Stochastic Processes-1
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Lecture – 13
Problems in Sequence of Random Variables

So this is a stochastic processes model-1 probability theory refresher lecture 4, problems in sequence of random variable.

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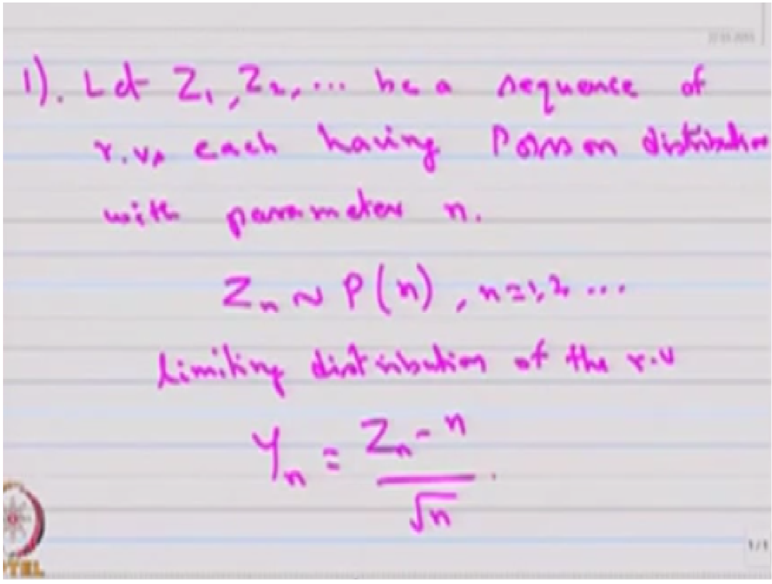

As a illustrative examples we are going to discuss four problems in this lecture. The first problem, let z1, z2 so on be a sequence of random variables each having poisson distribution with parameter n, that is zn is poisson distribution with a parameter n. For n is equal to 1, 2, 3 and so on. Our interest is to find the limiting distribution of the random variable that is defined as y suffix n that is zn minus n divided by square root of n.

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$$M_{z_n}(t) = E\left(e^{z_n t}\right)$$
$$= \sum_{k=0}^{\infty} \frac{e^{kt}\, e^{-n}\, n^{k}}{k!}$$
$$= e^{-n} \sum_{k=0}^{\infty} \frac{(e^{t}.n)^{k}}{k!}$$
$$= e^{-n}\, e^{ne^{t}} = e^{n(e^{t}-1)}$$
$$M_{y_n}(t) = M_{\frac{z_n-n}{\sqrt{n}}}(t)$$

So given zn is poisson distribution of the parameter n we can find out the MGF of zn is nothing but expectation of e power zn of t. That is same as summation k is equal to zero to infinity e power k times t, e power minus n, n power k by k factorial because it is a expectation of e power zn of t where zn is poissson distribution of the parameter lambda therefore this is going to to be nk is equal to zero to infinity this one.

So you can take e power minus n outside so the remaining term becomes k is equal to zero to infinity, e power t multiplied by n the whole thing power k by k factor. That is same as e power minus n, e power n times e power t. That can be rewritten as e power n times e power t minus 1. Now we will find out the MGF of the random variable yn where yn is a zn minus n divided by square root of n. There the MGF of the random variable yn as a function of t that becomes MGF of zn minus n divided by square root of n function of t.

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$$= E\left(e^{\frac{Z_n - n}{\sqrt{n}} \cdot t} \right)$$

$$= e^{-t\sqrt{n}} \, M_{Z_n}\left(\frac{t}{\sqrt{n}}\right)$$

$$= e^{-t\sqrt{n}} \, e^{n\left(e^{t/\sqrt{n}} - 1\right)}$$

$$= e^{-t\sqrt{n}} \, e^{n\left(1 + \frac{t}{\sqrt{n}} + \frac{t^2}{2n} + \frac{t^3}{3!\, n^{3/2}} + \cdots - 1\right)}$$

That is same as expectation of e power zn minus n divided by root n multiplied by t. You know the rules of a (()) (04:53) generating function, the constant is out, so you can use that logic, so it becomes e power minus t times root because nt by root n therefore it becomes a t times root n. Then MGF of the random variable zn use a another rule of a moment generating function instead of t becomes a t divided by square root ofn.

So that is same as e power minus t times root n, just now we found what is a moment generating function of a zn. So use the same thing but replace t by t divided by square root of n. Therefore, this becomes e power n times wherever the t, you replace t by t by square root of n, so t by square root of n minus 1.

Therefore, you can further simplify by expanding e power t by n. That means you keep this e power n, you expand only e power t by square root of n that is 1 plus t divided by square root of n, then the next term will be t square by two times n and next term will be t cube divided by three factorial n power 3 by 2 and so on. And the last term is so this is a expansion of e power t by square root of n minus 1, so close the bracket.

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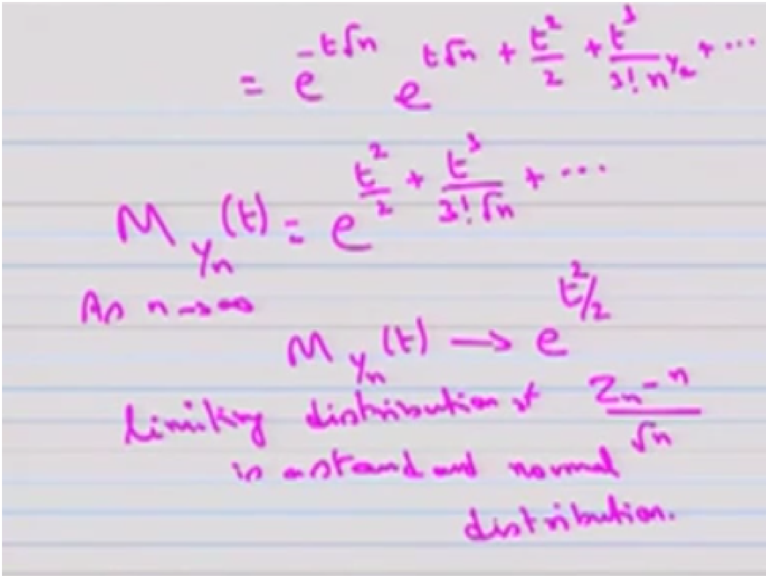

That is same as e power t times square root of n multiplied by, so this 1 and plus 1 and minus 1 will be cancelled. So you will get e power n times t by square root of n that becomes t of square root of n and the next term becomes t square by 2 then it becomes tq by 3 factorial n power 1 by 2 and so on. Therefore, this becomes e power t square by 2 plus tq by 3 factorial square root of n and so on.

Our interest is to find out the limiting distribution of yn. So this is the moment generating function of yn for n. So as n tends to infinity because our interest is to find out the limiting distribution as n tends to infinity the moment generating function of yn becomes e power t square by 2. If you recall the moment generating function for standard distributions, one can conclude this is the MGF of a standard normal distribution.

Therefore, we conclude the limiting distribution of yn is standard normal distribution. That is the limiting distribution zn minus n divided by square root of n is a standard normal distribution. So this problem is very important in the renewal processes therefore we discuss this example as a how to find the limiting distribution of a some standard variables.