Stochastic Processes - 1
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Lecture - 52
Infinitesimal Generator Matrix

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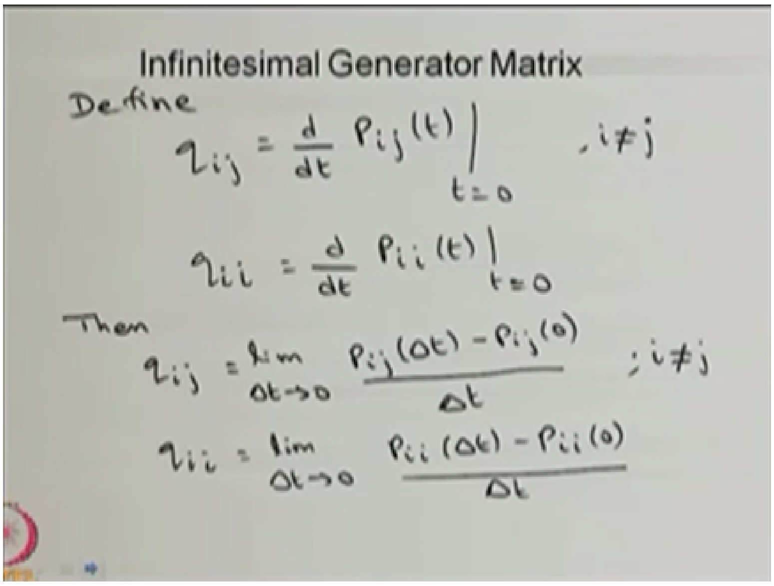

I am going to define the quantity called a Qij and later this is going to form a matrix that is going to be called it as a Infinitesimal Generator Matrix. So let me start with the definition Qij that is nothing, but take a derivative of Pij of T that is a function of T. You can find out the derivative. It is differential function only, so you take a derivative then substitute a T equal to 0 for all I is not equal to J.

Then you define Qii that also in the same way separately because the Qii the diagonal element is going to be different from all other elements therefore I am defining separately. You know how to find out the derivative, derivative of Pij of T with respect to T that is nothing, but the limited delta T times to 0 the difference divided by the delta T. Since a Pij of T is a transition probability of system moving from I to J.

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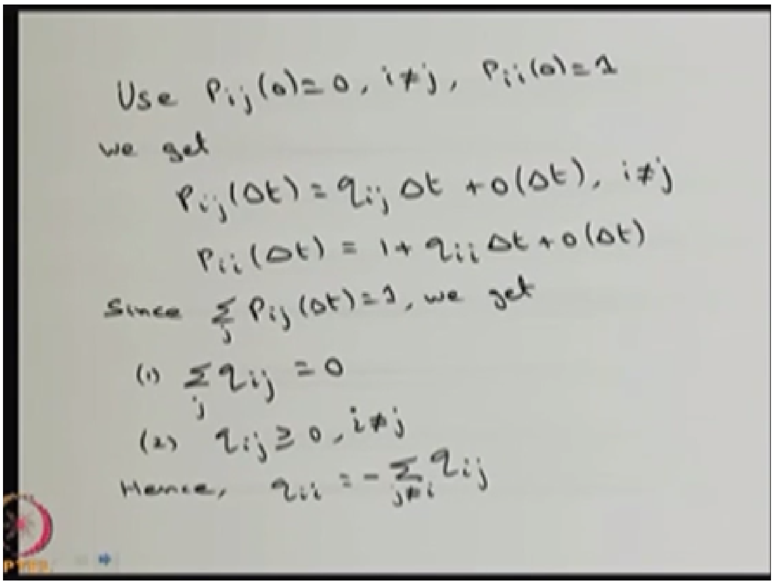

You can use Pij of 0 is equal to 0 for I is not equal to J for J is equal to I that is PI of 0 that is equal to 1 that means what is a transition probability of system moving from the state I to I in the interval 0 that is same as 1 that probability is 1. So use this in the previous limit in this Pij of 0 is equal to 0 and PI of 0 is equal to 1 you substitute then the limit delta T times to 0. Therefore, the Pij of delta T this will go to this side.

So Qij times delta T therefore this is going to be Pij of delta T is nothing, but the Qij multiplied by delta T plus small o order of delta T that means as delta T times to 0 this whole quantity will times to 0. Similarly, you substitute PII of 0 is equal to 1 here therefore PII of delta T that is same as this will come to this side. So 1 plus Qii delta T plus order of delta T. So this order of delta T that also times to 0 as a delta T times to be 0.

You know that the summation of Pij even at the time point of delta T the small negligible time point of delta T at that time also over the I that is equal to 1. Therefore, if you sum it up you can conclude the left hand side is a probability, right hand for I is not equal to J you have Qij whereas that is second expression you have 1 plus Qii therefore using the property of summation of Pij is equal to 1 you will get a summation of Qij for all J that is going to be 0.

When you add both the equations for all J you will get a summation over J Qij is equal to 0 as well as all the Qij quantities are going to be greater than or equal to 0 from the first one because the left hand side is a probability and this is multiplied by the delta T. A delta T is always greater than 0. Therefore, the Qij is going to be greater than 0 for all I not equal to J whereas if I add over all the J that is going to be 0.

Therefore, you will get the Qii that is nothing, but you make the summation for all Qij for all J except I then you make a minus sign so that is going to be the Qii. That means the diagonal element is nothing, but make the row sum except the diagonal term and put the minus sign that is going to be diagonal term. Therefore, when you make a row sum that is going to be 0. The details of the proof can be found in the reference books.

So the quantity Qij that has the property the row sum is going to be 0 and other than the diagonal elements are greater than or equal to 0. Therefore, the diagonal element is going to be summation of all the other terms with the minus sign. So using this we can make a matrix that is going to be Q matrix. The entity is a Qij such that satisfies the property Qij is always greater than or equal to 0 for I is not equal to J whereas the diagonal element is minus of summation.

Therefore, it has the property the row sum is going to be 0. So the difference between this matrix and the one step transition probability matrix in the DTMC that is a probability matrix so the entries are probability values from 0 to 1 and the summation row sum is 1 whereas here because Qij are obtained by differentiating the Pij. These are all the rates and this rates are always greater than or equal to 0 other than the diagonal elements and the diagonal elements are a minus with the summation of all other row elements.

So this matrix is called Infinitesimal Generator Matrix. Some books they use the word rate matrix also and whereas here the rates are placed in other than the diagonal elements and sum of the rates could be 0. That means the probability of system moving from that particular state is not possible that probability is 0 or there is a in a small interval of time there is the transition is not possible.

So whenever the rates are greater than 0 that means there is a positive probability that the system can have a transition of system moving from I to J.

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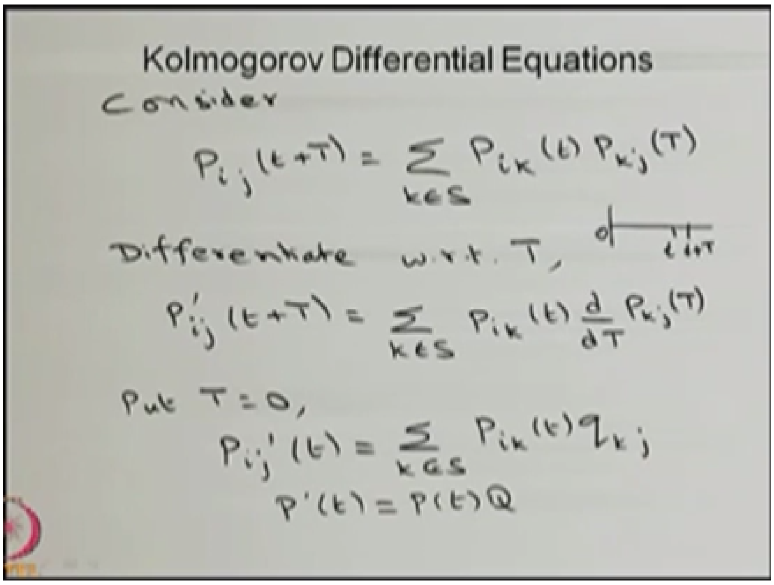

So we have defined the Q matrix. Now using the Q matrix we are going to find out the Pij of T. So let me start with the Chapman–Kolmogorov equation. Now I am going to differentiate with respect to capital T that means I make the interval 0 to small t plus capital T as a 0 to T then I make a T to T plus capital T differentiate with respect to capital T. Therefore, the left hand side is going to be I have written with a dash so the derivative come inside the PKJ of T.

Then I am substituting T equal to 0. So basically I am making a system to move from state 0 to small t then there is a smaller interval of time from t to t plus capital T that is the interpretation of this. Then substituting T equal to 0 I will get the left hand side is going to be Pij of dash T that is same as the summation over this. Whereas this is nothing, but the way we have defined the Infinitesimal Generator Matrix entities.

So this is nothing, but the QKJ that s the rate in which the system is moving from the state K to J. In a matrix form I can make it as a Pij of T is going to form a matrix. So the P dash of T that is same as a P of T times Q. So this is the matrix and P of T is also matrix and this is the P dash of T means each entities are differentiated with respect to time T. So this is in the matrix form and this equation is called a forward Kolmogorov differential equation because the derivation goes from 0 to T.

Then T to T plus T were considering as a very small interval of time. Therefore, this equation is called forward Kolmogorov differential equation.

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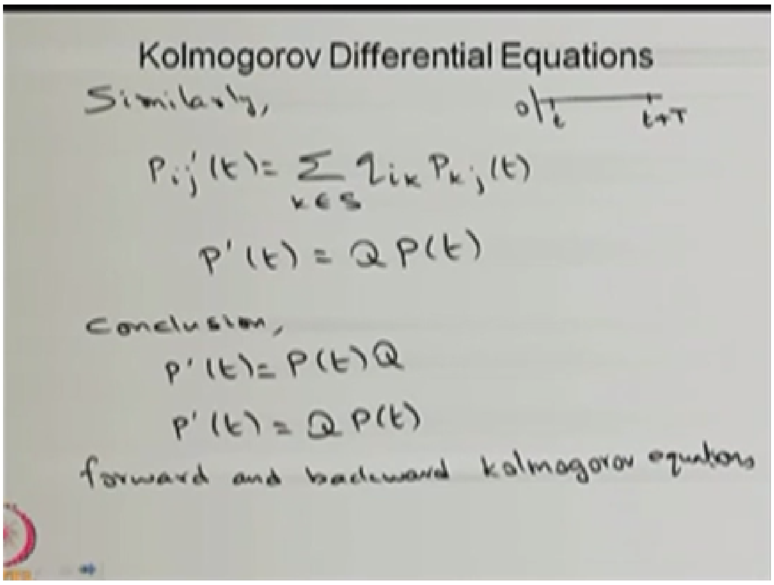

In the same way, if you do 0 to small t that has a that has a small interval of time and a T to T plus capital T then I will get the P dash of T is equal to Q times P of T that is called the backward Kolmogorov differential equation. Whether you frame a forward equation or a backward Kolmogorov equation if you solve that equation you will get the Pij of T. If you solve P dash of T is equal to P of T into Q that is a forward equation.

The P dash of T equal to Q times P of T that is a backward equation. If you solve the equation with the initial condition because it is a differential equation so you need a initial condition what is a probability, what is a transition probability of system moving from I to J at time 0. If you know the initial condition by supplying that solving this equation you will get the Pij of T. Once you know the Pij of T then you can get the distribution of X of T.

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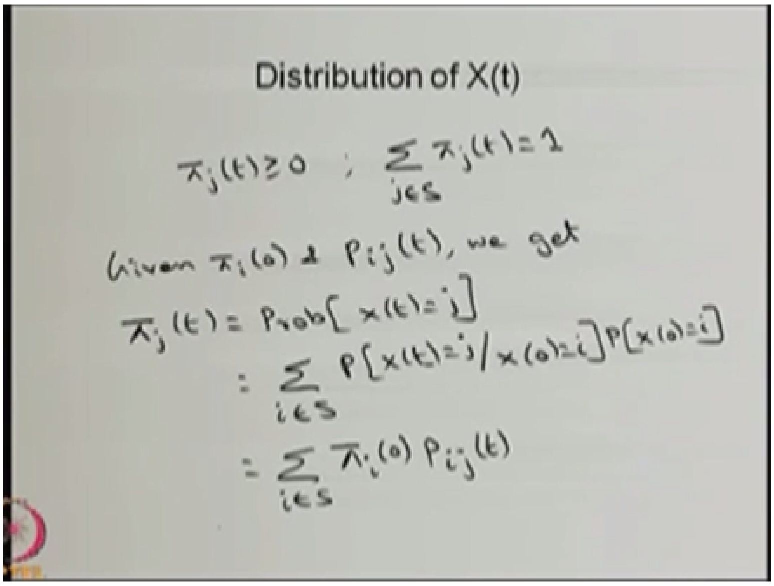

So once you know the Pij of T the given is pi I of 0 and by solving that forward or backward Kolmogorov differential equation you will get the Pij of T using this two you can get the Pi J of T. So for a given Pi I of 0 and Pij of T that means the transition probability and the initial state probably vector one can find out the distribution of X of T. So in this lecture I have started with the Markov process then I have discussed the definition of a Continuos Time Markov Chain and also I have given what is the distribution of time spending in any state before moving into any other state.

And also I explain the Infinitesimal Generator Matrix using that how to find out the transition probability of Pij T from the Chapman-Kolmogorov equation and we got a forward as well as the backward Kolmogorov differential equations by solving a forward or backward Kolmogorov differential equation one can get the Pij of T that is the transition probability. Using this equation, you can get the pi J of T that is nothing, but the distribution of X of T.

With this, let me stop this lecture and in the next lecture I will go for simple example of a Continuous Time Markov Chain as well as the stationary limiting distribution and steady state distribution in the next lecture.