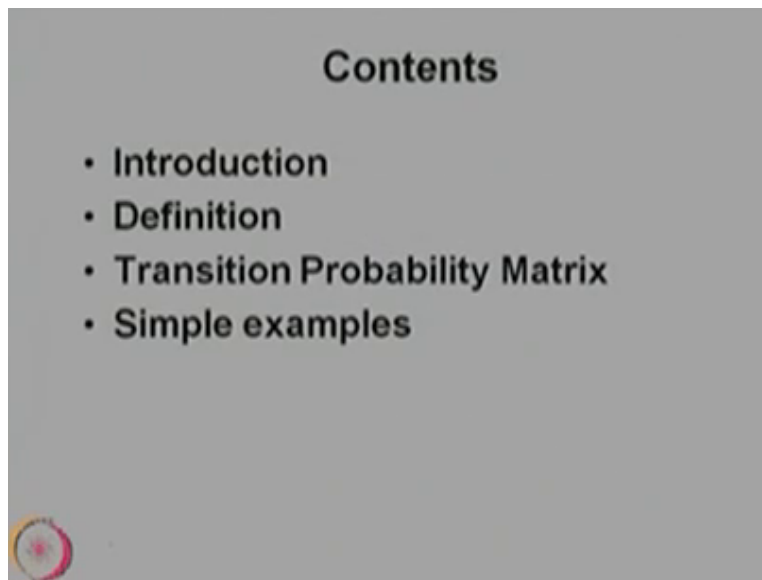


Stochastic Processes - 1
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Lecture - 25
Introduction to Discrete time Markov Chain

This stochastic process in this, we are going to discuss the Module 4 Discrete time Markov Chain and this is a lecture 1, in this lecture, I am going to discuss the introduction about the discrete time Markov chain, then followed by the definition and the important one concept called one step transition probability matrix.

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So this lecture I am going to cover the Introduction, Definition, Transition probability matrix and few simple examples also, consider a random experiment of tossing a coin infinitely many times each trial there are two possible outcomes namely head or tail.

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$P(\text{Head}) = p$
 $P(\text{Tail}) = 1-p$ $0 < p < 1$
 for n^{th} trial,
 X_n
 $P(X_n=0) = 1-p$
 $P(X_n=1) = p$
 X_1, X_2, \dots $X_i, i=1, 2, \dots$ mutually independent random variables

Assume that the probability of head, that probability you assume that - that is p and the probability of tail occurring in each trial that you assume it as $1 - p$, you assume that the p lies between 0 to 1, denote for the n th trial, because you are tossing a coin infinitely many times for the n th trial, you denote the a random variable X_n is the random variable whose values are 0 or 1 with the probability.

The probability of X_n takes a value 0 that is same as in the n th trial you are getting the tail that probabilities $1 - p$ and the probability of X_n takes a value 1 that probability is make it as p for the head appears and already you assume that and the probability is lies between 0 to 1, thus you have a sequence of random variable $X_1 X_2$ and so on and this will form a stochastic process.

And assume that all the X_i 's are mutually independent random variable - independent random variables, so this is a random experiment in which we are tossing a coin infinitely many times and for any n th trial, you define the random variable X_n with the probability it takes a value 0 the probability 1 minus p and takes a value 1 with the probability p and that is equivalent of appearing head at the probability p and occurring the tail with the probability $1 - p$.

Now I am going to define another random variable that is a partial sum of first n random variables.

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$$S_n = X_1 + X_2 + \dots + X_n$$

$$S_{n+1} = S_n + X_{n+1}$$

$$\{S_n, n=1, 2, \dots\} \text{ is a stochastic process}$$

$$P(S_{n+1} = k+1 / S_n = k) = P$$

$$P(S_{n+1} = k / S_n = k) = 1 - P$$

An n X_i 's, so the S_n will be sum of a first n random variables therefore the sum S_n gives the number of heads appears in the first n trials, it can be observed that S_{n+1} is same as $S_n + X_{n+1}$, since S_n is the partial sum of a first n trials outcome so the S_{n+1} is nothing but $S_n + X_{n+1}$. you can also observed that since S_n is the sum of a first n random variables and S_{n+1} is X_{n+1} and also all the X_i 's are mutually independent variables.

S_n is a independent with X_{n+1} , that means here the S_{n+1} th random variable is a the combination of a two independent random variables whereas the S_n is the till n th trial how many heads you appear plus whether it is a head or tail accordingly these values is going to be 0 or 1, therefore if you see the sample path of S_{n+1} , it will be incremented by 1 if X_{n+1} takes a value 1 or it would have been the same value earlier if this S_{n+1} takes a value 0.

And also you can observe that S_{n+1} is a depends on S_n and only on it, it is not a depends on S_{n-1} or S_{n-2} so on, because it is accumulated the number of trials values over the n , therefore S_{n+1} is depends on S_n and only on it, the S_n for different values of n this will form a stochastic process - this will form a stochastic process and now you can come to the conclusion the probability of this is a stochastic process.

The probability of S_{n+1} suppose this values is $k+1$ given that S_n was k that means the S_{n+1} value would have been 1, therefore the appearance of a the head appears in the $n+1$ th trial and

probability is going to p. similarly you can make out suppose S_{n+1} value will be k such that S_n is also k then that is possible with a $n+1$ th trial you got the tail, therefore that probability is $1-p$, this is satisfied for all n.

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$$\begin{aligned}
 & P(S_{n+1} = k+1 / S_1 = i_1, S_2 = i_2, \dots, S_n = k) \\
 &= \frac{P(S_{n+1} = k+1, S_n = k, \dots, S_2 = i_2, S_1 = i_1)}{P(S_1 = i_1, S_2 = i_2, \dots, S_n = k)} \\
 &= \frac{P(S_{n+1} = k+1 / S_n = k) P(S_n = k, \dots, S_2 = i_2, S_1 = i_1)}{P(S_1 = i_1, S_2 = i_2, \dots, S_n = k)} \\
 &= P(S_{n+1} = k+1 / S_n = k) = p, \quad n \geq 1
 \end{aligned}$$

So you can make out this is satisfied for all n is greater than or equal to, even I can go for n is greater than or equal to 1 not only these.

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$$\begin{aligned}
 & P(S_{n+1} = k / S_1 = i_1, S_2 = i_2, \dots, S_n = k) \\
 &= P(S_{n+1} = k / S_n = k) = 1-p
 \end{aligned}$$

"Memoryless" property
 or
 Markov property
 $\{S_n, n=1, 2, \dots\}$ - Markov process
 - discrete time discrete state
 stochastic process.

Similarly, I can come to the conclusion the probability of S_{n+1} is equal to k given that S_1 was i_1 , S_2 was i_2 and so on S_n was k that is also can be proved the probability of S_{n+1} is equal to

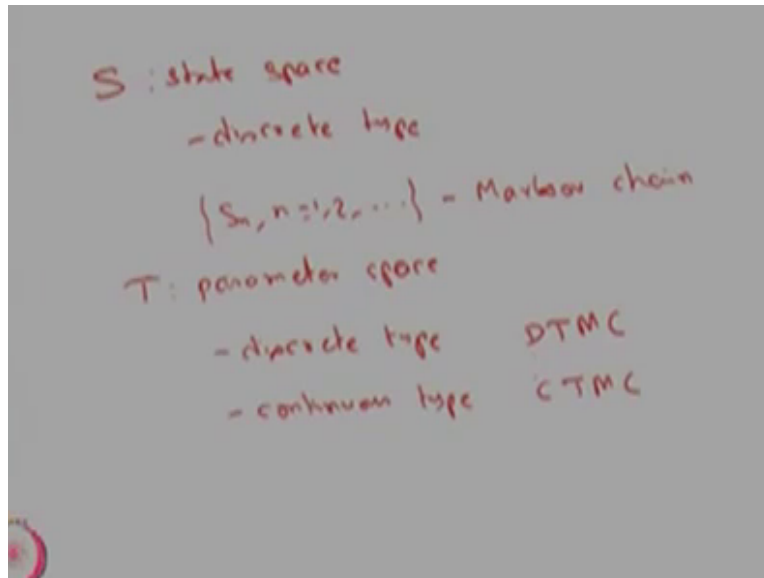
k , given that S_n is equal to k that is same as what is the probability that value was the same k in the subsequent trials that is possible of appearing a tail in the $n + 1$ th trial.

Therefore, that the appearance of the tail in the $n + 1$ th trial the probability is $1 - p$ or q or i can use the notation q , that means the probability of $n + 1$ th trial that distribution given that I know the value till the n th trial that is same as the distribution of $n + 1$ th trial, given with the only the distribution n th distributions not the earlier distributions and this property is called memoryless property.

The stochastic process the S_n satisfies the memoryless property or the other word called Markov property, the distribution of $n + 1$ given that the distribution of 1 first random variable, second random variable, the n th random variable that is same as the conditional distribution of $n + 1$ th random variable given that with the n th random variable only and this property is called a memoryless or Markov property.

The stochastic process the S_n satisfying the Markov property or memoryless property is called Markov process, the stochastic process satisfying the memoryless property or Markov property is called Markov process, in this example the stochastic process S_n is the discrete time, discrete state stochastic process. Now I can give based on the state space and the parameter space I can classify the Markov process or I can give the name of the Markov process in an easy way based on the state space as well as the parameter space.

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So when the state space S , the S is the state space this is nothing but the collection of all possible values of the stochastic process, if this is of the discrete type - discrete type that means the collection of a elements in the state space S is going to be a finite or countably infinite then we say the states spaces of the discrete type.

So whenever the stochastic process satisfying the Markov property then the stochastic process is called the Markov process or you can say whenever the state space is a discrete then you can say the corresponding stochastic process we can call it as a Markov chain, whenever the state spaces are discrete.

Now based on the parameter space T , parameter space is nothing but the possible values of T whether it is going to be a finite or countably infinite then it is going to be a discrete parameter space or discrete time or it is going to be uncountably many values then it is going to be called it as a continuous type, so whenever T is going to be a discrete type then the Markov chain is going to be called it as discrete time Markov chain.

Whenever the parameters which is going to be of the continuous type that means the possible values of T is going to be uncountably many then we say continuous time Markov chain, so in this example the S_n , the possible values of S_n is also going to the state space is going to be

discrete type and the parameter space is also going to be discrete type therefore the given example the sun is going to be the discrete time Markov chain.

So in this module, we are going to study the discrete time Markov chain then next module, module four, module five we are going to discuss the continuous time Markov chain, so in general whenever the stochastic process satisfying the Markov property it will be called Markov process. So based on the state space the Markov process called as Markov chain and the based on the parameter space it is called discrete time Markov chain or continuous time Markov chain accordingly, discrete type or continuous type.