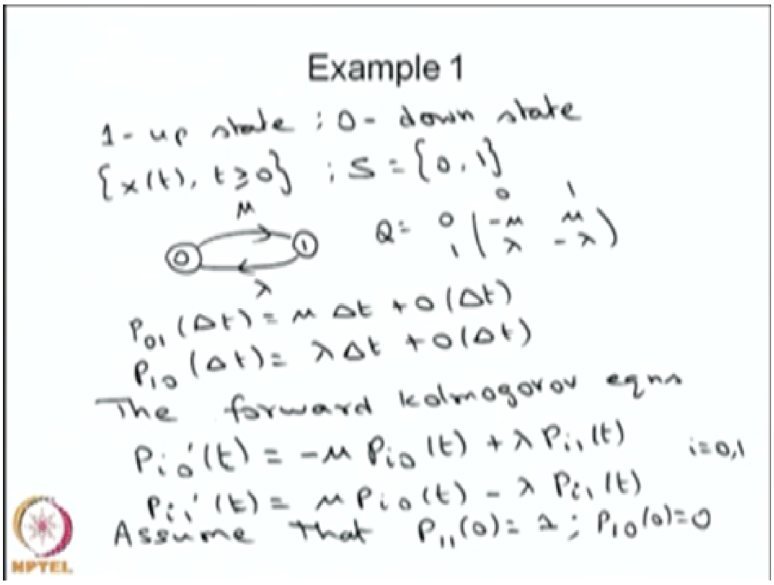

Before I go to the limiting distribution, let me give the example for the Continuous Time Markov Chain to get the time dependent solution. This example is the very simplest example that is a two state Continuos Time Markov Chain the default one is a Time Homogenous. The state space are 1 and 0. One you can consider as upstate or operational sate and 0 is a downstate non operation state.

So this can be visualized for any model in which the whole dynamics can be described with the two state and Markov property is satisfied. The system going from the state 1 to 0 or the time spend in the state 1 before moving into the state 0 that is exponentially distributor with a parameter lambda. Once it is failed that means the systems is in the downstate. The time spent in the repair time that is exponentially distributor.

The parameter mu so once the repair is over the system is operation state therefore it is in the upstate. So 0 is related to the downstate and 1 is related to the upstate and mu is nothing, but the mean I by mu is the Mean time for the repair and one by lambda is the mean time of a failure and the failure time is exponentially distributed with the parameter lambda and the repair time is exponentially distributed with a parameter mu.

This is a state transition diagram for the two states CTMC. The corresponding Q matrix the Infinitesimal Generator Matrix that it consists of a two cross two matrix. The system going from the state is 0 to 1 that rate is mu. The system going from the state 1 to 0 that rate is lambda and the diagonal values are minus of summation of other values that row sum. So 0 to 0 is minus mu and 1 to 1 is minus lambda. Therefore, the rates are other than the other than diagonal elements.

And the diagonal elements are minus of sum of the row values other than that diagonal element. So this is nothing, but in a very small interval of time delta T the system is moving from the state 0 to 1 that probability the probability of system moving from the state 0 to 1 that is nothing, but the downstate to the upstate in a very small interval of time delta T why you are finding the probability of delta T.

Since the model is a Time Homogeneous only the interval is matter not the actual time or you can visualize this as the sometime T to T plus delta T also. So this is the interval of delta small negligible interval delta T the system is moving from the state 0 to 1 that probability is nothing but the rate means the rate. The rate is nothing but the repair rate so the rate mu times the delta T plus order of delta T it's a small row.

Order of delta T means as delta T times to 0 the order of delta T will be 0. Similarly, you can visualize the probability of system moving from the state 1 to 0 in the interval delta T in a small interval delta T that is same as the failure rate lambda times the delta T that is the small interval of time plus order of delta T. So this order of delta T also times to 0 as delta T times to 0. So using this I can make the forward Kolmogorov equation.

I can go for writing a forward Kolmogorov equation or backward Kolmogorov equation, but forward Kolmogorov equation is easy to make out so if the system is in the state I at time 0 what is the net rate the system will be the state 1 at the time T that net rate is nothing, but what are all the inflow that probability rate minus what are all the outflows that is the way you can visualize the right hand side.

So all the positive terms are related to the incoming rates and all the negative terms related to the outgoing rates. So since it is a two state model if the system is in the state 0 at time T there is a possibility it has not moved anywhere from the state 0 or it would have come from the state 1. Therefore, the incoming will be state 1. Therefore, the system will be in the state 1 at a time T and given that the starting from the state I that probability multiplied by the rate sort of inflow minus because we are writing the equation for the state 0.

Therefore, it is not moved from the state 0 that is a with the rate mu it can move to the state 0 to 1. Therefore, minus mu times it does not move from the state 0 therefore minus mu times the probability of being in the state 0 are time T given that it was in the state I at time 0 that probability multiplied by minus mu that is an outflow and the lambda time PI 1 T that is a inflow.

Therefore, in the left hand side, it is a derivative of the function T it is a probability function. So PI 0 dash T that is nothing but the net rate being in the system at time T in a state 0 given that it was in the state I at time 0 that net rate is same as the inflow minus outflow with the corresponding rates. Similarly, you can write the equation for the state 1 that means you start from the state 1 either you would have come from the state 0 to the 1 or you did not move from the state 1.

Therefore, minus lambda times PI1 of T plus mu time PI0 of T that is the net rate corresponding to the state 1. So now we are able to write the forward Kolmogorov equation. So this is a interpretation of the forward Kolmogorov equation. You can write easily by making a matrix PIJ of T dash that is equal to P of T times Q where Q is the Infinitesimal Generator Matrix then also you will get the same things so I am just giving the interpretation.

Now my interest is to find out the time dependent or transient solution for this two state CTMC for that this is a difference differential equation we need initial condition to solve these equations. So I make the assumption at time 0 the system is in the state 1. Therefore, the transition probability of system the P11 of 0 that is equal to 1 since I made the assumption the system was in the state 1 at time 0 therefore being in the state 0 that is going to be 0. So I need this both the initial condition there to solve the equation.

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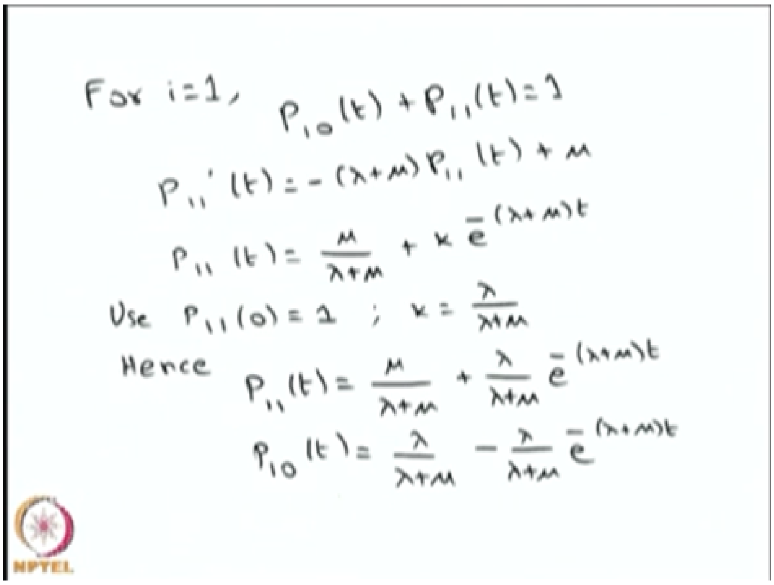

So let me start since I made the initial condition state is 1 therefore I is equal to 1 so I will have the first equation that is I always have the summation of the probability at time T. This is a transition probability are going to be 1 the summation and also I have a two difference of differential equations. So what I can do I can take the second equation in this then instead of P10 of T I can use the summation of probabilities is equal to 1.

Therefore, instead of P10 of T I can use the P10 of T is nothing, but 1 minus P11 of T I can substitute in the second equation therefore I will get a P11 dash of T is equal to minus lambda plus mu times P11 of T plus mu substituting P10 of T is equal to 1 minus P11 of T in the second equation the previous slide. Now I have to solve this a differential equation the unknown is P11 of T conditional probability.

And I have to use a initial condition P11 of 0 is equal to 1 using that I will get P11 of T is equal to mu divided by lambda plus mu plus sum constant E power minus lambda plus mu times T that constant I can find out using this initial condition therefore K is equal to lambda divided by lambda plus mu. So the P11 of T is equal to substituting K is equal to lambda divided by lambda plus mu in this equation I will get the P11 of T.

Once I know the P11 of T use the first equation so I will get P10 of T is equal to 1 minus P11 of T therefore P10 of T that is equal to this expression. You can cross check this now if you add both the equations you will get a 1 and if you put T equal to 0 you will get the initial condition also correctly and if you put a T times to infinity that we are going to discuss in the limiting distribution if you put a T times to infinity in this equation.

You will get a mu divided by lambda plus mu, lambda divided by lambda plus mu. So this is for the T times to infinity. Therefore, if you make a matrix the limit n times to infinity of the limit.

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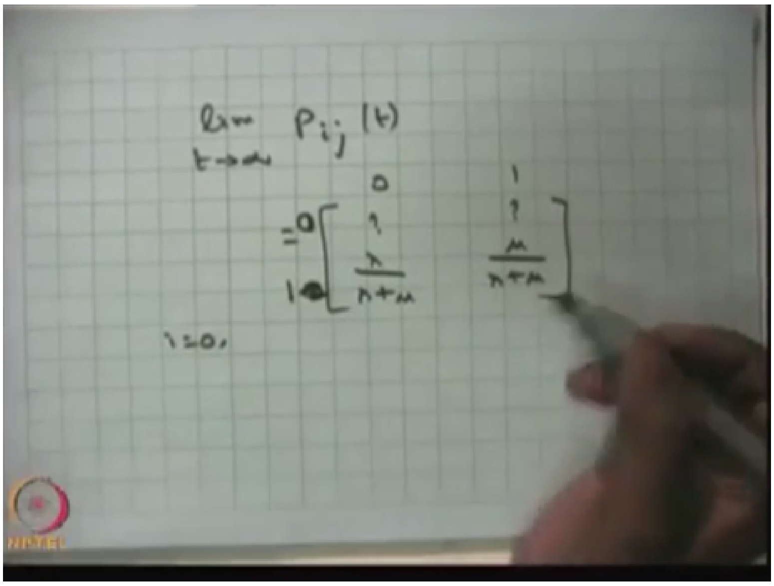

If you find out the limiting distribution of a limit T times to infinity of P of T so you will get the matrix and this matrix as T times to infinity for this example is set to cross two matrix. And that consists of a for different values you will have a now you are doing for the second row therefore that is equal to lambda divided by lambda plus mu and this is equal to mu divided by lambda plus mu.

So if the system starts from the state 1 at a time times to infinity the system will be in the state 0 with a probably lambda divided by lambda plus mu and the system will be in the state 1 with a probability mu divided by lambda plus mu. Similarly, if you go for I equal to 0 you will get the same derivation and you can fill up what is the element here. So this is the limiting distribution of probability matrix and if you see the rows are going to be identical.

So you will have a same identical row in this row also so that means you will get the limiting distribution. I will discuss the limiting distribution in the after giving one more example I will explain in detail. So this is the transition probability system starting from the state 1 and being in the state 1 or 0 at a time T.