Stochastic Processes - 1
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Lecture – 44
Time Reversible Markov Chain and Examples

Good morning. This is Model-4 Discrete-time Markov Chain. Lecture-6. Time Reversible Markov chain. And then Application of irreducible Markov Chain in Queuing models.

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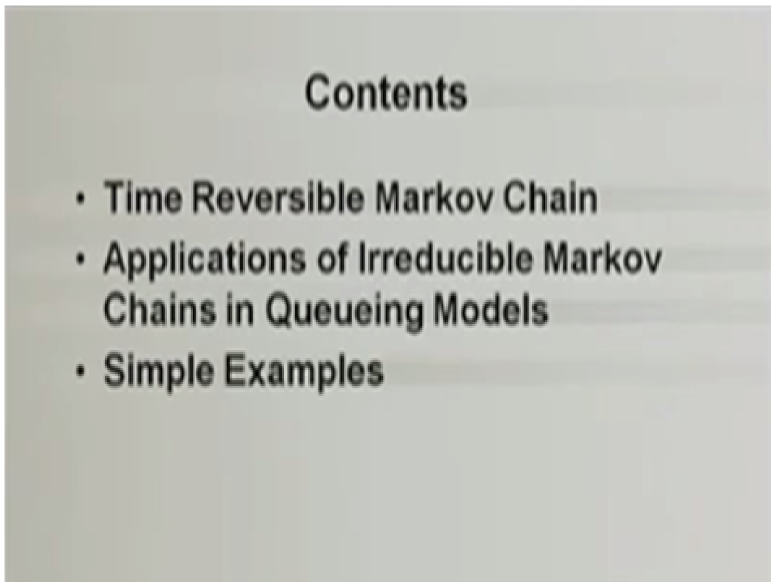

So in the lecture I am planning to give the Time Reversible Markov Chain and how to compute the Stationary Distribution in a easy way. Then I am going to give Applications of Irreducible Markov Chains in Queuing Models. Here irreducible Markov Chain mean it is a DTMC model because later we are going to give the applications of irreducible continuous Markov chain in Queuing Models also in the later lectures. And also I am going to give few simple examples.

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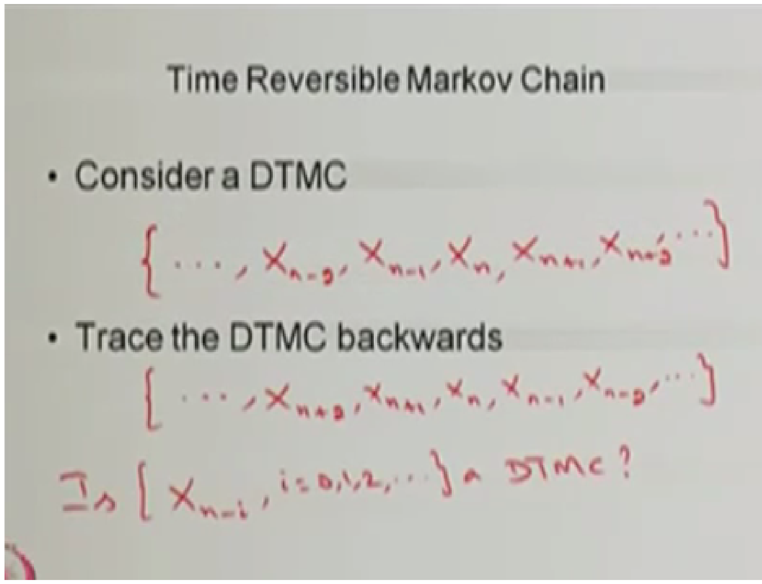

What is the meaning of Time Reversible Markov Chain? First let me explain how to construct the time reversible Markov Chain. Consider the DTMC it is of course time homogenous Discrete-time Markov chain. You see the collection it is usual way Xn-2, Xn-1, Xn+1, Xn+2 and so on. Now you trace the DTMC backwards. That means you know the first all Xn+2 then Xn+1 then you know what is Xn then you collected Xn-1, Xn-2 and so on.

Now the question is whether if you make a DTMC backwards and that sequence that sequence of random variable of course it is a stochastic process whether this is going to be a DTMC. Any Stochastic process is going to be a DTMC if it is satisfies the Markov property and the state spaces discrete and the time space is also discrete. But if you see this-- the DTMC backward that is also going to satisfy the Markov property the way the given situation the future depends only one the present and not past history of the Markov property.

The same thing is going to be satisfied in the reverse also. The Markov property satisfied by the reverse the backward DTMC therefore this is also going to be a time homogenous discrete-time Markov chain.

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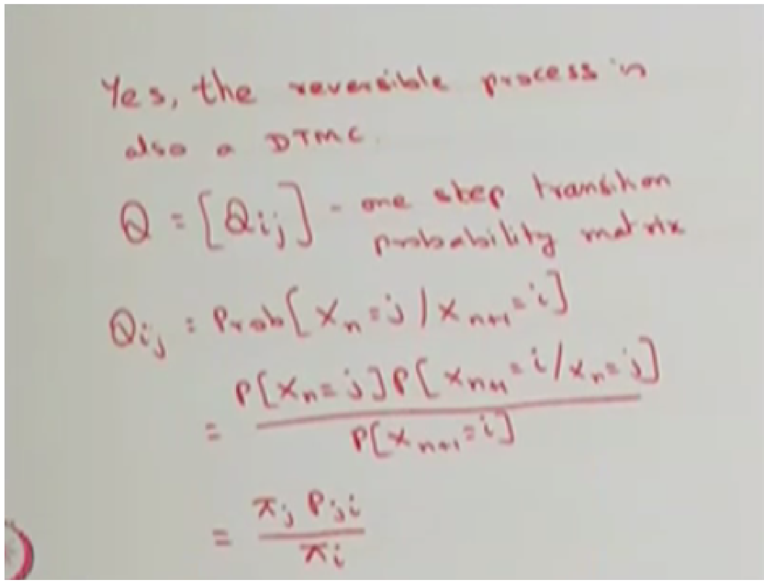

Now, I am going to define I am going to give how to find out the one step transition probability for the reversible process that you take it has the matrix Q that consists of elements Qij. You can find out the entries the Qij that is nothing but what is the probability that the system will be in the state j at the nth step given that it was in the state i at the n+1th step. That is a different from the original DTMC.

The original DTMC it is a what is the probability that the system will be in the Xn+1 in the state j given that Xn was i whereas here it is Xn is equal to j given that Xn+1 is equal to i. This conditional probability you can compute in this way the product of probability of Xn is equal to j multiplied by Xn is equal to i given that Xn is equal to j divided by what is the probability that Xn+1 is equal to i.

That is same as the probability of Xn is equal to j is nothing but in the steady state sorry in a what is the probability that at the nth stage in a system in the state that is equal to pi j multiplied by what is a probability that this is the one step transition probability of system is moving from j to i divided by i,i that means the Qij is going to be pi j Pji divided by pi i. Assuming that the stationary distribution exists otherwise I,j is equal limit entrance to infinite of Pij n.

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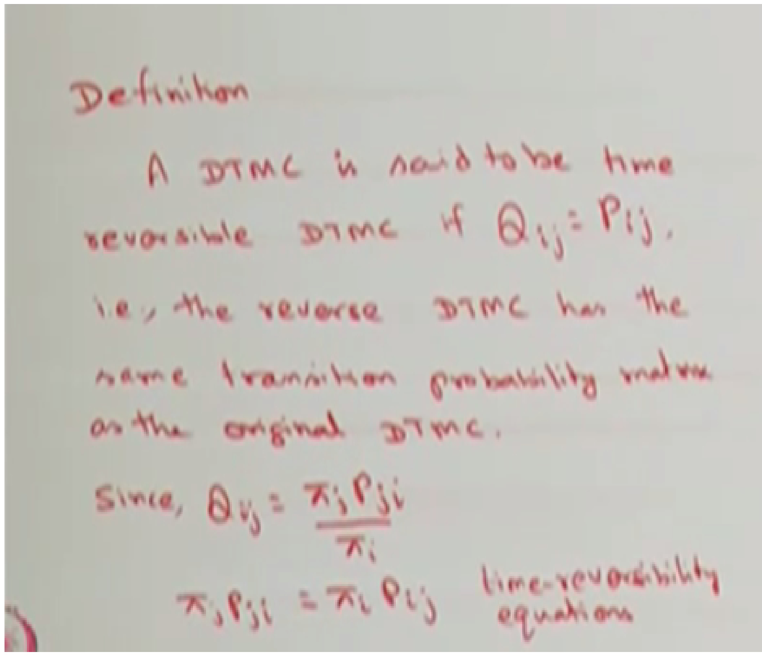

Now I am going to give the definition of time reversible. A DTMC is said to be a time reversible DTMC if both the transition probabilities are one and the same. That means the one step transition probability of the new or the time reversible process Qij is same as what is the one step transition probability of the original DTMC that is Pij. That is the reverse DTMC has the same transition probability matrix has the original DTMC.

Now I am going to equate this the Qj is equal to this much therefore that is same as Pij of Pji is equal to pi i Pij. And if this equation is going to be satisfied then that DTMC is going to be call it has a Time Reversible Markov Chain. And this collection of equation for all Pij that equation is called Time-Reversibly Equations.

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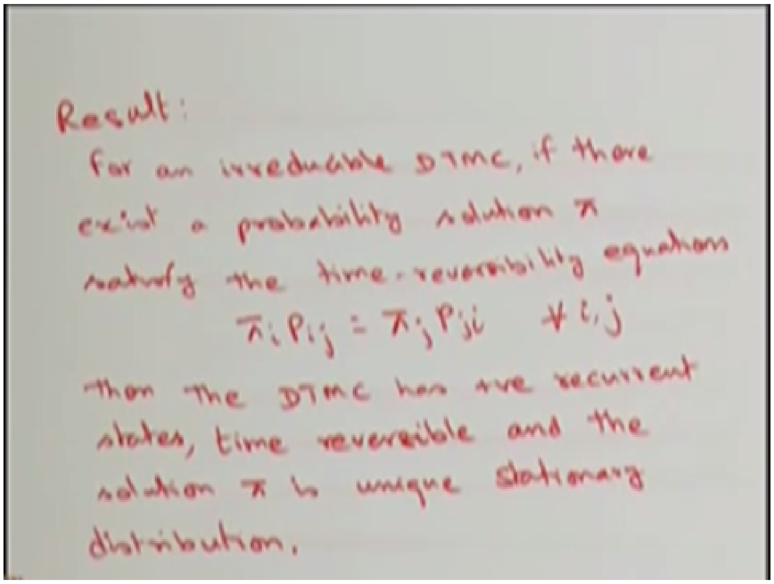

Now I am going to give the few results on Time Reversible Markov Chain. Let me take, irreducible DTMC, if there exist a probability that is a pi is a vector satisfies a time reversible equations that is the vector pi consist of a pi one, pi two and so on. So if that entries satisfies the time reversibility equation that is pi a is equal Pij is equal to Pji for all pairs of i,j then the DTMC has a positive recurrent states and also it is a time reversible and the solution pi is a unique stationary distribution.

That means that whenever you have an irreducible DTMC and if you have there exist probability solution vector pi satisfies the time reversibility equation then you can conclude the DTMC has a positive recurrent states as well as the DTMC is a time reversible Markov chain also the vector pi that satisfies that time reversibility equation that vector pi is a unique stationary distribution. So how one can use the time reversible concept in finding the stationary distribution that I am going to explain in the next example.

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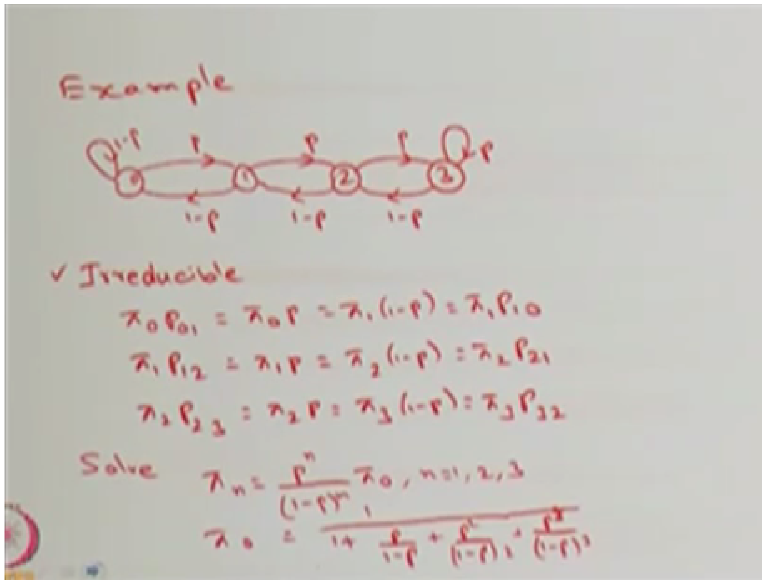

Let me take a simple example which consists of four states it is finite model finite state space. Also it is irreducible because each state is communicating with each other states. And I assume that the probability P that is lies between 0 to 1 therefore this is going to be a aperiodic states. So this Markov chain is a finite reducible and you had the result for finite irreducible Markov chain at least one state is going to be a positive recurrent.

So since it is irreducible all the states are of the same type therefore all the states are going to be a positive recurrent states and also it is a aperiodic. So you can use a result of irreducible aperiodic positive recurrent and also the finite states going to give the unique stationary distribution and that can be computed by solving pi p is equal to pi where pi is the stationary probability vector.

Here you can use the time reversibility concept therefore you do not want solve actually by this pi but you can start from the time reversible equation from that you can get the solution that is what I have done it in this example. First I checked it is a irreducible then I check whether the time reversible equation is going to satisfied by this irreducible Markov chain. So since it has the four state I am just checking all the states whether the time reversible reversibility equations are going to be satisfied.

Since it is you see the previous result for irreducible DTMC if they are exist a probability solution that means I started with their existed solution but since I know the result is irreducible aperiodic or positive recurrent in stationary distribution exist therefore I started with probability solution and I have checked time reversibility equations for the example also then I am concluding it is going to have a unique solution.

So I have verified the time reversibility equations. After the time reversibility equations from that I am getting the pi n terms of pi not because the way that recurrent relation goes you can make out pi p1 from the first equation you can get p1 in terms of pi not then the second equation pi 2 you can get it in terms of pi 1 then n terms you can get pi 2 in terms of pi not. Similarly, you can get a pi 3 in terms of pi not.

Now you have to find out what is pi not? pi not you can use the normalization equation that is summation of a pi i is equal to 1. That is a pi not plus pi 1 pi 2 plus pi three that is equal to 1 from that you can get a pi not is equal to one divided by 1 + p divided by 1 - p, p square one minus p whole square plus p cube by one minus p whole power three. So this is going to be the pi not substitute the pi not in this pi n therefore you got the-- you get the pi n also.

So you are getting the unique stationary distribution because this DTMC is a time reversible therefore, without solving the pi p is equal to pi, you are using the time reversibility equation itself and summation of a pi i is equal to one you are getting the pi i's, so that is an easy way whenever the DTMC is going to be a time reversible Markov chain. So in this example we have used times reversibility property to get the unique stationary distributions.

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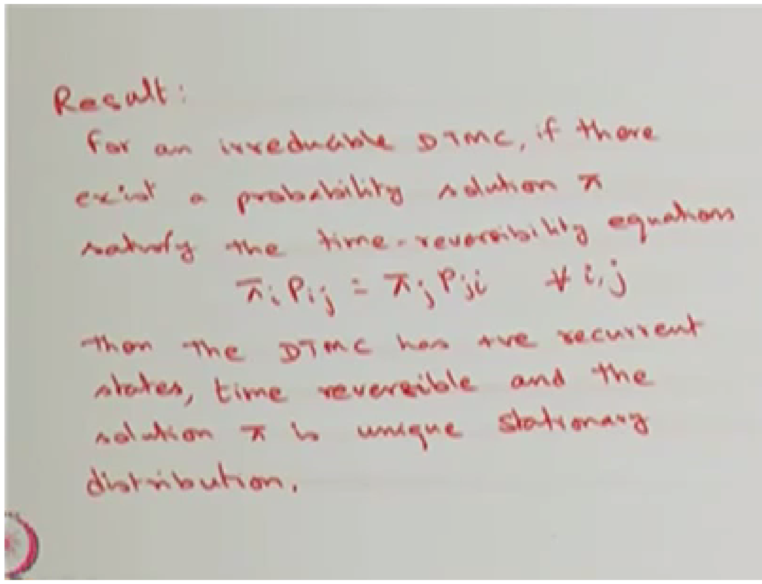

The result we said there is a and the solution pi is a unique stationary distribution so in the results means only the proof whether the pi is going to satisfies the equation pi p is equal to pi.

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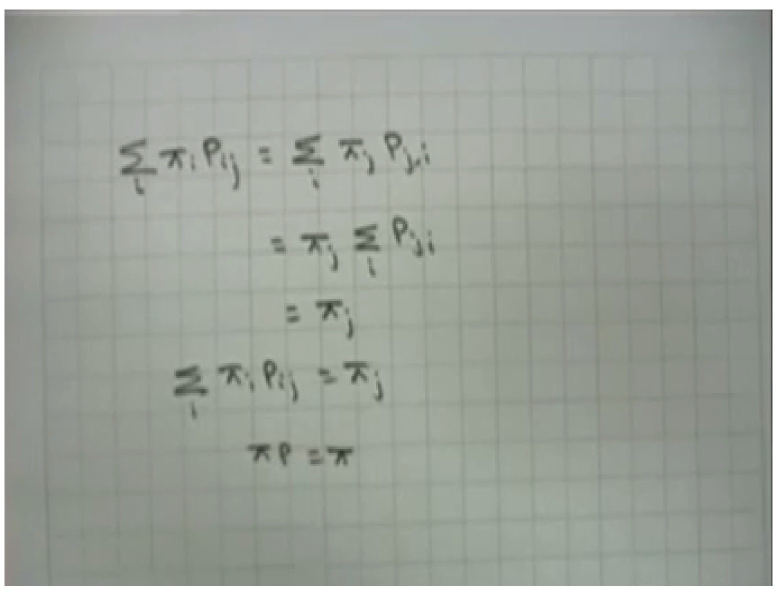

So that can easily provide by taking summation of pi Pij that is same as summation over i pi j of pji because it satisfies the time reversible equation you can write summation i pi times pij is same as the summation pi j pj,i that is same as you can take out the pi j outside that is same as the summation over i the pji. And you know that the summation over i the pji that is going to be one therefore this is going to be a pi j.

So hence we get summation over i the pi i of pij that is equal to pi j. So this is nothing but in the matrix form pi p is equal to pi. So whenever you have irreducible DTMC and satisfy the time reversibility equations then you have a unique solution pi and that pi unique solution is a stationary distribution. So and also you can prove easily it has the positive recurrent state and the time reversible Markov chain also.

So with this prove we have got the result that pi is going to be a unique stationary distribution and also I have given the example how to use the time reversibility equations to get the unique stationary distribution.