Stochastic Processes - 1

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Lecture - 26
Introduction to Discrete time Markov Chain (Contd.)

Discrete time Markov chain, I am going to give the formal definition of a discrete time Markov chain.

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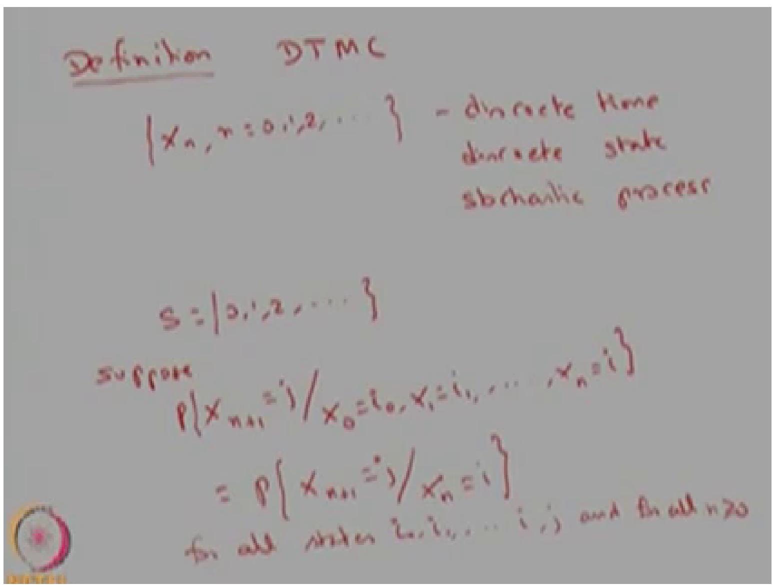

Formal definition of discrete time Markov chain in notation we - in short we call it as a DTMC, consider a discrete time discrete state stochastic process, consider here discrete time this is a discrete time discrete state stochastic process, assume that Xn takes a finite or countably countable number of possible values unless otherwise mentioned this set of possible values will be denoted by the set of non-negative integers S is equal to 0 1 2 and so on.

Unless otherwise measured you can mention you can always assume that the states space S consists of the element 0 1 2 and so on, even if you take a other values also you can always make a one to one correspondence and make the state space is going to be a S is equal to 0 1 2 so on. Suppose the probability of the Xn + 1 will be taking the values j given that it was taking the value X naught is equal to i naught, X1 was i 1 and so on.

And Xn was i, that probability is same as the probability of Xn + 1 will be j, given that Xn was i, for all states - for all states i naught whatever be the value of i naught, i 1 and j and also for all n greater than or equal to 0, if this property is satisfied by for all states i naught, i 1, i, j as well as for all n greater than or equal to 0. Then this stochastic process that is a discrete time discrete state stochastic process is going to be known as a discrete time Markov chain.

So basically this is a Markov property and the Markov property is satisfied by all the states as well as all the random variables so if this Markov property is satisfied by any stochastic process then it is called a Markov process and since it is the time space is discrete and parameter space is discrete therefore it is called a discrete time Markov chain.

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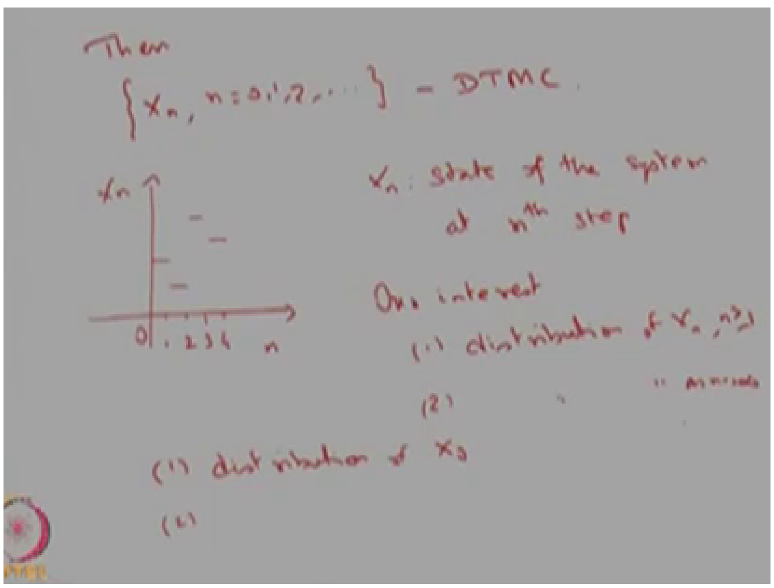

Then we see this stochastic process Xn takes value n starting with 0 1 2 and so on is the discrete time Markov chain, we can just have a look of how the sample path look like for different value of n and the y axis is Xn, suppose at n is equal to 0, it started with some value at X equal to n equal to 1, it would have been a different value and n is equal to two it would have been the different value.

So these values are the either it could be a finite value or countably infinite number of values therefore the state space is going to be discrete and the parameter space is going to be discrete, so like that it is taking a different value over the n, so this is going to be the sample path or trace of the stochastic process Xn, suppose you assume that Xn is the state of the system at n th step or n th time point and this Xn satisfies the previous this Markov property.

Then the stochastic process is going to be call it as a discrete time Markov chain and our interest will be suppose the stochastic process satisfies the Markov property our interest will be to know the two things one is what is the distribution of Xn for n is greater than or equal to 1, you know where the system starts so X naught you - you know, your interest will be what could be the distribution of Xn that is nothing but what is the probability that the Xn will be in some state j.

And also what could be the distribution of Xn as n tends to infinity, as n tends to infinity our interest will be finding out the distribution of Xn, so at any finite n as well as n tends to infinity that will be of our interest, to compute this you need two things, one is you need what is the distribution of X naught that is a initial distribution vector, where the system starts at the zero th step, what is the distribution of X 0.

And also second things of your interest will be, what is the transition distribution or how the transition takes place, what is the distribution of a transition from any n th step to n + 1 th step for all n, so if you know the two things the initial distribution vector as well as the distribution of the transition from n th step to n + 1 th step, using these two quantity you can find out what is the distribution of Xn for any n as well as you can find out the distribution of Xn as n tends to infinity.

For that I am going to define few conditional probability distribution as well as the marginal distribution for the random variable Xn through that we are going to find out the distribution of Xn for any n as well as n tends to infinity.

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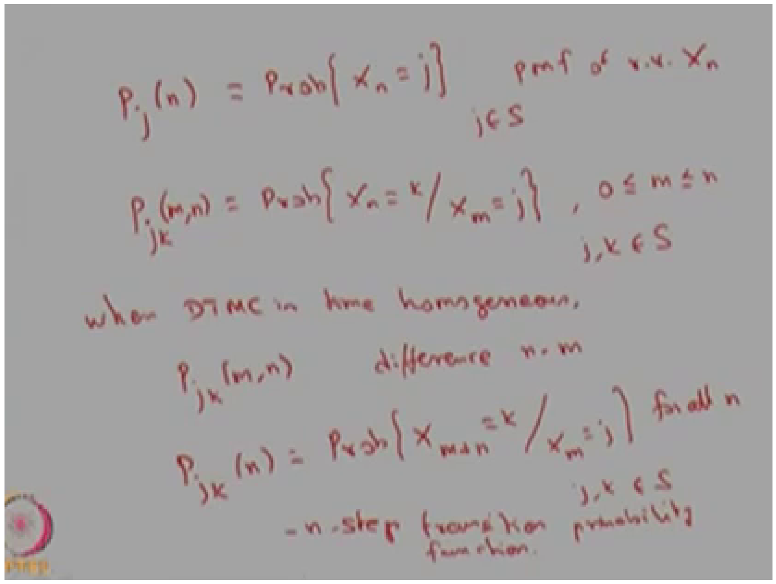

So the first one, I am going to define the probability mass function as the P suffix j of n that is nothing but what is the probability that Xn takes a value j, so this is the probability mass function of the random variable Xn, what is the probability that a Xn can takes a value j that I am going to denote it as the P suffix j of n, where here the j is belonging to the state space S, this is the probability of the mass function of the random variable X n.

Similarly, I am going to define the conditional probability mass function as a P suffix j k of n, that is nothing but what is the probability that Xn takes a value k given that X sorry, I need two suffixes, two index for here with the two variables m, n what is the probability that Xn will be the state k given that Xm was j, obviously the m is lies between 0 to n, whatever m and every n and the j, k is belonging to S.

so this is a conditional probability distribution of the random variable Xn with Xm and the m th step the system was in the state j, and the n th step the system is in the state k and this is the conditional probability with the two arguments m, n so this is the probability that the system makes a transition from the state j at step m to the state k at step n, this is called transition probability function of the discrete time Markov chain.

When the DTMC is a time homogeneous, this is very important when DTMC is a time homogeneous that means it satisfies the time invariant property, that means the P j k of m, n depends only on the time difference n minus m, whenever the DTMC is a time homogeneous that means in the time invariant so the actual time is not a matter only the time difference is the importance therefore this is going to depends only on the time difference n minus m.

In this case I don't want to the two arguments m, n I can go for writing P j k of n, that is nothing but what is the probability that the m + n th step, the system will be in the state k given that the m th step it is in the - it was in the state j, for all n and here j, k belonging to S, so the m does not matter only the interval or the interval length of step n is matter, so that means the system was in the state j and it is a transition into the state k in n steps.

Because the DTMC is a time homogeneous so the Xm to Xm + n, it is valid for all m, for all n we are finding out for the n step transition, this is called n step because the DTMC is a time homogeneous and this is called n step transition probability function, this is a n step transition probability function, using this we can define the one step transition probability.

That is denoted by P suffix j k of 1 or we can avoid the bracket 1 also so you can write it as the P j k that is nothing but what is the probability that the Xn + 1 is equal to k, given that Xn is equal to j, for all n greater than or equal to 1, obviously for j, k belonging to S, if you find out the zero step transition probability that values is going to be 1 for j equal to k, otherwise it is going to be 0.

This one step transition probability I can make it in the matrix form as the P is the matrix and that consists of P i j, where the P i j is nothing but one step transition probability mat - elements of Xn + 1 is equal to j given that Xn is equal to i, here i, j belonging to the states space S, you should remember that states space S is consist of finite elements or countably infinite number of elements.

Accordingly, this matrix is going to be either when S is going to be finite elements then the P matrix is going to be a square matrix, since the P i j is the one step transition probability of a system moving from the state i to j in one step and since it is a time homogeneous this is valid for all n this is valid for all n greater than or equal to 1.

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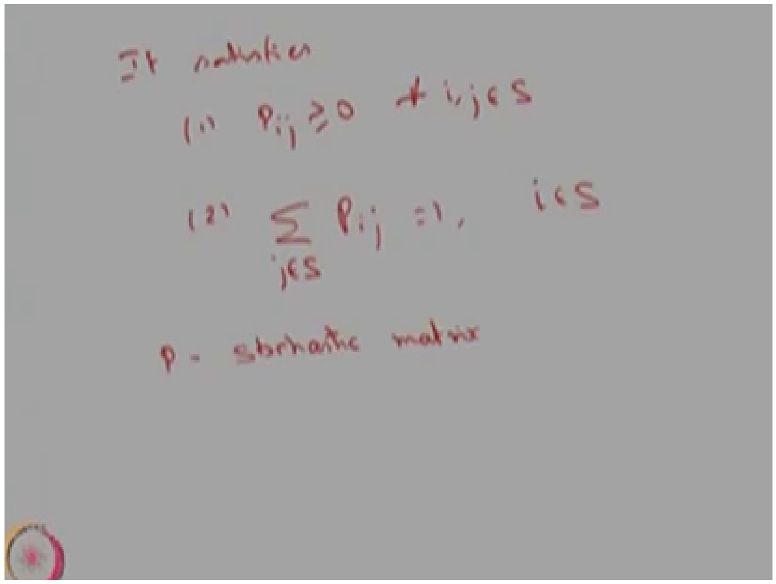

And this satisfies the one step transition probability matrix satisfies two properties the each entity will be greater than or equal to zero for all i, j belonging to S, because these are all only the conditional probability of system moving from state i to j in one step, therefore either it will be a zero or greater than zero for all possible values of i, j.

The second condition if you make the summation over j for fixed i then that is going to be 1, i belonging to S, that means the row sum is going to be 1, because it is a conditional probability of system moving from one state to another states, if you add all the other possible probabilities then that is going to be 1, since these one step transition probability matrix satisfies these two properties and this matrix is P is known as the stochastic matrix.

Because of satisfying these two conditions the matrix one step transition matrix is also called stochastic matrix. Now I am going to explain what is the pictorial way of viewing the one step transition probability matrix or the stochastic matrix.

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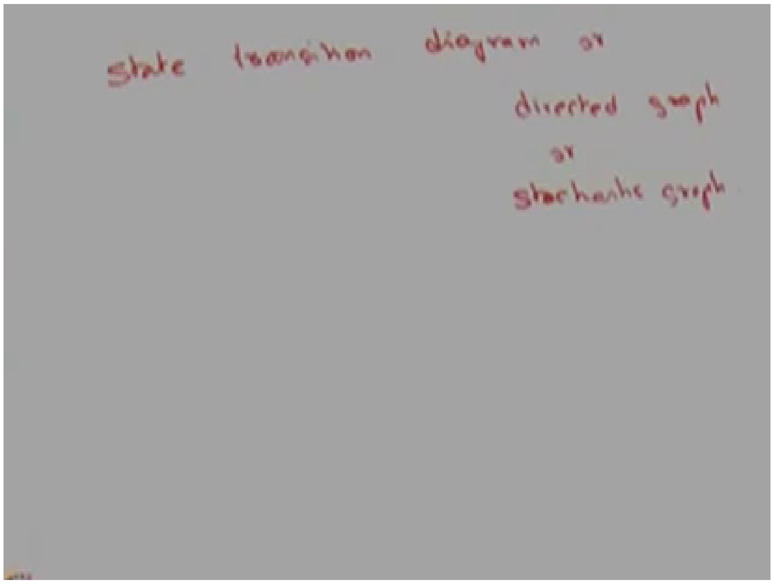

That is provided by state transition diagram or the other words it is called a directed graph the DTMC can be viewed as a directed graph such that the state space S is a set of vertices or nodes and the transition probabilities that is a one-step transition probabilities are the weights of the directed arcs between these vertices or nodes, since the weights are positive and the sum of the arc weights from the each node is unity this directed graph is also called stochastic graph.