Stochastic Processes - 1
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Lecture - 40
Introduction and Limiting Distribution

Good morning, this is a module 4, lecture 5, Limiting Distributions, Ergodicity and Stationary Distribution. In the last 4 lectures we have discussed the discrete time Markov chain starting with the definition, transition probability matrix then in the second lecture we have discussed the Chapman-Kolmogorov equations.

Then we have discussed the one step transition probability matrix followed by that we have discussed the end step transition probability matrix in the lecture 3 we have classified the states of the discrete time Markov chain as a recurrent that is a positive recurrent and null recurrent transition states absorbing state and periodicity then we have in the fourth lecture we have given a simple examples.

In the fifth lecture, we are going to discuss the limiting distributions, ergodicity, stationary distributions.

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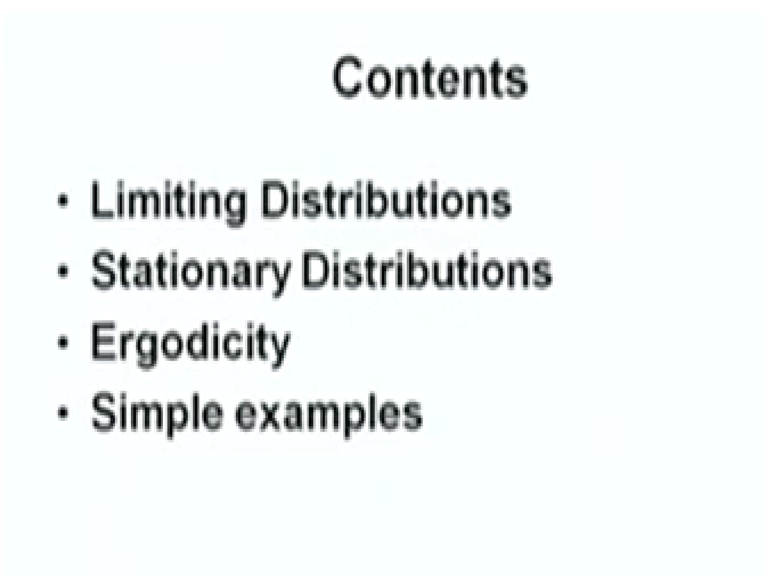

If I am not able to complete limiting distributions and the ergodicity then I will discuss the stationary distribution in the next lecture, and followed by the limiting distribution and ergodicity I am going to give some examples also. So the introduction what is the meaning of a limiting distribution? It is very important concept in time homogeneous discrete time Markov Chain.

And the limiting distribution is going to give some more information about the behavior of the discrete time Markov Chain and before I move into the limiting distribution let me discuss the some of the important results then I am going to give the limiting distributions.

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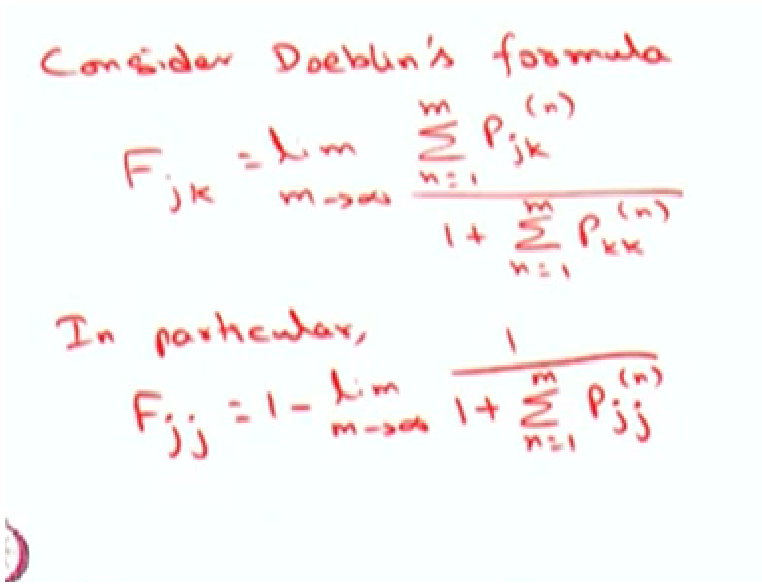

So consider the Doeblin's formula that is F j k in terms of a limit m tends to infinity of summation.

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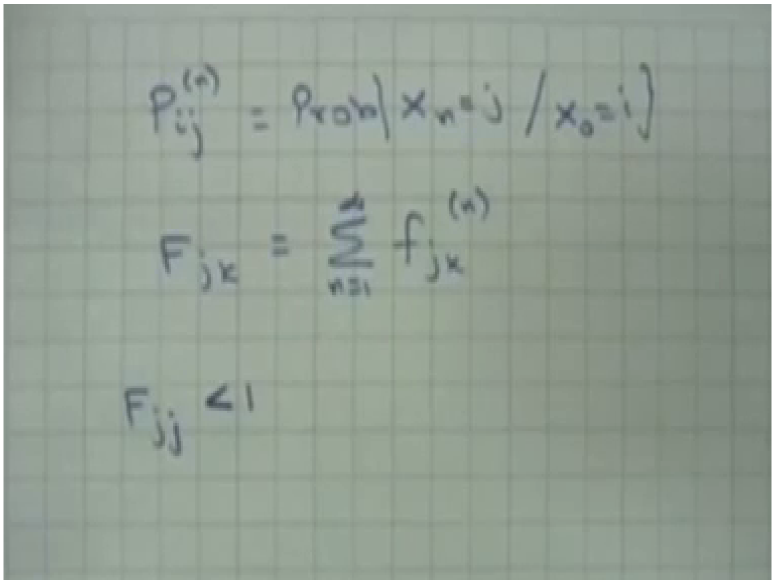

You we - we know that the P i j of n is nothing but what is the probability that the system will be in the state j given that the system was in the state i, whereas the f - F j k can be written as in terms of F j k n where n is running from 1 to infinity, here the f j k of n is nothing but the first visit to the state k starting from the state j in n-th step. And all the combination of n steps that will give F j k.

So now, you see the F j k is nothing but the limit m tends to infinity the summation divided by 1 + the summation in particular we can go for k equal to j, so that is nothing but 1 minus this, now based on the state is recurrent transient and so one I can discuss the further results.

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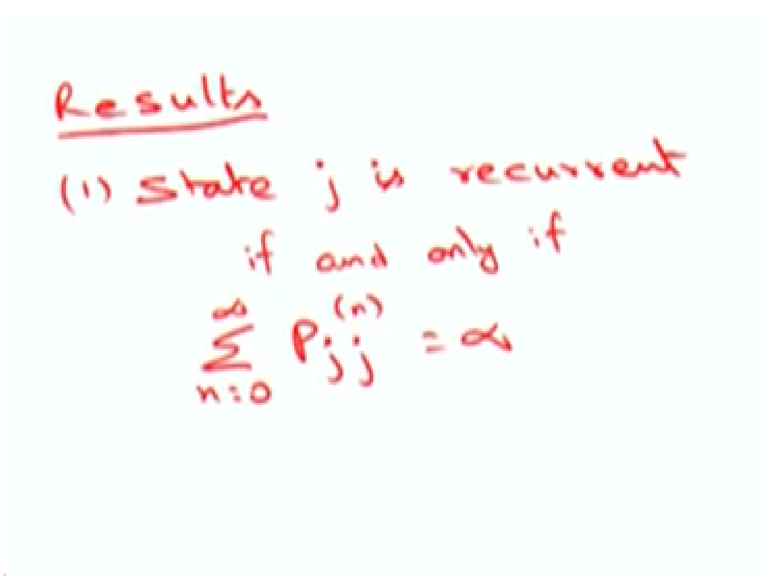

The first result the state j is going to be a recurrent if and only if the summation of P j j of n has to be infinity, the if and only if means, if the recurrent the state is recurrent then you can come to the conclusion this summation of the probability not the first visit starting from the state j to j in n steps, that summation is going to be infinity, if for any state j the summation is going to be infinity then that state is going to be recurrent.

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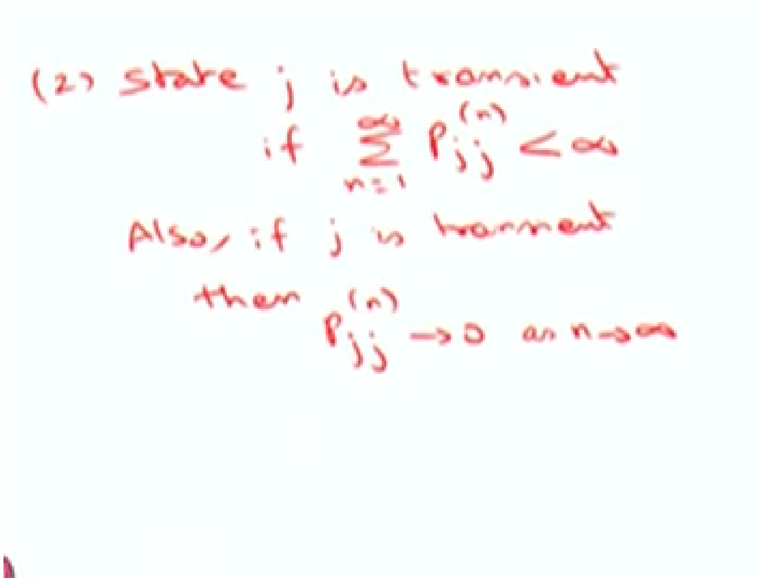

The second result suppose the state is a transient then you can have the P j j of n tends to infinity as n tends to infinity, this you can conclude easily, if the state is transient then you know that the F j j is going to be less than 1, the probability of the system coming back to the state is going to be less than 1, therefore the P j j of n tends to infinity as n tends to infinity for the transient state. And also if the state is a transient then sorry, if the summation is going to be a finite quantity then you can conclude the state is going to be a transient.

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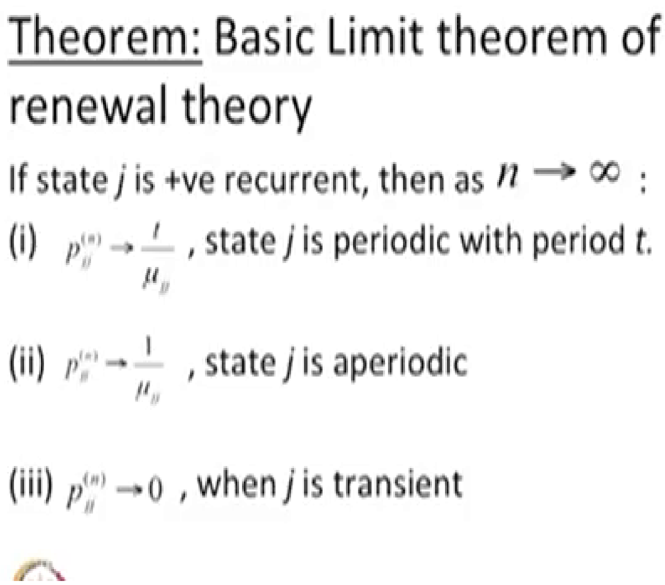

Based on this I am going to give the next theorem that is basic limit theorems of renewal theory. I'm not giving the proof here I am just only stating the theorem if the state j is a positive recurrent that means the state is going to be a recurrent as well as it satisfies the positive recurrent property that means the mean recurrence time is going to be a final value for that state j, then the P j j of n that will tends to t divided by mu j j.

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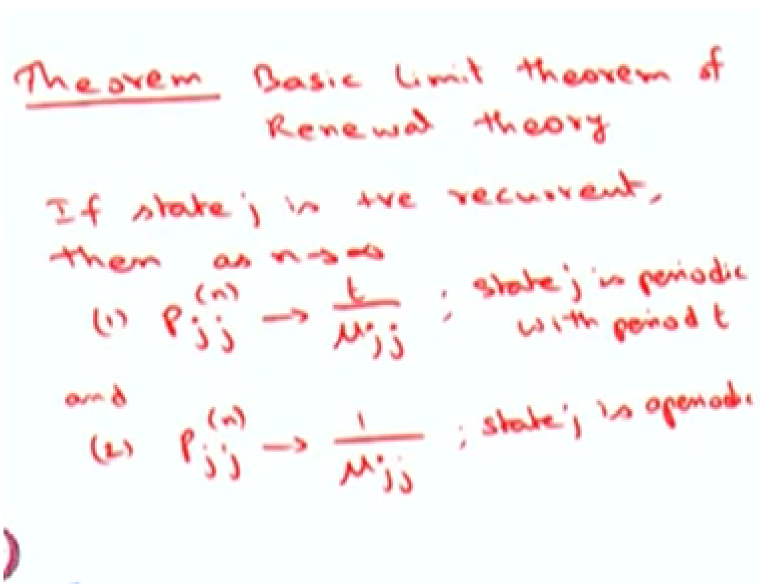

Where mu j j is nothing but the mean recurrence time for the state j and the t is nothing but the periodicity for the state j, if the periodicity is going to be 1, then as a n tends to infinity the P j j of n that is nothing but what is the probability that the system start from the state j and reaches the state j in n steps will tense to the 1 divided by the mean recurrence time for a positive recurrence state with the aperiodic, if state j is transient then limit P j j as n tends to infinity is 0.

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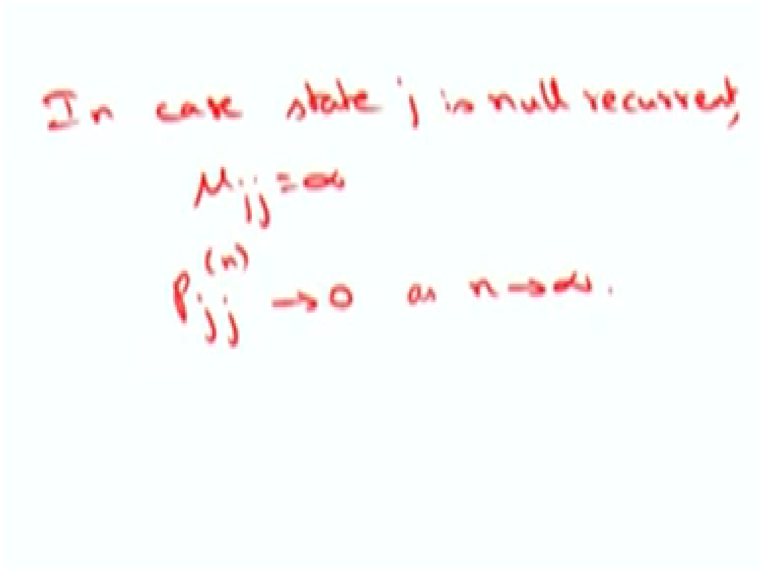

In a case of null recurrent if the state j is an null recurrent then you know that for a null recurrent the mean recurrence time is going to be infinity, therefore as n tends to infinity the P j j of n will tends to 0.

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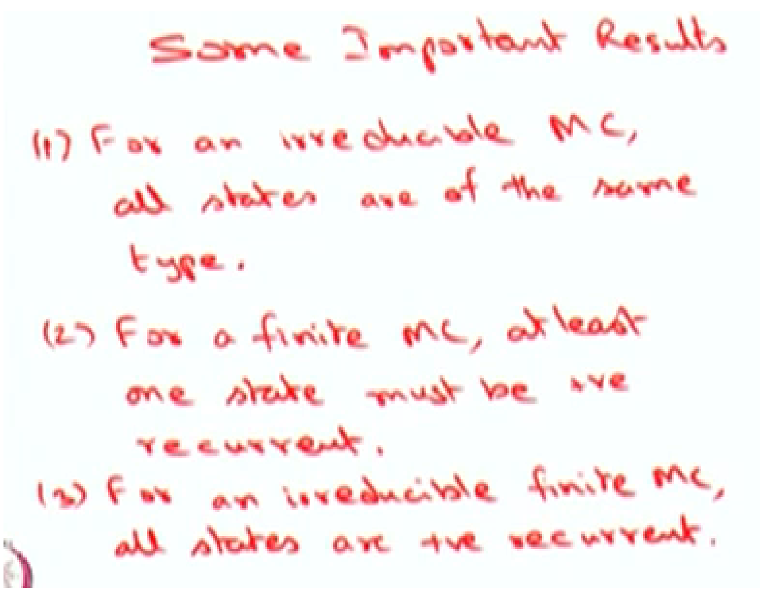

Now I am going to give some more important results for a discrete time Markov chain, here I am considering a time homogeneous of discrete time Markov chain only, so for irreducible Markov chain all the states are of the same type, that means if the Markov chain is going to be irreducible, that means which state is communicating with each other state then only the Markov chain is going to be call it as a irreducible Markov chain.

That means for our irreducible Markov chain all the states are of the same type that means if one state is going to be a positive recurrent then all the states are going to be positive recurrent, if one state is going to be a null recurrent then all the states are going to be null recurrent. The second result for a finite Markov chain, the discrete time Markov chain with the finite state space at least one state must be a positive recurrent.

This can be proved easily but here I am not giving that proof at least one state must be a positive recurrent, because it is a finite Markov chain that means it has a finite states, therefore the mean recurrence time that is nothing but on average time spending in the state starting from the state j and coming back to the states j that mean recurrence time that is going to be always a finite value at least for a one state.

Now I am combining the result one and two gives the third result that means the finite Markov chain has a at least one positive recurrent state and the first results state - first result states that if the Markov chain is irreducible then all the states are of the same type therefore the third result is for irreducible finite Markov chain. That means it is a time homogeneous discrete time Markov chain with a finite state space.

And all the states are communicating with all other states that is a irreducible, then all the states are going to be a positive recurrent.

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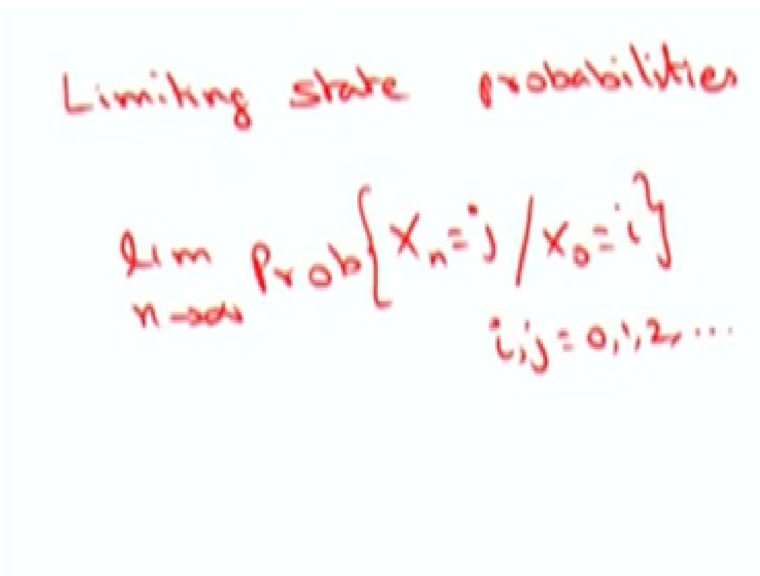

Now I am describing the limiting distributions, the limiting distributions means what is the probability that the system starting from the state i and reaches the state j as a n-th steps as n tends to infinity, so this is nothing but this is the definition of a Limiting state probabilities, we are only considering a time homogeneous discrete time Markov chain if this limit is going to exist then it is going to be unique.

So what is the limiting state probability for any time homogeneous discrete time Markov chain whether it will exist if it exists what is a value, so that's what we are going to discuss in the further in this class, in this lecture.

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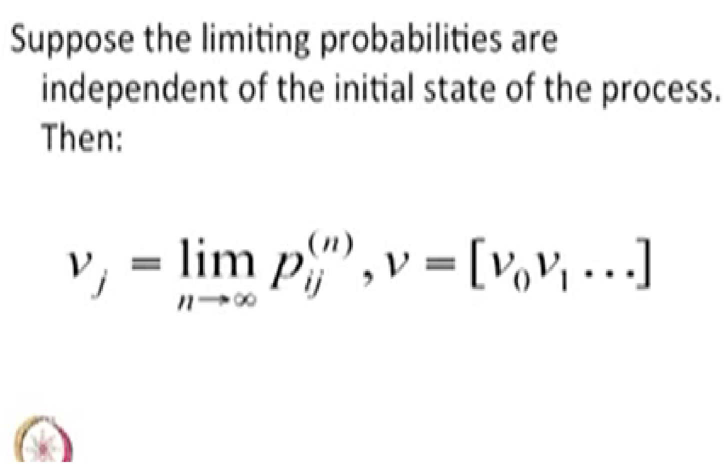

Suppose limiting probability is independent of initial state of the process V naught vector, suppose I am just making the assumption if the limiting probability is going to exist as well as if it is a independent of a initial probability distribution we can write as a V j, because that is nothing to do with I so V j is nothing but what is the limiting state probability of system being in the state j as n tends to infinity, that is nothing but limit n tends to infinity P i j of n.

So now I can write a vector V consists of V naught, V 1 so those entries are nothing but the limiting state probabilities.

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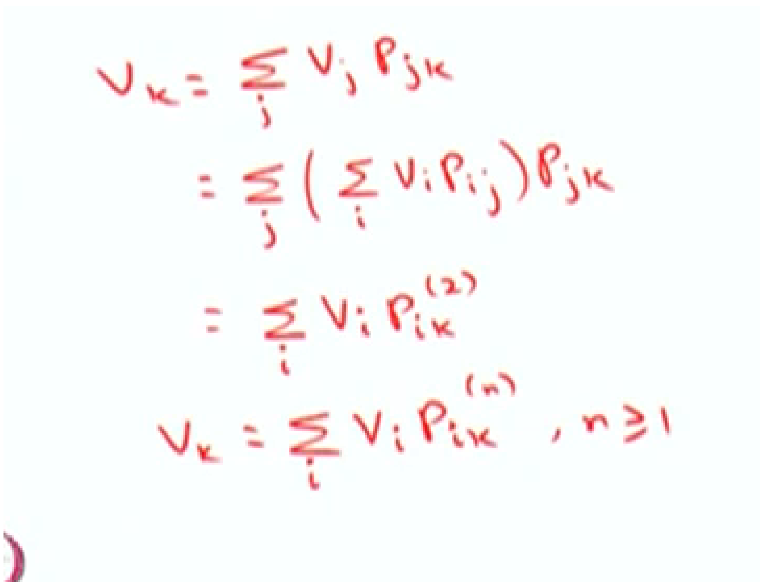

$$V_k = \sum_j V_j P_{jk}$$

$$= \sum_j \left(\sum_i V_i P_{ij} \right) P_{jk}$$

$$= \sum_i V_i P_{ik}^{(2)}$$

$$V_k = \sum_i V_i P_{ik}^{(n)}, \quad n \geq 1$$

So this I can compute as a V k is equal to summation j, V j, P j, k that means the P j k is nothing but the one step transition probability so that possibility summation will give V k, now I can replace V j by again the submission over i, V i, P i, j, I can do simple calculations it lands up V k is equal to summation i, V i, P i, k of 2.

Again I can repeat the same thing for V I, so I will get a V k is equal to submission over i, V i, P i, k of n, for n is greater than or equal to 1, that means that this is the entry of a n step transition probability matrix having the probability that is a probability of the system is moving from the state i to k in n steps for n is equal to 1 2 3 and so on.