Stochastic Processes - 1
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Lecture – 29
Introduction to Chapman-Kolmogorov equations

Good morning. This is Stochastic Processes Module-4. Discrete-time Markov Chain. And this is the Lecture 2. And in this lecture we are going to discuss about the Chapman-Kolmogorov Equations then we are going to discuss N-step Transition Probability Matrix and we are going to discuss a few more examples in this lecture.

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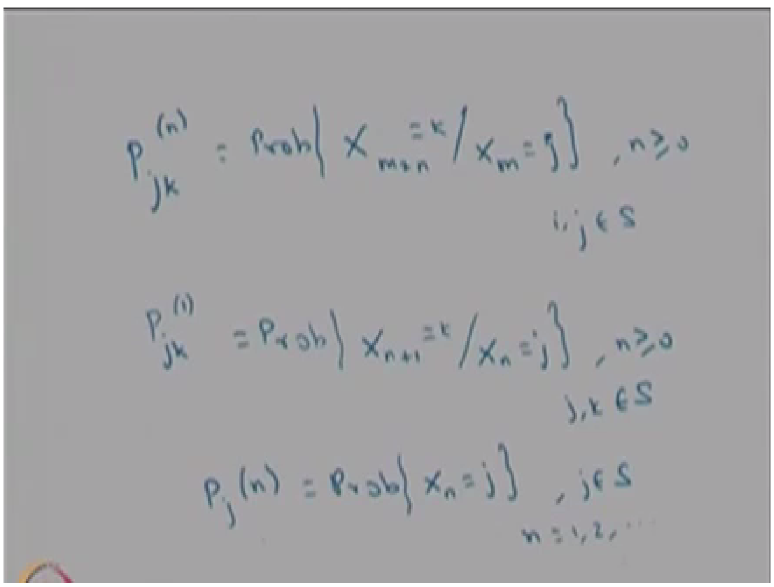

In the last class, we have discussed that transition probability of j to k in n steps has the probability that the xm+n takes a value k given that xm was j for m is greater than equal to 0 and j belonging to S. Since the underlying DTMC is a time homogenous this is the N-step Transition Probability of system is moving from the state j to k in n steps. So this we denoted as the conditional probability of pj, k in N-step Transition Probability where i,j is belonging to S where S is the state space and n can take the value greater than or equal to 0.

Also we have discussed in the last class what is the one step transition probability of pj, k it is-- we can write it within the bracket one or we can remove the bracket one in the Stochastic also that I nothing but what is the probability that the system will be in the state k in n+1th step given that it was in the state j in the nth step. Here also j, k belonging to capital S. So this is the one step transition probability.

So our interest is to find out what is the distribution of Xn. Another basic sequence of random variable xn is time homogenous a DTMC our interest is to find out the distribution of xn. So it has the probability mass function the pj of n that is nothing but what is the probability that the system will be in the state j at the nth step. So if the j is belonging to S and the n can be 1 or 2 and or so on because you know the distribution of in is equal to 0 that means you know the initial probability vector of x0.

So our interest is to find out what is the distribution of xn for n = 1,2,3 and so on. So how we are going to find out this distribution?

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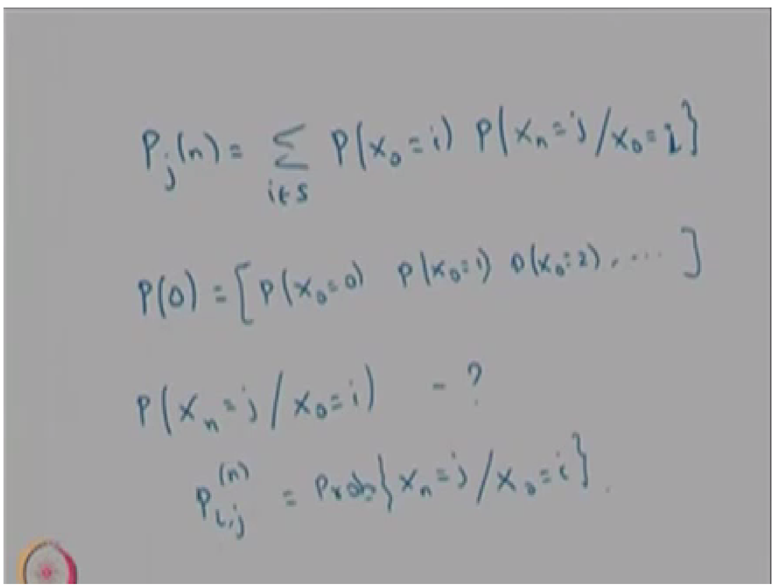

So this distribution can be written using the pj of n is nothing but the submission over i belonging to S such that the system was in the state i at 0th step and multiplied by what is the probability that it will be in the state j given that it was in the state i at nth -- at 0th step. So this is nothing but what is the probability that the system will be in the state j the nth step that is same as what are all the possible ways the system would have been moved from the state i from the 0th step the state j in the nth step.

So this is the product of one marginal distribution and one conditional distribution for all possible values of i that gives the distribution of a xn in the nth step. So for that you need to compute this distribution of xn you need N-Step Transition Probability as well as the initial distribution vector or initial probability vector or the distribution of x not. So the distribution of x not can be as a vector p of 0, this vector p of 0 it consists of the element what is the probability that x not takes the value 0.

What is the probability that x not takes the value 1? What is the probability that x not takes value 2 and so on? So this is the initial probability vector. Why we have taken the state 0, 1, 2 and so on unless otherwise as mentioned the set of state space that is going to be the possible values of 0, 1, 2 and so on unless otherwise it is assume it you can take always this values. So this is the initial probability vector or initial distribution vector.

So what we need, what is the N-Step Transition Probability of the system will be in the state j given that it was in the state i at the 0th step. This is what we want to find out. What is the additional probability mass function of N-Step Transition Probability vector? So that we can write it in the form of pi, j of superscript n that is nothing but the probability of the system will be in the state j given that the system was in the state i at the 0th step.

That is that we need to compute the N-Step Transition Probabilities that is that pi j of n. So this can be computed by using the method called Chapman-Kolmogorov Equations. So this Chapman-Kolmogorov Equations provide a method for computing this N-Step Transition Probability-- probabilities. So how we are going to derive this Chapman-Kolmogorov Equation that I am going to do it in the—do it now.

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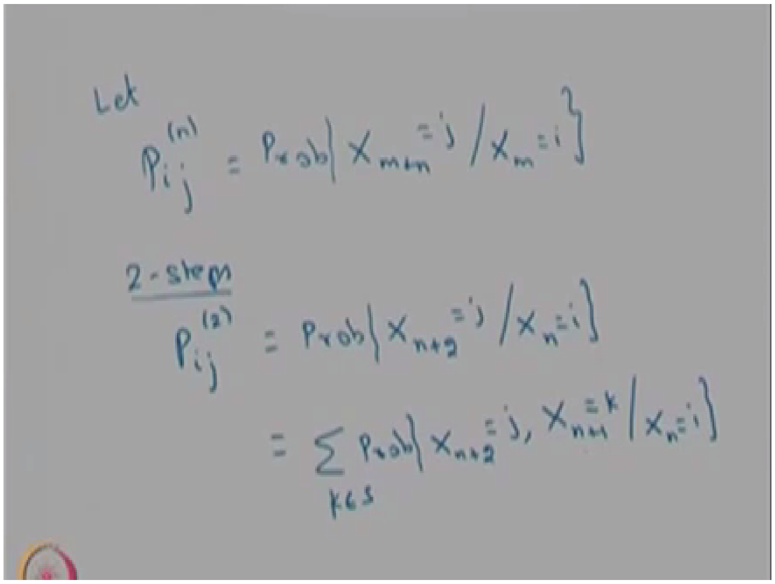

So we are going to derive the Chapman-Kolmogorov Equations for the time homogenous Discrete -Markov Chain. So let the pi j of superscript n that is nothing but what is the probability that the x m + n takes the value j given that x m was i. Since the Discrete-Markov Chain is the time homogenous so this is the transition probability of system moving from the state i to j from the mth step to m + nth step.

Therefore, this transition is the nth transition probability matrix for the time homogenous Discrete-Markov Chain. Let us start with the 2-Steps. The 2-Step is nothing but what is the probability that system is moving from the state i to j in two steps. So n + 2 takes the value j given that xn was I, this is for all n it is 2 because discrete DTMC is time homogeneous.

So this probability you can write it as-- this 2-Step transition probability of system moving from i to j the state i to the state j in 2-Steps that that you can write it as what are all the possible ways the system is moving from the state i to j by including one more state in the first step the state is k given that the system was in the state i in the nth step for all possible values of k belonging to S.

I can write this conditional 2-Step-- conditional probability mass function from the nth step to n+ second step that is same as I can include one more possible state of k in the n+ 1th step.

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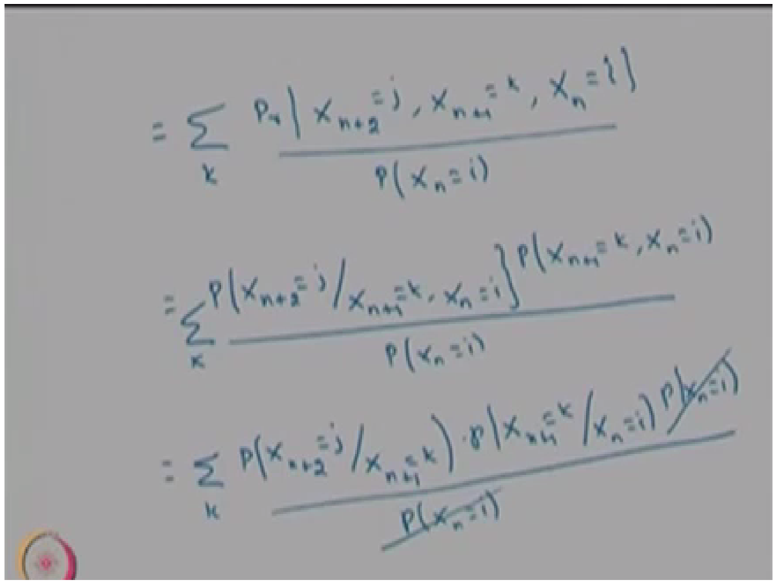

Now I can expand this as that is same as possible values of k and what is the probability that the system was in the state j in the n+ second step and the system was in the state k in the n+ 1th step; the system was in the state i in the nth step divided by what is the probability that in the nth step it is the state i. The numerator join distribution of this three state-- this three random variable that I can write it as in the form of conditional.

What is the conditional probability that the xn+2 takes the value j given that xn+1 takes the value k and xn takes the value i product of xn+1 takes the value k, xn takes the value i divided by what is the probability that xn takes the value i and here submission is over the k. So basically I am writing the numerator join distribution of this three random variable as the product of conditional distribution with the marginal distribution of those two random variable.

Since the Xi's are time homogenous Markov Chain this conditional distribution by using the Markov property is same as the conditional distribution of xn+2 takes the values j given that only the latest value is important the latest value is needed not the previous history therefore-- because of the memory less property xn takes the value i is removed therefore this conditional-- distribution is the conditional distribution only xn+1 with xn+2.

And similarly, I can apply the join distribution of this two random variable xn+1 and xn I can again write it as the probability of xn+1 takes the value k given that xn takes the value i and the probability of xn takes the value i whole divided by probability of xn and takes the value i. So this and this get canceled; so this is nothing but the conditional probability; this is nothing but the one step transition probability of system moving from k to j and the second term is a one-step transition probability of system is moving from i to k.

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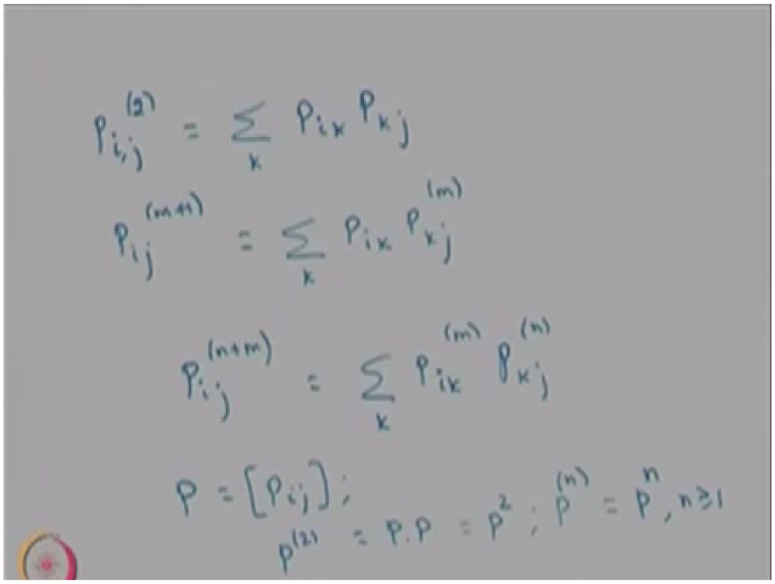

Therefore, the left hand side we have what is the 2-Step transition probability of i to j k is same as all possible values of k. What is the one step transition probability of system is moving from i to k and one step transition probability of k to j? So this product will give the 2-Step transition probability of system is moving from the state i to j that is same as what is the possible values of k the system is moving from state i to k and k to j. So this is for the 2-Step.

Similarly, by using the induction method one can prove i to j of m+1 steps that is same as what is the possible values of k the system is moving from one step from i to k and m steps from k to j. This is the 2-Step. So this is one step from i to k and one step from k to j by induction I can prove the m+1 step will be i to k and the k to j in m step. Similarly, I can make it the other way round also. It is i to k in m steps and k to j in one step also.

That combination also land up a the m+1 step the system is moving from i to j. In general, we can make the conclusion the system is moving from i to j in n+m steps that is same as the possible values of k of probability of system is moving from i to k in the m steps and by n step the system is moving from k to j that will give for all possible values of k that will give the possibilities of system is moving from i to j in n+ m steps.

So this equation is known as Chapman-Kolmogorov Equation for the time homogenous Discrete-time Markov Chain. So whenever you have a Stochastic Processes time homogenous Discrete-Markov Chain then that satisfies this equation and this equation is known as the Chapman-Kolmogorov Equations. In the matrix form you can write the capital P is the matrix which consist of the element of one step transition probabilities.

In that case if you make m=1 and n=1 then the matrix of P of superscript 2 that is a matrix form of 2-Step transition probability that is nothing but if you put n=1 and m=1 you will get P into P and that is going to be P square. So the right-hand side P superscript within bracket 2 means it is a 2-Step transition probability matrix and the right hand side the P square that is the square of the P matrix where P is the one step transition probability matrix.

So in this form in general you can make the N-Step Transition Probability matrix is nothing but P of n for n is greater than or equal to 1 for n=1 it is obvious for n= 2 onwards the P power n that is same as the N-Step Transition Probability matrix.

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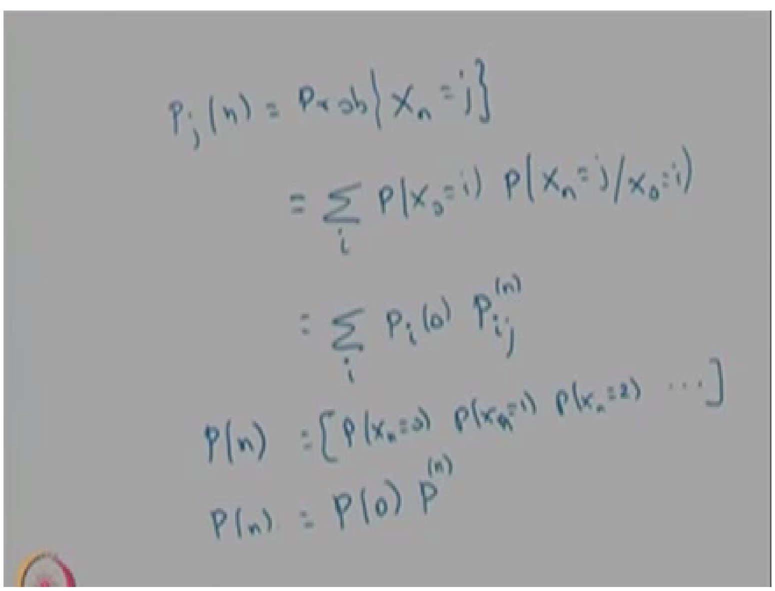

Hence, so now we got the N-Step Transition Probability is nothing but the P power n where P is the one step transition probability matrix therefore in matrix form I can given the P of n is nothing but in the matrix form of the distribution of xn or this is nothing but the vector which consist of a the nth step where the system will be. So this is nothing but what is the probability that in the nth step the system will be in the state 0 or in the nth step.

The system will be in the state one and in the nth step the system will be 2 and so on. This is the vector. So P of n you can find out in the matrix form by using the above equation; it is going to be P of 0 that is also vector initial probability vector multiplied by P power P of within bracket n that is the N-Step Transition Probability matrix. But the N-Step transition Probability matrix is nothing but the P power n therefore this is same as the P of 0 into P power m.

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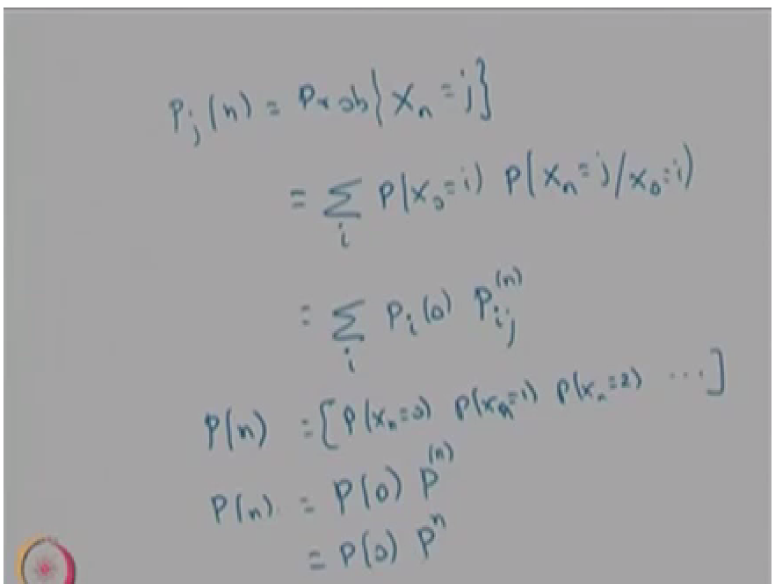

In the last slide, we got P of superscript within bracket n that is the N-Step Transition Probability matrix is same as the one step transition probability with the power n therefore this is going to be the distribution of xn in the vector form that is same as the P0 multiplied by the P power n where the P is nothing but the one step transition probability matrix. That means if you want to find out the distribution of xn for any n you need only the initial probability vector and one step transition probability metrics.

Because the Discrete-time Markov Chain is the time homogenous we need only the one step transition probability matrix and the initial probability vector that gives to find out the distribution of xn for any n. So with the help of one step transition probability and the initial probability vector you can find the distribution of xn for any n.