Stochastic Processes - 1

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Lecture - 22
Poisson Process

So, till now we have discussed what the discrete time arrival process is. Now we are going to discuss the continuous time arrival process that is a Poisson process.

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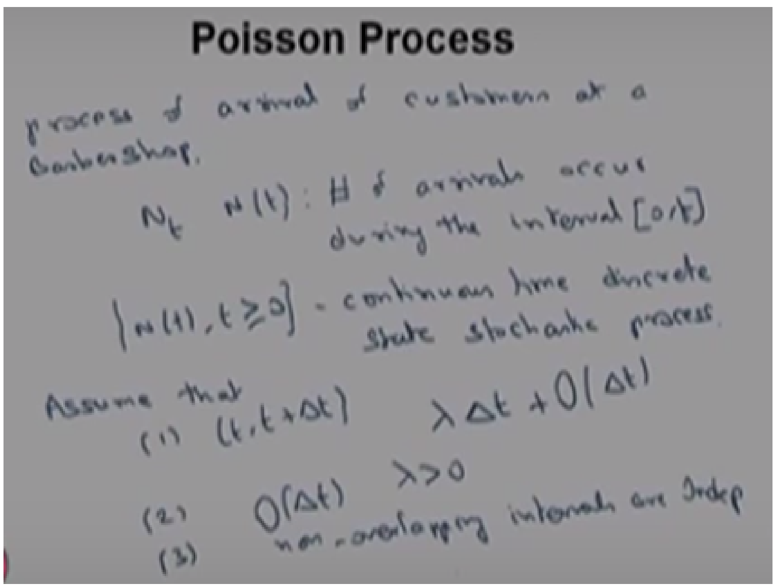

In this lecture, I am going to develop what is the Poisson process and how we can get the Poisson process from the scratch. Suppose you consider the process of arrival of customers at a barber shop. So this is the same example, we have discussed in the beginning of this course also. So over the time, how many arrivals is going to take place? That is going to be a random variable. So let N t, N suffix t or in some books they uses N (t).

So the N (t) denotes number of arrivals occur during the interval zero to the closed interval zero to t. That means we are defining a random variable N (t) that denotes the number of arrival occurs during the interval zero to t. For fixed t, N (t) is going to be a random variable. Therefore, N (t) over the time, because t is greater than or equal to zero, this is going to be a since the possible values of a capital T that is the parameter space is going to zero to infinity, therefore this is going to under the classification of a continuous parameter or continuous time.

And the possible values of N (t) for different values of t that is going to be takes the value zero or one or two, therefore it is going to be a countably infinite. Therefore, this is going to be a continuous time or continuous parameter discrete state stochastic process. So, this is the N (t) over the t greater than or equal to zero that is going to be a continuous time discrete state stochastic process. Now, we are going to develop the theory behind Poisson process.

To create the Poisson process, you need few assumptions so that you can able to develop the Poisson process. The first assumption, in a small negligible interval, if the interval is t to t plus delta t, then the probability of one arrival is going to be lambda times delta t plus capital O of delta t. The probability of one arrival occurs during the interval t to t plus delta t is going to be lambda times that smaller interval delta t plus capital order of capital O delta t.

Here the lambda is going to be strictly greater than zero and we are going to discuss what is lambda and so on in the later, after explaining the Poisson process. So here the lambda is going to be a constant and which takes the value greater than zero. And the capital O delta t means as delta t tends to zero the order of delta t that is going to be tends to zero, as delta t tends to zero. So this is the first assumption.

The second assumption, the probability of more than one arrival is going to be an order of delta t, in a same interval t to t plus delta t, more than one arrival in this small negligible interval that probability is an order of delta t. That means, as delta t tends to zero, this values is going to tends to zero. Then the third assumption, occurrence of arrivals in a non-overlapping intervals are mutually independent. Non-overlapping intervals are independent. So this is very important assumption.

That means, what is the probability that the arrival occurs in a non-overlapping intervals that probability is same as the product of a probability of an arrival occurs in the each interval. Therefore, it is going to satisfy the independent property, occurrence of events in non-overlapping intervals are mutually independent. Therefore, the probability is going to be probability of intersection of all those things is same as the probability of individual probability and their product.

So with these three assumptions, we are going to develop the Poisson process.

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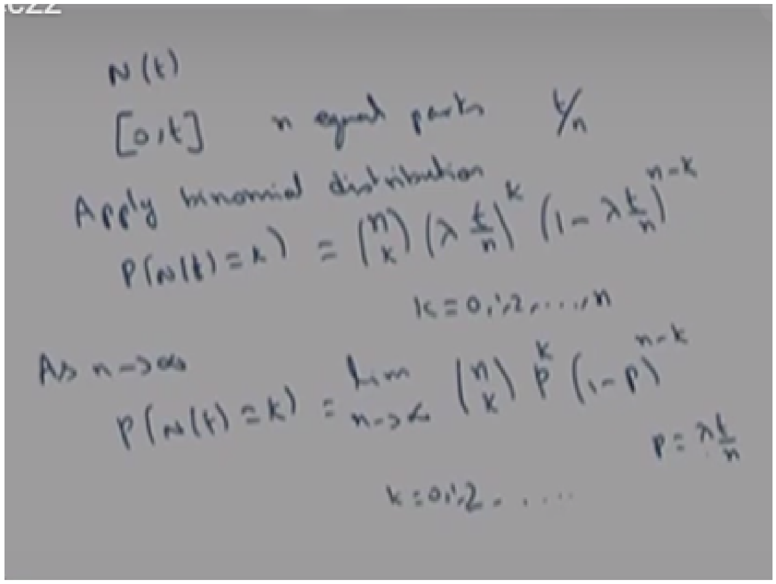

So what I am going to do, since I started with the random variable N (t) is the number of arrivals in the interval zero to t, I am going to partition the interval zero to t into n equal parts. I am going to partition the interval zero to t into n equal parts. Since I made it the interval zero to t into n equal parts, then each will be of the length t/n. And since I made the assumption the non-overlapping intervals are independent.

And the probability of one arrival is lambda times delta t and the probability of more than one arrival is order of delta t and so on, therefore I can apply binomial distribution. The way I have partitioned the interval zero to t into n pieces, therefore this is going to be a, of a n intervals of interval length t/n. Therefore, I can say what is the probability that I can able to find out, what is the probability that K arrivals takes place in the interval, n intervals of each length t/n.

What is the probability that K arrivals takes place, therefore the possible values of K is going to be zero to n and I can able to find out by using the binomial distribution, what is the probability that n of t takes the value K. Since non overlapping intervals are independent and each probability of one arrival is lambda times delta t, where delta t is a t/n. So each interval behaves as a Bernoulli trial, whether the arrival occurs or there is no arrival and like that you have n such independent trials.

Therefore, the sum of n independent Bernoulli trials land up a binomial trials. Therefore, by using the binomial distribution, I can able to get what is a probability that N (t) takes a value K that is what is the possible nck way. And what is the probability of arrival takes place in one interval that is lambda times, this interval length is t/n, lambda times t/n power k. And what is the probability of no arrival takes place in each interval that is one minus lambda times t by n power n minus K.

So this is the way, I can able to get what is the probability that K arrival takes place in the interval zero to t by partitioning np, n intervals, so this is the probability. But the way I made a partition n equal parts, so now I have to go for what is the result as n tends to infinity. That means my interest is, what could be the result, if n tends to infinity of k of, what is the probability that N (t) takes a value K, as N tends to infinity.

Therefore, the running index for K is going to be 0, 1, 2 and so on. What is the probability of N (t) takes a value K. That means in the right hand side, I had to go for finding out, as N tends to infinity what is the result for the right hand side, what is the probability of N (t) takes the value K. We take N tends to infinity because we need to study the limiting behavior of the stochastic process.

So that is same as limit N tends to infinity of nck, I can make it as p power k, where p is going to be lambda times t/n and one minus p power n minus k. Now I have to find out what is the result for limit N tends to infinity of this expression nck p power k one minus p power n minus k, where p is going to be lambda times t/n.

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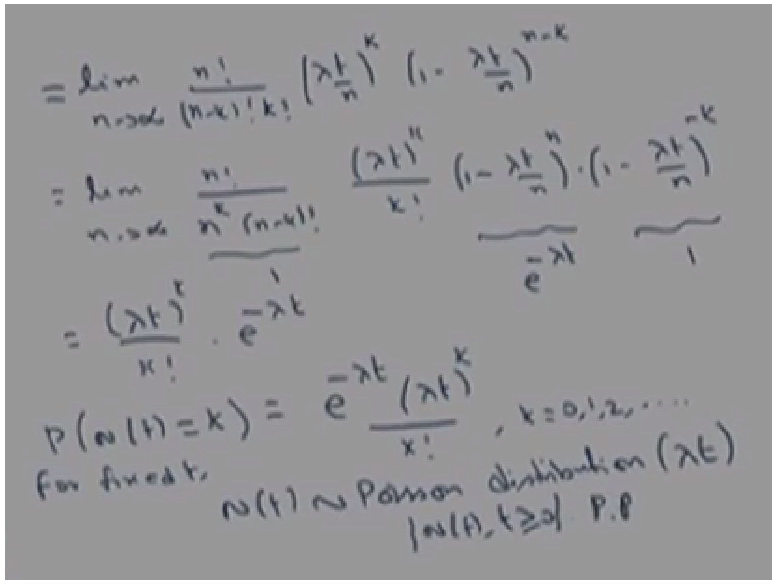

If I do this simple calculation, let me explain, the limit n tends to infinity that is same as limit tends to infinity of nck, I can make it as a n factorial, n minus k factorial and k factorial and that is lambda t by n power k and that is one minus lambda t by n power n minus k and that is same as the limit n tends to infinity of n factorial. And here this n power k, I can take it outside. And n minus k factorial and lambda t power k and divided by k factorial. So this k factorial I take it inside.

And the power one minus lambda t by n power n minus k, I split it into one minus lambda t by n power n into one minus lambda t by n power minus k. So, now I can look as n tends to infinity this is nothing to do with n. Therefore, lambda t power k by k factorial will come out. So this result is going to be lambda t power k by k factorial. And this will land up as n tends to infinity. This is going to be e power minus lambda t and this will land up one and this is also land up one as n tends to infinity. Therefore, I may land up it is e power minus lambda t.

Hence the final answer of what is the probability that k arrival takes place in the interval zero to t that is going to be e power minus lambda t and the lambda t power k by k factorial. And the possible values of k can be 0, 1, 2 and so on. For fixed t, if you see this is same as, for fixed t it is going to be a random variable. For all possible values of t, it is going to be a stochastic process.

So for fixed t, the N (t) is a random variable and that probability mass function is e power minus lambda times t, lambda t power k by k factorial. So lambda is a constant. For fixed t, lambda into t, that is going to be a constant. Therefore, the right hand side look like the probability mass function of the Poisson distribution. Therefore, for fixed t, the N (t) is Poisson distribution. The random variable N (t) for fixed t, it is going to be a Poisson distribution with the parameter lambda times t. Lambda is a constant.

And for fixed t, t is a constant. So lambda multiplied by the t, again this is going to be a constant. Therefore, for fixed t, it is going to be a Poisson distribution with the parameter lambda multiplied t. Therefore, for possible values of t, the N (t) is going to form a stochastic process. And since for fixed t, it is going to be a Poisson distribution, the collection of a random variable and each random variable is a Poisson distribution.

Therefore, this is going to be call it as the Poisson process. The way I have, we have explained earlier, each random variable is a Bernoulli distributed random variable, the collection of random variable is a Bernoulli process, similarly each sn is going to be a binomial distribution, therefore the collection is going to be a binomial process. The same way, for fixed t, it is going to be a Poisson distribution, therefore the collection is going to be call it as the Poisson process.

So now we have developed N (t) is going to be a Poisson process.