Stochastic Processes - 1
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Lecture - 51
Chapman-Kolmogorov Equation

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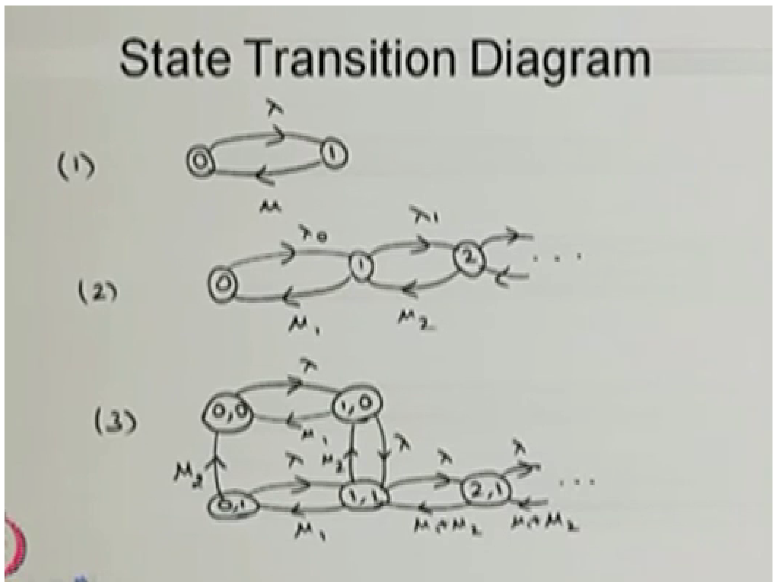

Now I am going to give few state transition diagrams for the time homogeneous continuous time Markov chain, you see the first example it has only two states 0 and 1 so the state space S is 0, 1 and the time spending the state 0, before moving into the state 1, that is exponentially distributed with the parameter lambda, once the system come to the state 1 the time spent in the state 1 before moving into the state 0 that is exponentially distributed with the parameter mu.

Lambda is strictly greater than 0 and mu is also strictly greater than 0, that means you know the exponential distribution has been 1 divided by the parameter, therefore the average time spent in the state 0, before moving into the state 1 that is 1 divided by lambda, the average time spending in the state 1 before moving in to the state 0 that is 1 divided by mu, since it is a two state so over the time the system will be in the state 0 or 1.

And you can classify the states also, the way we have discussed in the continuous discrete time Markov chain, since both the states are communicating, both the states are accessible from each other - each other direction therefore both the states are communicating each other, since the state space is 0 and 1 and both the states are communicating each other therefore this is a irreducible Markov chain, for a irreducible Markov chain all the states are of the same type.

For a finite Markov chain we have at least 1 positive recurrent state therefore both the states are going to be a positive recurrent state, but here there is no periodicity for the continuous time Markov chain, therefore we can conclude the first example both the states are a positive recurrent and the Markov chain is a irreducible Markov chain.

So the continuous amount of time system spending in state 0 and 1 that is exponentially distributed with the parameters which I discussed earlier. Now I am moving into the second example in the second example we have a state space is countably infinite and the system spending in this state 0 before moving into the state 1 that is exponentially distributed with a parameter lambda naught.

Whereas the state 1, the system can spend exponential amount of time, amount of time spending in the state 1 before moving in to the state 2 that is exponentially distributed with parameter lambda 1, and similarly, the system spending in the state 1 before moving into the state 0, that is exponentially distributed with the parameter mu 1.

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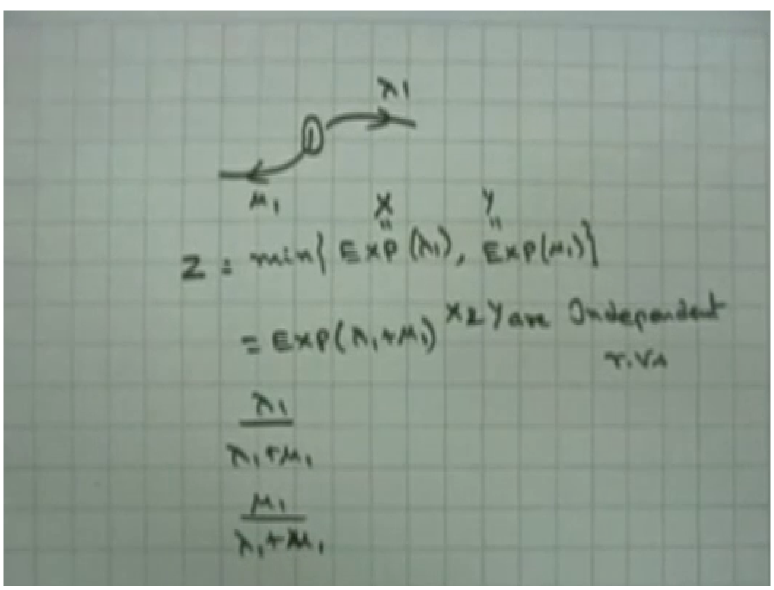

Therefore, this is mu 1 and this is lambda 1, therefore the time spending in the state 1 before moving into any other state that is going to be minimum of the exponentially distributed with the parameter lambda 1, 1 random variable you can call it as X and the you can call it as a another random variable that is exponentially distributed with the parameter mu 1.

Therefore the amount of time spending in the state 1 before moving into any other state that just now we have concluded that waiting time distribution is exponentially distributed that will come from here also, so here these two random variables are independent, x and y are independent random variables both the random variables are independent.

Therefore the time spending in the state 1 before moving into any other state that is going to be minimum of the random variable with exponential distributed parameter lambda 1 and the random variable which that which follows exponential distribution with the parameter mu 1, you know that the minimum of two exponential as long as both the random variables are independent random variable.

Then this is also going to be exponential distribution with the parameters - with the parameter lambda 1 + mu 1, as long as both random variables are independent and both are exponential you can do it as a homework minimum of two exponential are going to be exponentially with the parameter lambda 1 + mu 1, therefore the time spending in the state 1 that is exponential distribution with the parameter lambda 1 and mu 1.

Also one can discuss what is the probability that the system moving into the state 2 before moving into the state 1 that is a lambda 1 divided by lambda 1 + mu 1, similarly what is the probability that the system moving into the stage 0, before moving into the state 2, that is mu 1 divided by lambda 1 + mu 1, that also you one can find out, so what is the conclusion here is the time spending in the state 1, that is exponential distribution with a parameter lambda 1 + mu 1.

Similarly, the time spending in the state 2 that is suppose if it is lambda 2 then lambda 2 + mu 2, so this is the one type of a continuous time Markov chain. The third example this is also continuous time Markov chain is sort of a two dimensional Markov chain with the labelling with a 0, 0 1, 0 2, 0 and so on, so all the labelling which is parameters for the exponential distribution. So the change from the discrete time Markov chain state transition diagram.

And the state transition diagram of a continuous time Markov chain, here there is no self-loop and the labels are the parameters for exponential distribution, whereas the discrete time Markov chain it is a one step transition probability going from one state to other states. Here the labels the arrow gives the - the time spending in the state exponential distribution with the parameter lambda naught and moving into the state 1 and so on.

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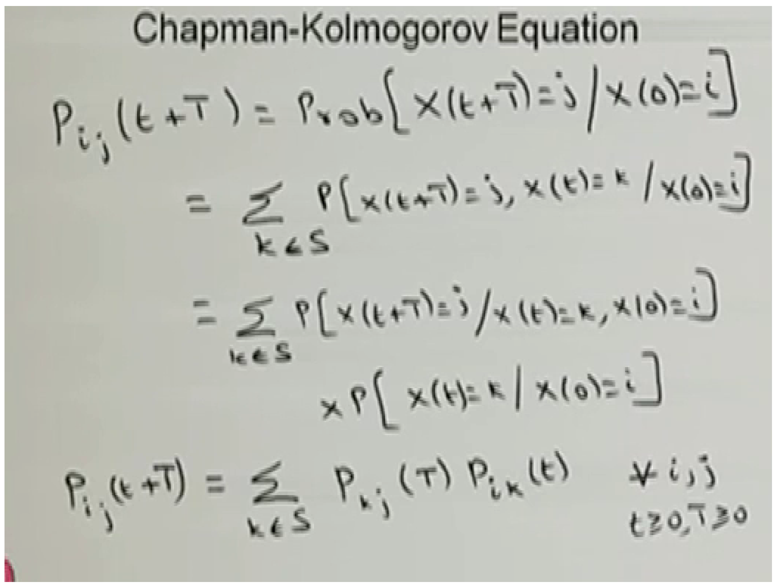

Now I am going to find out how - now I am going to find out the P i j of t, for that I am going to do the derivation starting with the Chapman Kolmogorov equation. let me start with what is the transition probability of system is moving from i to j, during the time 0 to t + capital T that is nothing but what is a transition probability system will be in the state j at that time point t + capital T given that it was in the state at time 0.

That is same as I can in between make a some other state I can make a one more state k at time point t for all possible values of k also I will get the same result, that is same as I can make a summation over k, k belonging to S, S is state space that is same as what is the conditional probability of system will be in the state j at the time point t + capital T given that it was in the state i at time 0.

As well as it was in the state k at small t also multiplied by what is the transition probability of system moving from 0 to t, from the state i to k, that is same as, the first conditional probability you see this is same as the Markov property in which we have discussed in the definition of a continuous time Markov chain there I have discussed the CDF Cumulative Distribution Function here it is the probability mass function where this is a conditional probability mass function.

What is the conditional probability mass function of the system will be in the state j at time point small t + capital T given that it was in the state i at the time point 0 as well as it was in the state k at the time point t, and you know that 0 less than t less than t + capital T, because the way we made it as all these values are greater than 0, therefore by using the Markov property of a continuous time Markov chain.

So this is same as what is the probability that the system was in the state k at time small t and move into the state j at the time point t + capital T, again you use the time homogeneous property, first you use the Markov property, therefore this is a transition probability of t to t + T moving from the state k to j, then use the time homogeneous property therefore only the length matters therefore t to t + capital T that is same as 0 to capital T.

Therefore the system is moving from the state k to j, from 0 to capital T, that is P k j of t, the second one, it's a transition probability system is moving from state i to k during the interval 0 to capital T, therefore this is i to k of t, so this is valid for all i, j with the t greater than or equal to 0 and capital T is also greater than or equal to 0, therefore the left hand side is the transition probability of system is moving from the state i to j, from 0 to t + capital T.

That is same as submission over I can rewrite in a different way i to k in the interval 0 to small t, k to j instead of small t to small t + capital T, because of the time homogeneous and I am just making 0 to capital T, therefore this is valid for all values of k summation, this equation is called the Chapman Kolmogorov equation for a time homogeneous continuous time Markov chain.

Because here for this transition probability we have used Markov property as well as the time homogeneous property also, therefore this is a Chapman Kolmogorov equation of a transition probability of system moving from i to j, in small t + capital T can be broken into product of these for all possible values of t, so like this you can break it many more ways with the summation for all for different state of k.

Using this, we are going to find out the transition probability of P i j of t, you remember to find out the distribution of X of t you need an initial state probability vector as well as the transition probability P i j of t, the initial state probability vector is always given you have to find out the P i j of t, once you know the P i j of t, you can find out the distribution of X of t for any time t