Stochastic Processes - 1
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Lecture – 45
Definition of Reducible Markov Chains and Types of Reducible Markov Chains

Module 4 – Discrete-time Markov Chain. Lecture-7, Reducible Markov Chain. The last three lectures we have discussed Irreducible Markov Chain and this lecture we are going to discuss Reducible Markov Chain.

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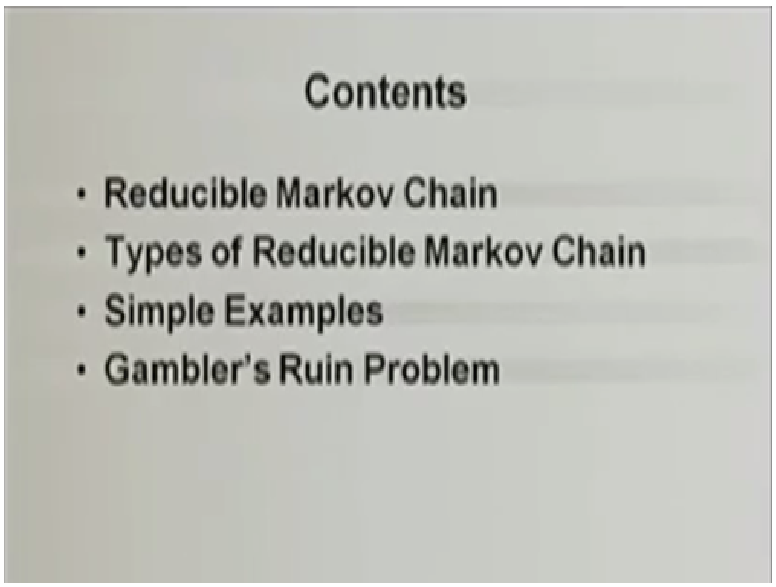

So, in this lecture, I am going to start with the concept of Reducible Markov Chain. Then I am going to give the different type of Reducible Markov Chain. And also I am go to present some simple examples then finally one important application of reducible Markov that is a Gambler's Ruin Problem- Gambler's Ruin Problem going to be discussed.

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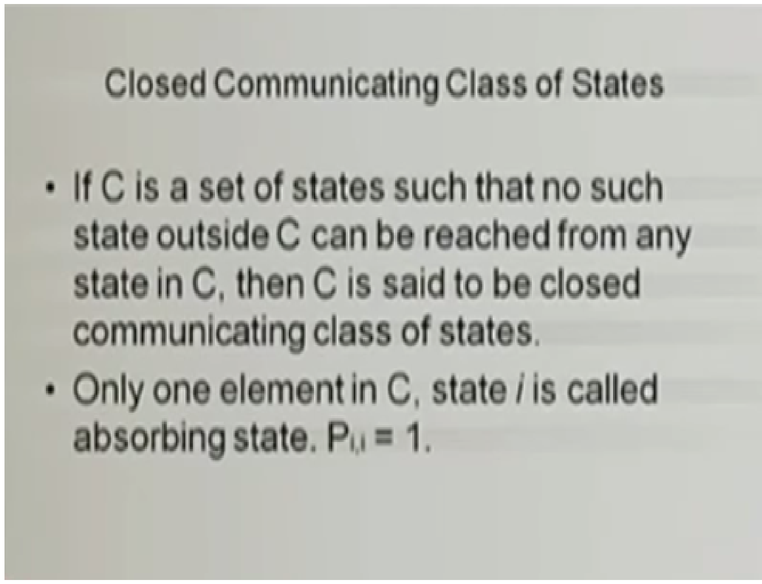

Before we discuss the Reducible Markov Chain, let me explain the Closed Communicating Class of States. This definition we have already given in the few lectures earlier also. Again I am giving. Using this, we are going to conclude the Markov Chain is a reducible Markov chain or irreducible Markov chain. The closed communicating class, suppose you collect the set of states that you label name with the it the C.

If that collection of set of states is going to be call it has a closed communicating class of states, if it satisfies no such state outside C can be reached from any state in C. Then C is said to be closed communicating class of states. If in a set of states forming a closed communicating class and it has only one element only one state, you cannot include one more state. So that it is going to be a closed communicating class of states then that class is a – in that class the state is going to be call it as an absorbing state.

And you know the definition of absorbing state that means one step transition probability i to i that is equal to one. So there are two ways you can have an absorbing state either P i,i = 1 is are the closed communicating class has only one element then tax rate is going to be absorbing state. So using a closed communicating class of states we are going distinguish or we are going to make the Reducible Markov Chain and Irreducible Markov Chain. How?

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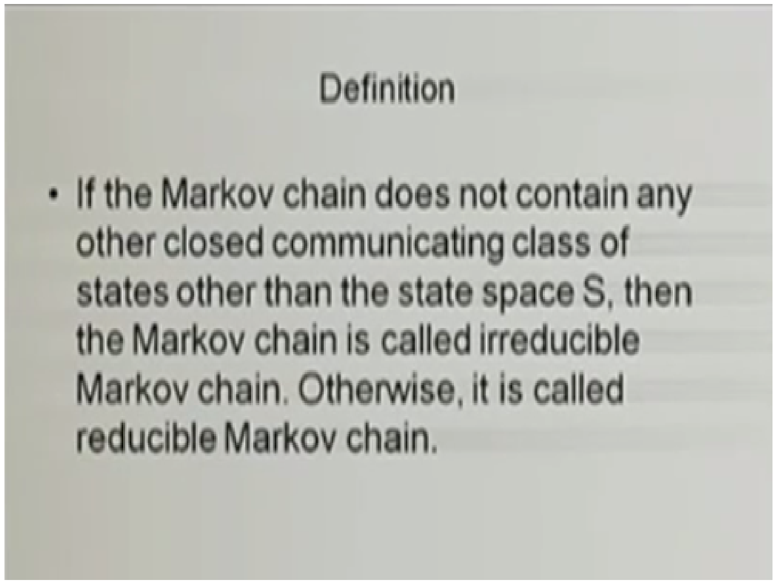

Let me see the definition of a Irreducible Markov Chain. If the Markov chain does not contain any other closed communicating class of states other than the state space S, then the Markov chain is called irreducible Markov chain otherwise, it is a reducible Markov Chain. That means, you have a Markov chain with the state space capital S. You are trying to create a closed communicating class.

If that class and the state space S both are one and the same that means all the states are going to form one closed communicating class that means each communicating with each other state and that is must state space, then that Markov chain is a reducible Markov chain. Otherwise, that Markov chain is going to be called as a reducible Markov chain. Before we go to the various reducible Markov chain, I am going to give few examples.

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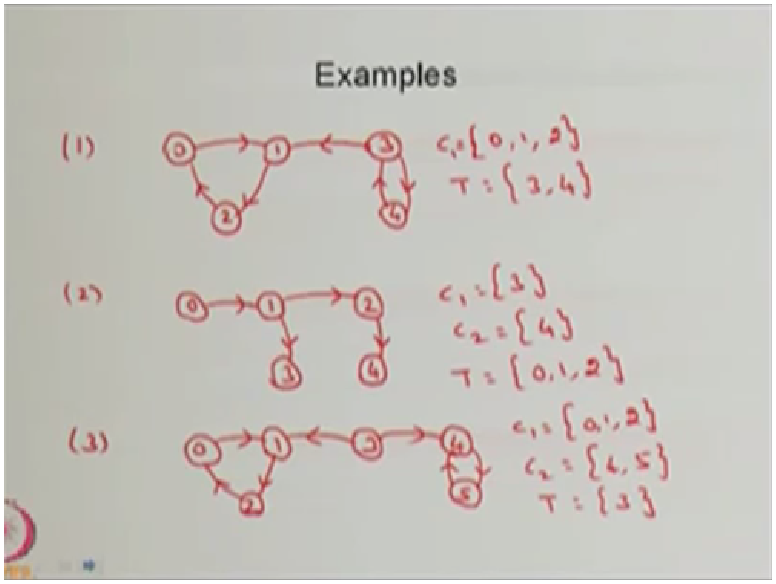

So through this example, we can make the classification over the reducible Markov chain. You see the first example; it has five states instead of the one step transition probability matrix I have drawn the state transition diagram so using this you can easily conclude whether it is going to be a reducible Markov chain or irreducible Markov chain. If you see the arc from 3 to 1 and the states 0, 1 and 2. All three are connected.

Therefore, you can conclude 0, 1 and 2 is going to form a closed communicating class because all the states inside that class are communicating each other there is no state is going away from this collection to outside that is why it is the closed communicating definition. Or else the 3 and 4 even though there is a communication between 3 and 4 states once the system goes from 3 to 1 it will not be back.

Therefore, the state 3 and 4 are going to be a transient state. The first visit if you find out F capital F i,i F for state 3 and 4 it is going to be less than one whatever will be the probability he have not assigned the probability. You can assign the probability to 0 to 1 and you will get the conclusion the states three and four are going to be the transient states.

So since this it satisfies the definition of a reducible Markov chain that means you have a closed communicating class which is rather than the state space, that means you have closed communicating class with the fewer elements then the state space 0, 1 and 2 and a few transient state therefore this Markov Chain as a reducible Markov Chain of some type, I am going to discuss later.

See the second example, this also has the five states. If you absorb you will conclude the states 0,1 and 2 are going to be the transient states whereas the state three as well as four are going to form two different closed communicating class but it consists of only one element in it only one state in it. You cannot include the state one along with three or you cannot include the state two along with four to create a closed communicating class.

Therefore, the states three and four will form a closed communicating class with one single state in each in it respectively and these two states are absorbing state also. So this is also going to form a reducible Markov chain. See the third example, this has a 3 + 3 six states, out of six states there is no backward arc to the state three therefore state three will be a transient state whereas state states 0, 1 and 2 form a closed communicating class.

Similarly, the states four and five will form another closed communicating class of states. So in the third example, we have a two closed communicating classes of states whereas as the first example you have one closed communicating class and transient states. You see that all these three examples you have a collection of transient states and close communicating class either one or many or the closed communicating class consist of only one element.

But all the states, all the model has the fewer transient state. Therefore, you can easily find out the reducible Markov chain whenever it is not going to form a only one closed communicating class with all the states if that is not there then all other things are going to be the – and all other things are going to be a reducible Markov chain. So based on this three examples-- there are some more example I can create with the infinite state.

But here I have not made it but based on these three examples you can have some idea how one can have a various types of a reducible Markov chain.

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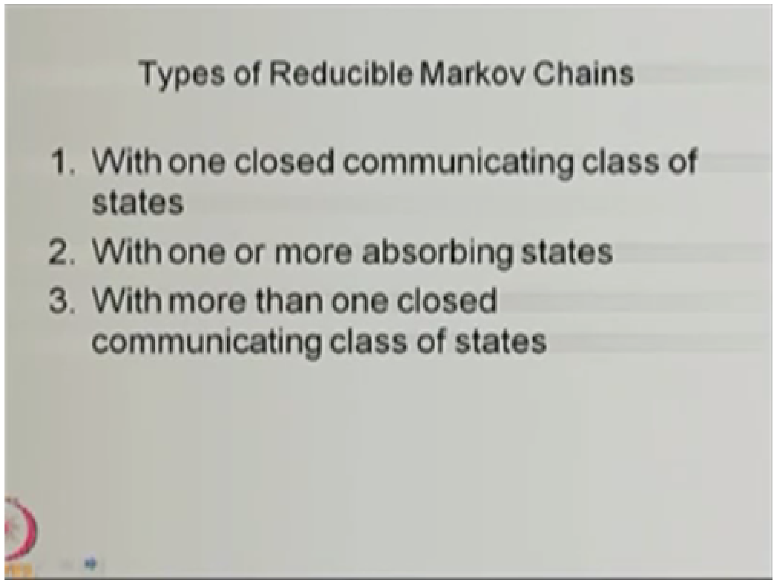

I am listing here in these the default is all the types has the few transient states along with that it has a one closed communicating class of states that is one type. One or more absorbing states that is similar to the example two. The first one is similar to the first example. The third one is with more than one closed communicating class of states that is related to the third example, but here I have not specified whether it is a finite state or infinite state Markov chain.

So immaterial of that the reducible Markov chain can be classified into these three in general. So we are going to discuss out of these three the first two we are going to discuss in detail and the third I am not going to discuss. So the way we have discussed in the first model the similar logic and we use to study the third type also.

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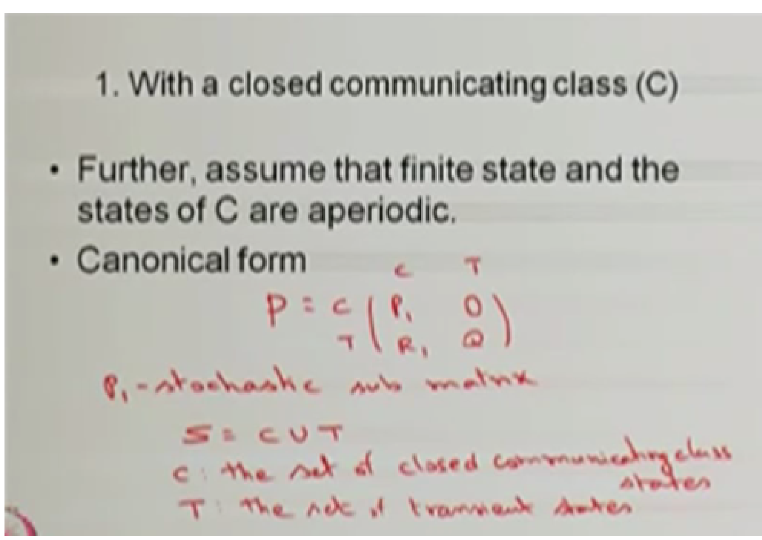

The first type which a reducible Markov chain that means it has the transient state and one closed communicating class. My interest is to study the stationery distribution. Therefore, I am making further assumption so that I can go for studying the stationary distribution, for that I am making the first assumption it is a finite state model, state spaces finite. And also this model state space has the one closed communicating class and the set of transient states.

So whatever the states in the closed communicating class that state I am making it a aperiodic. Aperiodic is important to study the stationary distribution therefore I am making the aperiodic. So this state space is the collection of the transient states as well as one closed communicating class therefore I am making a two notations C and T, C for the set of closed communicating class only one; the T is set of all transient states.

Therefore, the state space S is going to be the C union Capital T. Since it has one closed communicating class and set of transient states I am reordering the one step transition probability matrix such a way that the first few rows are corresponding to the states of the closed communicating class therefore I make it a C but inside suppose the state space the number of states in this reducible Markov chain is capital N.

There is a possibility some fewer elements fewer states maybe in the capital C. Therefore, fewer rows, that will make a sub matrix that is P1. That means C to C that sub matrix is or one step transition probability sub matrix P1. Whereas the one step transition probability going from closed communicating class that states to the transient states that probability is 0. Therefore, all the entities are 0.

Therefore, this 0 is nothing but a matrix, sub-matrix with the number of rows is number of states in the closed communicating class and the number of column that is same as the number of transient states. This is the way we reorder the one step transition probability matrix therefore C to capital T that is a sub matrix of zeros. The remaining elements are capital T that you reorder it in the other remaining rows.

Therefore, T to C will be some non-zero fewer elements that is a R1 matrix R1 sub matrix. And similarly, T to T there is possibility of therefore the probability is maybe greater than or equal to 0. Therefore, that matrix is a Q matrix. Therefore, the whole P matrix is divided partition into four sub matrices P1 0 matrix, R matrix and Q matrix. Since it is a 0 matrix entries of zeros therefore this P1 is also going to be Stochastic matrix the rows are going to be one.

And the integers are equal to 0 lies between 0 to 1. So these values are this sub matrix will form a stochastic, this is called a stochastic sub matrix. That means I am just reordering this P matrix that labeling the states such a way that first I am collecting all the states corresponding to the closed communicating class of states then set of transient states and this form is called a Canonical form.

For a reducible Markov chain this Canonical form is very important because once you are able to make a Canonical form then you can study the stationary distribution in an easy way.