Stochastic Processes - 1
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Lecture - 50
Waiting time Distribution

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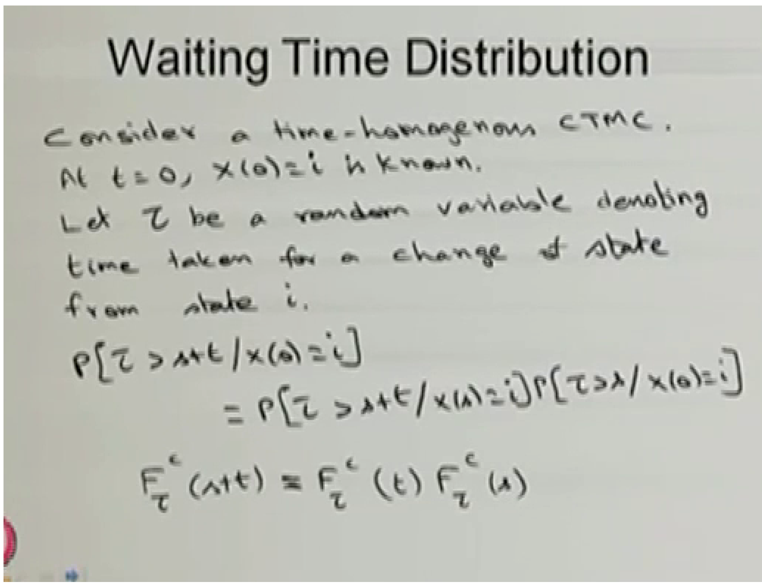

So before going to the p i j, you see the sample path of.

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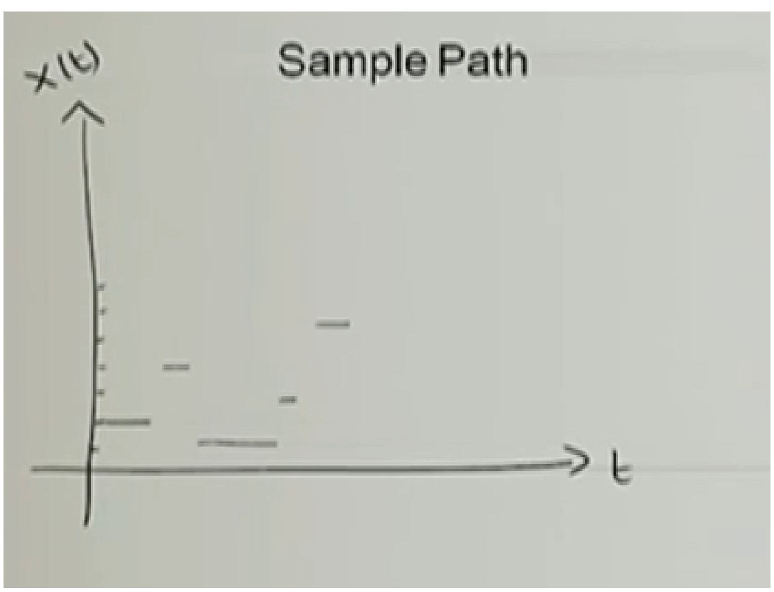

The sample path of a time homogeneous continuous time Markov chain as i said the system is staying for some positive amount of time in any state before moving into any other states, our interest is what is the distribution or what is the waiting time distribution of system being in any state before moving into any other states that is our interest to find out. So how we are going to find out that I am going to explain that is called the Waiting Time Distribution.

That means what is the distribution of a time spending in any state for a time homogeneous continuous time Markov chain before moving into any other states, I assume that at time 0 system was in the state i, that means X of 0 is equal to i, that is known or the probability of X of 0 is equal to i that probability is 1. Let me make out the random variable tau that is a random variable denoting the time taken for a change of state from the state i,

Change of state means it does not matter which state it goes my interest is to find out what is the waiting time distribution for the state i, the time spent in the state i, for that let me make a simple graph.

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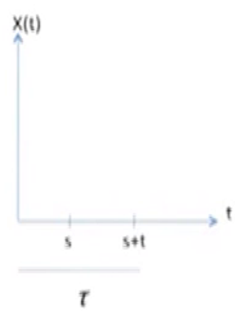

$$P[\tau > s + t \mid X(0) = i] = P[\tau > s + t \mid X(0) = i, \tau > i]$$

$$\times P[\tau > s \mid X(0) = i]$$

$$= P[\tau > s + t \mid X(s) = i]$$

$$\times P[\tau > s \mid X(0) = i]$$

$$F_{\tau}^{c}(s + t) = F_{\tau}^{c}(t) F_{\tau}^{c}(s)$$

So this is t and this is X of t, suppose you assume that the system was in the state i at the time point 0 after some time it moved into some other state, okay at the time point s it was in the state i, at the time point t also it moved into the some other state, so the tau here is nothing but the time spent in the state i from here to here so that is a random variable. So what I am going to do I am going to find out what is a compliment CDF for the random variable tau.

That is a - what is the probability that the tau greater than s + t given that X of 0 is equal to i, that is same as the probability of the tau is greater than s + t given that I can introduce one more condition tau is greater than s, than I can multiply by using the total theorem of probability tau greater than s given that X of 0 is equal to i. That is same as, the first one I can rewrite as a probability of tau greater than s +t given that X of s is equal to i.

Because X of 0 is equal to i, as well as tau is greater than s, where tau is the time spent in the state i, therefore I can make out X of s is equal to i, by combining these two concept multiplied by the probability of tau greater than s given that X of 0 is equal to i, that is same expression here. Now the probability of tau greater than s + t given that X of s is equal to i, that I can rewrite because the - this Markov chain is a time homogeneous Markov chain.

So the s to s + t, that is same as the complement CDF of the random variable tau for the time t, because it is s to s + t, since its Markov chain is a time homogeneous only the length is matters that is the interval of length t. Therefore, this is nothing but the compliments CDF for the random variable tau the time point t multiplied by, this is nothing but 0 to s, so this is the compliment CDF of the random variable tau the time point s.

Whereas the left hand side is the compliment CDF for the random variable tau for the point s + t. So what we got the result is the compliment CDF of the unknown random variable tau at the time point s + t that is same as the product of complement CDF at the time point s and t, so this is valid for all s and t greater than 0, so we have to find out what is the random variable or what is the distribution going to satisfies this compliment CDF at the time point s + t, same as the product of compliments CDF at the time point s and t.

If any distribution satisfies this compliments CDF property then we can find out the distribution further and the variable tau, so in this derivation we have use the time homogeneous property as well as the total probability rule as well as we have use the Markov property therefore it lands up the compliment CDF satisfying this equation.

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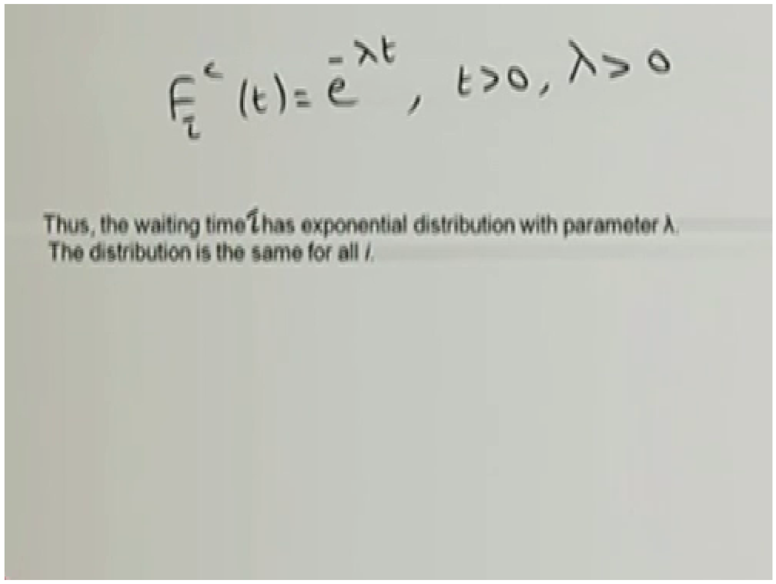

Now we have to find out what is the distribution going to satisfies this property so if you substitute any function with e power any - any - any parameter lambda with the exponential of e power minus lambda t the previous equation is going to be satisfied as long as the function is of the form e power minus lambda t, for lambda is greater than 0 and t is greater than 0, since the compliment CDF is e power minus lambda t.

Therefore, the CDF of the unknown random variable tau, that is 1 minus e power minus lambda t for t greater than 0, for some lambda and you know that if the CDF of the random variable is 1 minus e power minus lambda t for t greater than 0 and lambda greater than 0, then that random variable is a exponentially distributed random variable.

So we can conclude the amount of time or the time taken by the system staying in the state i, and that time is exponentially distributed that is a continuous random variable whose distribution is exponential distribution with the parameter lambda, even we can specify lambda suffix i, that means it is going to be a function of it depends on the i, that means if the random variable is going to spent in some state.

And that is always exponential distribution with some parameter lambda and that parameter lambda may depend on the state i, that means if I go back to the sample path I can say that the - that the time - that the time the system spending in this particular state that is exponential distributed with some parameters then it moved into some other state. The time spending in this state that is also exponentially distributed with some other it could be a some other parameter.

It depends on that particular state then it moved into the some other state and the time spending in this state that is also exponentially distributed and later we can conclude all these the time spending in each state because of it is a Markov property satisfied. The time spent in this state - the time spent in this state all are exponentially distributed which is independent of the other.

So all are going to be mutually independent random variables then only the Markov property is going to be satisfied that means whenever the system is moving from one state to another state we will have a exponentially distributed time spending in each state and they are form a mutually independent. And since the exponential distribution has the memoryless property, the system spending in this state.

If you just observe at any time t and what is the probability that the system will be for some more time in the same state given that it was spending already this much time in this state, then that is also exponentially distribution because of memoryless property of exponential distribution and which is independent of how much time spent in the same state already. Therefore, the Markov property is going to be satisfied throughout the time whether the system spending in this state or the other state and so on.

So the Markov property is going to be satisfied for all the time points and that time spending in each state is exponentially distributed and all the random variable spending in each state all are going to be mutually independent random variables. Now, we found out what is the time spending in each state and that is exponential distribution with some parameter lambda i and the distribution is same for all i, whereas the value of the parameter lambda maybe depends on the i.