Stochastic Processes - 1

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Lecture - 23
Poisson Process (Contd.)

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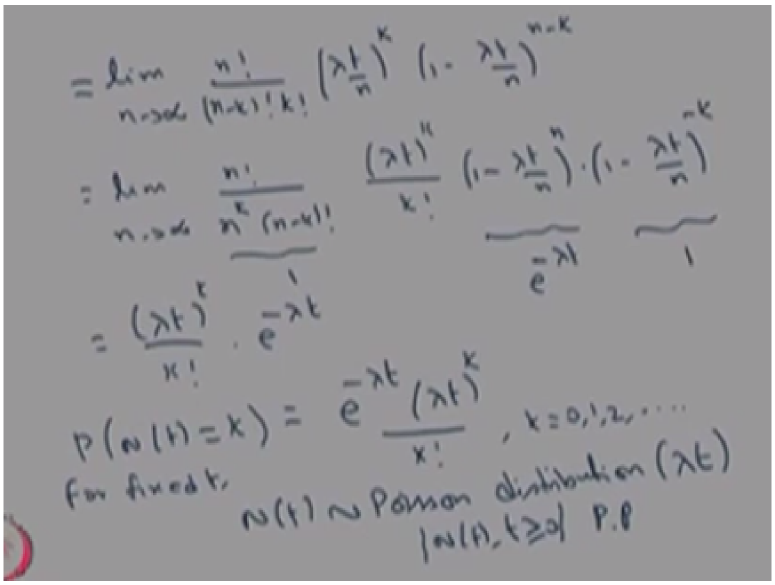

Here the lambda is a constant and there is another name for the default Poisson process is called the homogenous Poisson process because there is another one called non homogenous Poisson process in which the lambda need not be a constant. It can be a function of time t also. Therefore, the one we have derived now, it is a homogenous Poisson process in which the lambda is a constant, which is greater than zero.

When lambda is going to be a function of t, the corresponding Poisson process is called a non-homogenous Poisson process. So this is the one particular and very important continuous time or continuous parameter discrete state stochastic process that is a Poisson process or this is also we can say, this is going to be a very important continuous time arrival process that is a Poisson process.

The way we are counting N(t) is going to be a number of arrivals over the interval zero to t or number of occurrence of the event over the t, the way you are counting over the time. Poisson process is an example of counting process. So, the N(t) is also called a counting process. So the Poisson process is also called as the counting process.

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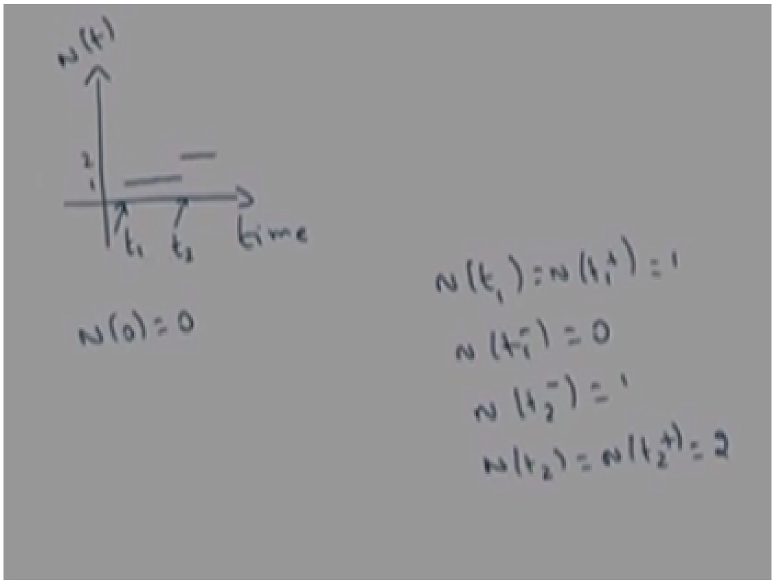

I can go for giving the sample path of N(t) over the time, what is the different values of N(t) is going to take. Obviously N (0) is equal to zero. Whenever some arrival occurs in some time, then the arrival is going to occur, therefore suppose the arrival occurs at this time, I make it as the up arrow. Then the value of N(t) is going to be incremented by one, till the next arrival comes. Suppose the next arrival takes place at this time point then the N(t) values is going to be one, till that time and it is going to be a right continuous function.

That means the time point in which the first arrival occurs, suppose you make it as t 1, so the n of t1 minus is going to be zero and the t1 and N (t1 +), t1 as well as N (t1+) that is going to be one. Whereas the left limit N (t1-) that is going to be zero. Suppose, the second arrival occurs at some point t 2, then the N (t2-) that is the left limit at the time point t2 that is going to be one.

And the N (t2) that is same as N (t2+) that is going to be two. So therefore it is incremented by one, so the values is going to be two. So this is the random amount, random time in which the arrival is going to occur and the way we have made the assumption in a very small interval only one, maximum only one arrival can occur. Therefore, the N(t) is going to be a non-decreasing right continuous and increased by jump of size one at the time a poke of arrival.

So whenever you see the sample path of the Poisson process, it is always going to be a non-decreasing right continuous and increased by a jumps of size one at the time a poke of arrivals. Now I am going to relate another random variable which involves in the Poisson process or I am going to discuss another stochastic process which involved in the Poisson process.

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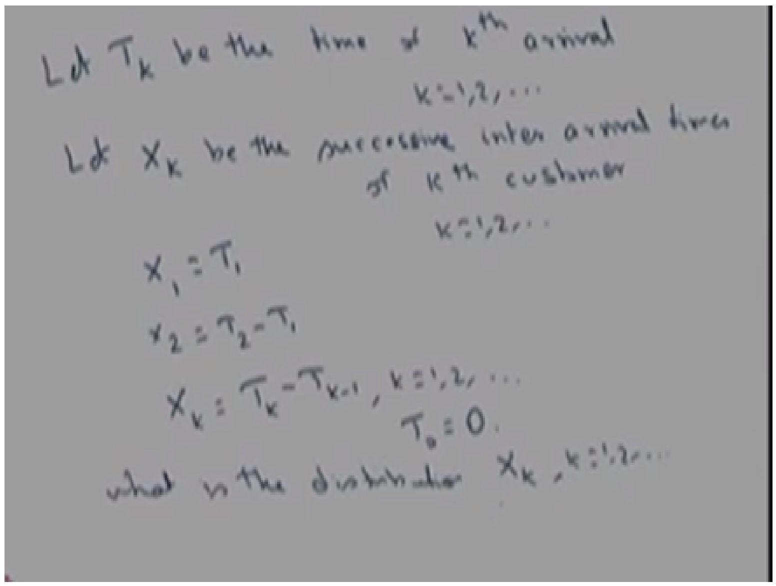

So for that, I am going to define the new random variable as let T suffix k be the time of k th arrival. So k can take the value one or two and so on. So therefore t be the random variable, takes what is the time point in which the k-th arrival occurs. That means the way I have given the sample path in the previous slide, the t1 and t 2, the small t1 and t2 are the different values of the capital T k.

I am going to define another random variable X suffix k be the successive inter arrival times of k-th customer. So now the k can takes the value one, two and so on. So the T k be the time point, whereas the S k be the inter arrival time. That means the X1 is nothing but T1 – T 0 and obviously T0 is zero, therefore X1 is same as T1. And X2 is nothing but T2 – T1. That means what is the inter arrival time for the second arrival, that inter arrival time is what time the first arrival occurs, that is the T1.

And what time the second arrival occurs, that difference is going to be the inter arrival of the second customer. So this is the way I can define Xk is going to be, T k – T (k – 1). So now the running index for K can take the value one and so on. Obviously T 0 is going to be zero. Our interest is to find out what is the distribution of Xk for all k 1, 2 and so on. Is it feasible to find out the distribution of Xk? It is possible.

First we can start with K equal to one, what could be the distribution of X1. Then once we get the X1 distribution, the same analysis can be repeated to get the distribution of X2 and X3 and so on because the scenario which we are going to take it for finding out the distribution of X1, that is the same as for X2 and so on.

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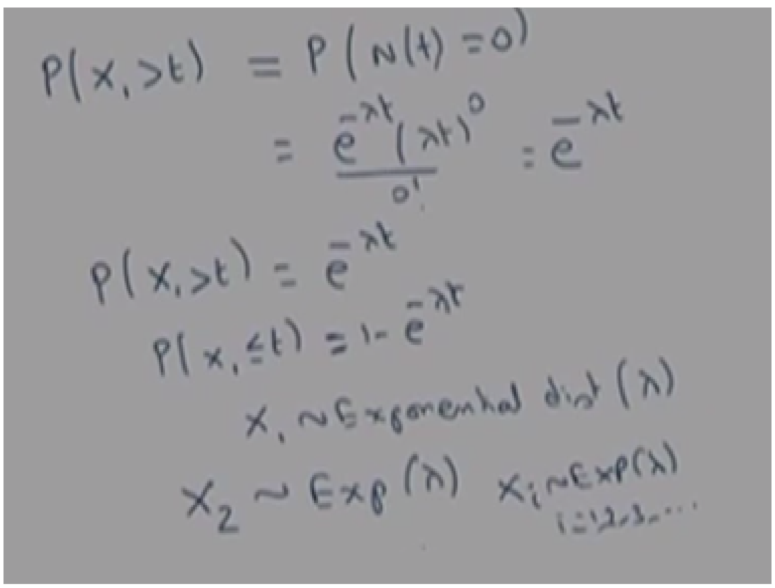

So now our interest is to find out what is the distribution of X1. First we will try to find out that X1 only. Now we will find out the distribution of X1. Since X1 is a continuous random variable, we can go for finding out what is the compliment cdf of X1. So this is the compliment cdf of X1. That is nothing but what is the probability that the first arrival occurs after time t. That is same as what is the probability that till time t, no customer enter into the system.

The left hand side is unknown, whereas the right hand side is the known one. So we are relating two different random variable. So here this is what is the probability that the first arrival occurs after time t. That is same as what is the probability that no arrival takes place during the interval zero to small t. But we know what is the probability of N(t) is equal to zero because just now we have made it.

For each t this is going to be a Poisson distribution with the parameter lambda times t. Therefore, the probability of N(t) equal to zero that is same as e power minus lambda t and lambda t power zero/zero factorial. And this is same as e power minus lambda t. So the left hand side is the unknown. The unknown is what the probability that X1 takes a value greater than t that is same as e power minus lambda t.

Therefore, we can get what is the probability of X1 less than or equal to t that is same as one minus e power minus lambda t. So this is going to be a, what is a cdf for the random variable X1. And the cdf of X1 is same as the cdf exponential distribution with the parameter lambda times t. Therefore, we can come to the conclusion, X1 is going to be a exponentially distributed. The X1 is exponentially distributed with the parameter lambda.

So the unknown distribution X1, first we are trying to find out what is the compliment cdf of X1 and that land up to e power minus lambda t. Therefore, the cdf of X1 is going to be one minus e power minus lambda t. From this we conclude the X1 is going to be exponential distribution with the parameter lambda, where lambda is greater than zero.

The way we have compute the, the way we get the distribution of X1, similarly one can show X2 that is the inter arrival time of the second customer entry into the system, that is also can be proved, it is exponential distribution with the parameter lambda. Not only X2, we can go for the further all the Xi's, so we can able to prove all the Xi's are going to be exponential distribution with the parameter lambda for 'i' takes the value one, two and so on.

Not only that, we can able to prove all the Xi's are independent random variable also and identical with the exponential, each one is exponential distribution with the parameter lambda.

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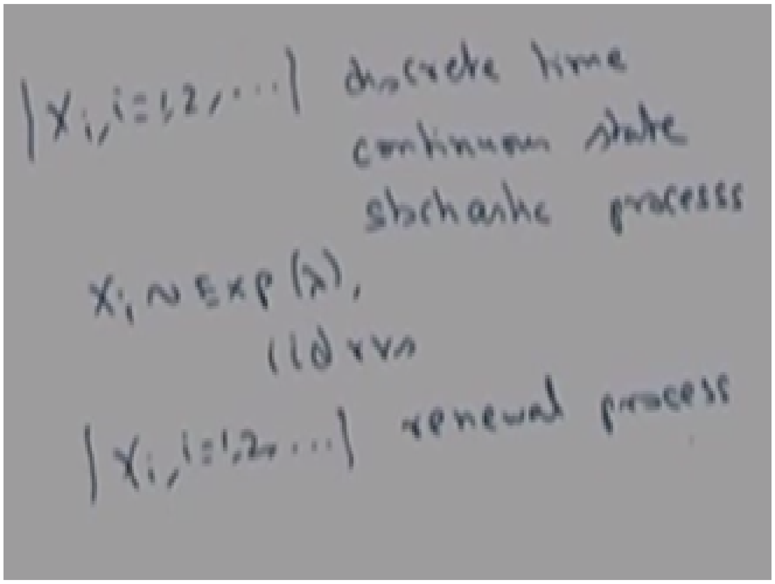

Therefore, the way we land up relating Poisson process with the inter arrival time, so this Xi's will form a discrete time or discrete parameter, continuous state stochastic process in which each random variable Xi is going to be an exponential distribution with the parameter lambda and all the Xi's are iid random variable also. And this each Xi's are nothing but inter renewal time. Therefore, this is going to be, call it as renewal process.

We are going to discuss the renewal process in detailed later of this course. But here, I am just explaining how you will land, create the renewal process from the Poisson process. And the N(t) is the Poisson process for different values of t, whereas the inter arrival time that is the time in which the renewal takes place or the arrival takes place. Therefore, the renewals will form a stochastic process and that corresponding process is called a renewal process.

Therefore, this is going to be a one particular type of renewal process in which the renewal takes place of an exponentially distributed time intervals and all the times are iid random variables also.