Stochastic Processes
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Module - 2
Definition and Simple Stochastic Processes
Lecture - 2
Simple Stochastic Processes

This is module 2, definition and simple **stochastic** process. Today is the lecture 2, simple stochastic process. In the lecture 1, we have seen the definition of a stochastic process and the classification of a stochastic process based on a time space and the parameter space and we have given few simple stochastic process, why are the classification.

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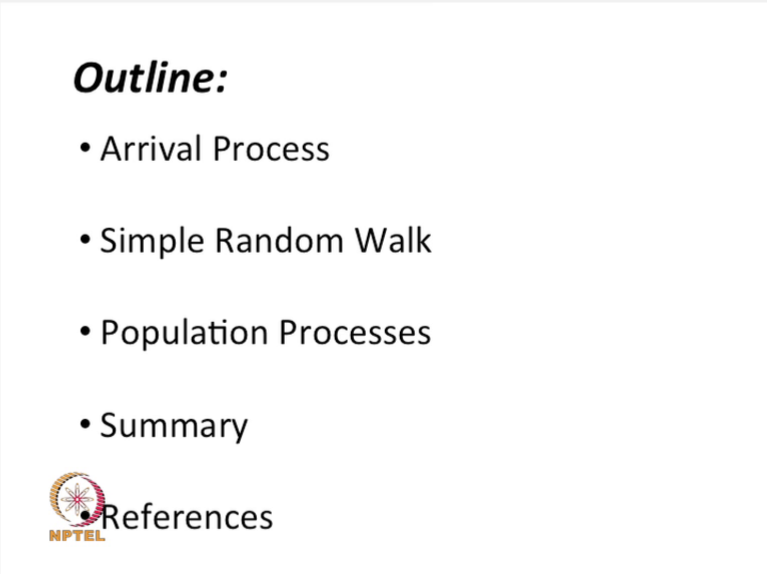

In this lecture, we are going to discuss some simple stochastic process starting with the discrete time arrival process, that is a Bernoulli process and continuous time arrival process, and that is a poison process. Followed by that, we are going to discuss the simple random walk and then, we are going to discuss one simple population process, which arises in the branching process. Then, we are going to discuss the Gaussian process. So, with that the lecture 2 will be over.

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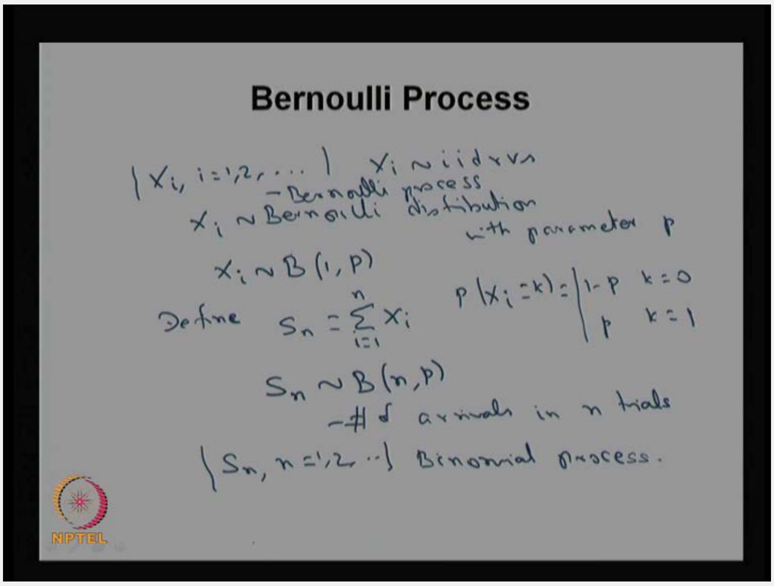

What is Bernoulli process? Bernoulli process can be created by the sequence of random variable. Suppose you think of a random variable x i, where i is belong I takes a value 1, 2 and so on. Therefore, this is going to be a collection of random variable and each random variable are x i's. You can think of x i’s are going to be i i d random variables and each is coming from the Bernoulli trials. That means, each random variable is a Bernoulli distributed. Each random variable is a Bernoulli distribution and with the parameter p.

So, the same thing can be written in the notation form, xi takes the xi’s are in the notation, it is a capital B 1 comma small p. That means, it is a binomial distribution with the parameters 1 and p, that is same as each x i’s are Bernoulli distributed with the parameter 1 and p.

So, now I can, so, this is going to be a stochastic process or you can say it is a stochastic sequence. Now, I can define another random variable for every n, s n is nothing but, sum of first n random variables. Suppose you think x i’s going to be the outcome of the i th trail, so, the x i can take the value 0 or 1. That means, with the probability, the x, each x i can take the value k, if k is equal to 0 with the probability 1 minus p and k taken, k can take the value 1 with the probability p.

Therefore, each, since each x i’s are i i d random variable, you can come to the conclusion s n is nothing but, binomial distribution with the parameters n comma p.

Suppose, you assume that xi is going to be number of, whether the arrival occurs in the i th trial or not, if xi takes the value 0; that means, no arrival takes place in the i th trail. If xi takes the value 1, that corresponding to the i th trail, there is a arrival. So, the s n represents, s n denotes the number of arrivals in n trials.

So, now you can create a stochastic process with s n, where n takes a value 1, 2 and so on. Therefore, this is going to be a binomial process. So, the xi's takes the value 0 or 1 with the probability 1 minus p and p each one is going to be Bernoulli distributed. Therefore, this is going to be a Bernoulli process. This xi's are going to form Bernoulli process. The way you have created s n is equal to sum of first n random variable and each s n is going to be a binomial distribution with the parameters n and p. Therefore, this s n, that sequence of s n for n is equal to 1 2 3 binomial process.

Therefore, since you have collected arrivals over the possible values of 1, 2 and so on, therefore, this is going to be one of the discrete time arrival processes. So similarly, we are going to explain what is the continuous time arrival process. Whereas here, binomial process, this is going to be a discrete time arrival process.

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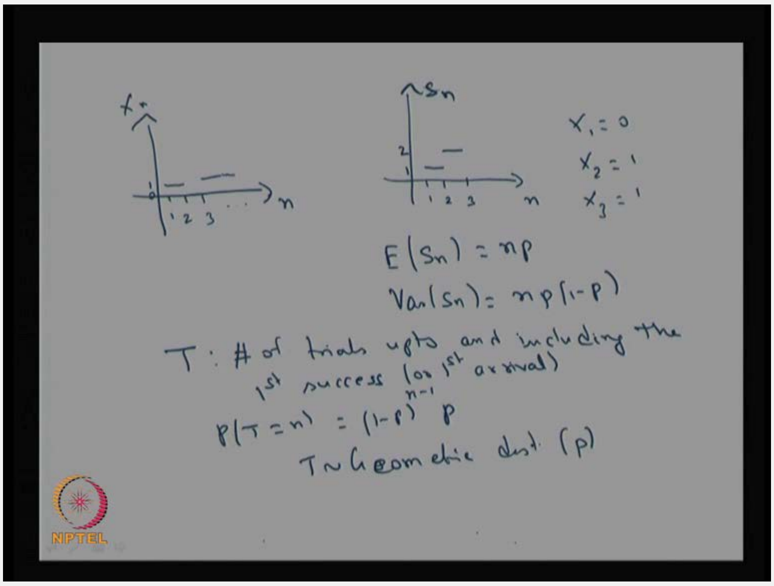

Suppose you would like to see the trace of s n. So, before you go to the trace of s n, we can go for what is the trace or sample path of x i for different values of n is equal to 1, n is equal to 2, n is equal to 3 and so on. If you see, each x i takes the value 0 or 1. Therefore, it can take the value 0 or x 1 can take the value 1 or x 2 can take the value 0 or this can take the value 1. Again it can take the value and 1 and 0. So, the possible values of x i’s are going to be 0 and 1. Therefore, each x i’s can take the value 0 in the horizontal line or it can take the one till you get the next trial.

Similarly, if you make the sample path or the trace of s n, since s n is going to be a sum of first n random variable, therefore, based on the x i, it takes the value. Suppose x 1 takes the value 0 and suppose x 2 take the value 1 and suppose x 3 takes the value 1 and so on. So, since x 1 is equal to 0, therefore s 1 is 0 and then at s 2 is same as x 1 plus x 2. Therefore, it takes a value 1 and s 3 is equal x 1 plus x 2 plus x 3. Therefore, that is going to be again, you are adding the values, therefore, it is going to be a 2. Therefore, this is 1 and this is 2.

So, based on the x 4, it is going to be 0 or 1. Either it can take the value 2 itself or it can go to the 3. Therefore, if you see the sample path of s n, it is going to be incremented, either incremented by 1 or it takes a same value till the next n. Therefore, not only you can find out the s n, not only you can find out the sample path of s n, you can get the mean and variance, because each s n is going to be a binomial distribution with the parameters n and p. Therefore, the expectation of s n is going to be n times p and the variance of s n is going to be n times p into 1 minus p.

So, you can be able to see the sample path of xi’s as well as s n over the different values of n. In discrete time, sample paths are sequences. I can also define the new random variable capital T, is nothing but, number of trails up to and including the first (()). That means, suppose it takes a value n; that means, for subsequent n minus 1 trials, I got the failures or no arrival takes place in the subsequent n minus one trial and the nth trial I get the first arrival. That means, the d is a random variable to denote how many trails to get the first success or the first arrival or the first arrival.

So, if it is going to take the first arrival in the nth trail, then the probability of t takes the value n, that is same as 1 minus p into n minus 1 into p, because all the trials are independent and subsequent n minus 1 trail gives no arrival and the n th trail, you get the first arrival. Therefore, this is going to follow a geometric distribution with the parameter p. So, since you know the distribution of t, you can find out the mean and variance, because the mean of geometric distribution is going to be 1 divided by p and the variance of t is going to be 1 minus p divided by the p square.

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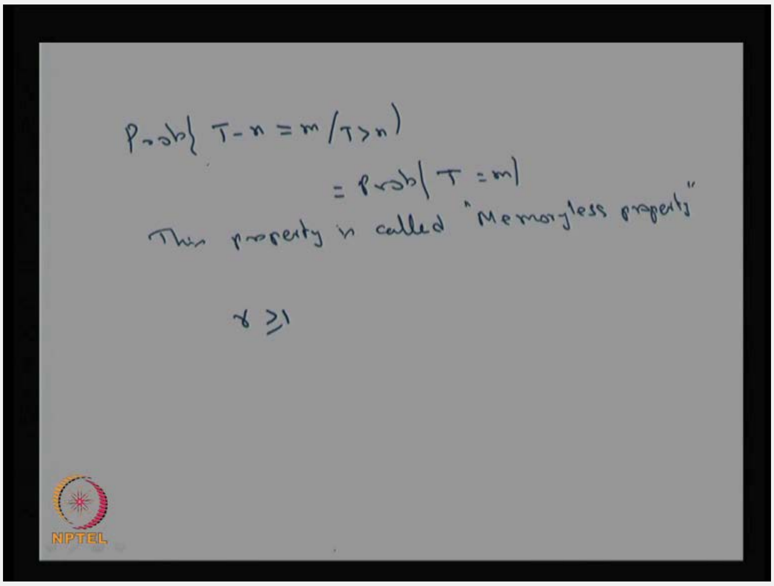

Similarly, I can go for finding out what is the probability that till n th trial, I did not get the first or I did not get the first arrival. So, if n plus m th trail, if I am getting the first arrival, what is the probability that it is going to take after m trails, it gets the first arrival and that probability, you can be able to get that is same as the probability of the t takes the value m. So, this property is called memory less property. Since t is geometrically distributed and the geometric distribution satisfies the memory less property, that can be visualized in this example.

The probability of t minus n is equal to m given that the t takes the value greater than n and that is same as what is the probability that the t takes the value small m. That means, the right hand side result is independent of n and it is the same as the distribution of; that means, the residual arrival, number of arrivals that is same as the original arrival distribution. Therefore, this satisfies the memory less property. So, this is the geometric distribution that satisfies the memory less property in the discrete time and there is another distribution that satisfies the memory less property in the continuous time and that is exponential distribution.

So, the way I have related the binomial distribution from the Bernoulli process, then I get the binomial process also and I was able to create the geometric distribution. You can create the or you can develop the Pascal distribution or negative exponential distribution the way I have defined the capital T is going to be the number of trails to get the first success or first arrival. Instead of that, if I make another random variable to go for how many trials are needed to get the r th success, where r can take the value greater than or equal to 1.

If it is a r th first success is going to happen in the n th trial, if r is greater than 1, then I can go for defining what is the negative binomial distribution for that particular random variable. If r is equal to 1, then that is land up to be the same random variable capital T.

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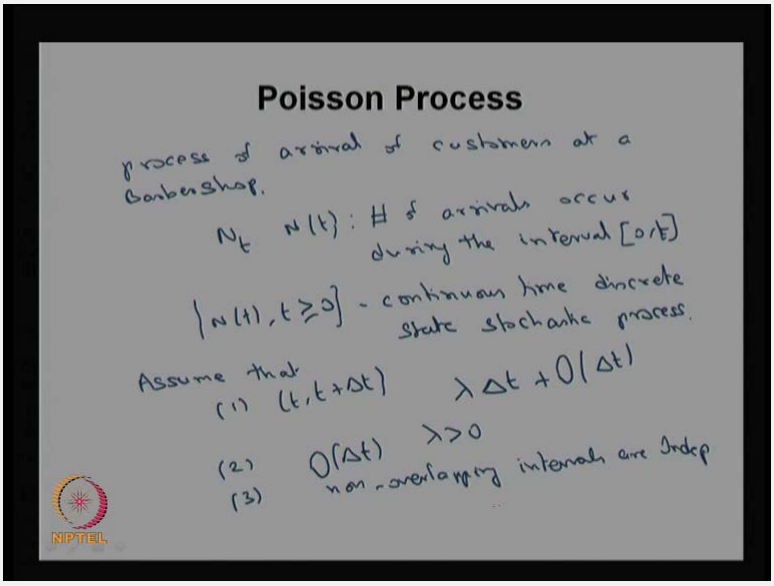

So, till now we have discussed what is the discrete time arrival process. Now, we are going to discuss the continuous time arrival process, and that is a poison process. So, in this lecture, I am going to develop what is the poison process and how we can get the poison process from the scratch. Suppose you consider the process of arrival of customers. Consider the process of arrival of customers at a barber shop.

So, this is the same example we have discussed in the beginning of this course also. So, over the time how many arrivals are going to take place, and that is going to be a random variable. So, let n t, n suffix t or some books they use as n of t. So, the n of t denotes number of arrivals occur during the interval 0 to the closed interval 0 to t. That means, we are defining a random variable n of t that denotes number of arrival occurs during the interval 0 to t for fixed t; n of t is going to be a random variable. Therefore, n of t over the time, because t is greater than or equal to 0 and this is going to be a, since the possible values of capital t that is the parameter space is going to 0 to infinity, therefore this is going under the classification of a continuous parameter or continuous time. The possible values of n of t for different values of t that is going to take a value 0 or 1 or 2, therefore, it is going to be a countably infinite. Therefore, this is going to be a continuous time or continuous parameter discrete state stochastic process.

So, this is the n of t over the t greater than or equal to 0 and that is going to be a continuous time discrete state stochastic process. Now, we are going to develop the theory behind Poisson process. To create the Poisson process you need few assumptions, so that, you can able to develop the Poisson process. The first assumption in a small negligible interval, if the interval is t to t plus delta t, if the small negligible interval is t to t plus delta t, then the probability of one arrival is going to be lambda times delta t plus capital O of delta t. The probability of one arrival occurs during the interval t to t plus delta t is going to be lambda times that smaller interval delta t plus capital order of capital O delta t. Here, the lambda is going to be strictly greater than 0 and we are going to discuss what is lambda and so on the in the later after this explaining the Poisson process.

So, here the lambda is going to be a constant and which takes a value greater than 0 and the capital O delta t means, as a delta t tends to 0, the order of delta t that is going to be tends to 0 as delta t tends to 0. So, this is the first assumption.

The second assumption, the probability of more than one arrival is going to be a order of delta t in the same interval t to t plus delta t. More than one arrival in this small negligible interval that probability is order of delta t. That means, as a delta t tends to 0, these values is going to tends to 0.

Then, the third assumption, occurrence of arrivals in a non-overlapping intervals are mutually independent non overlapping intervals are independent. So, this is a very important assumption. That means, what is the probability that the arrival occurs in a non-overlapping intervals, that probability is same as the product of probability of arrival occurs in the each interval. Therefore, it is going to satisfy the independent property occurrence of events in non-overlapping intervals is mutually independent. Therefore, the probability is going to be probability of intersection of all those things is same as the probability of individual probability and that product.

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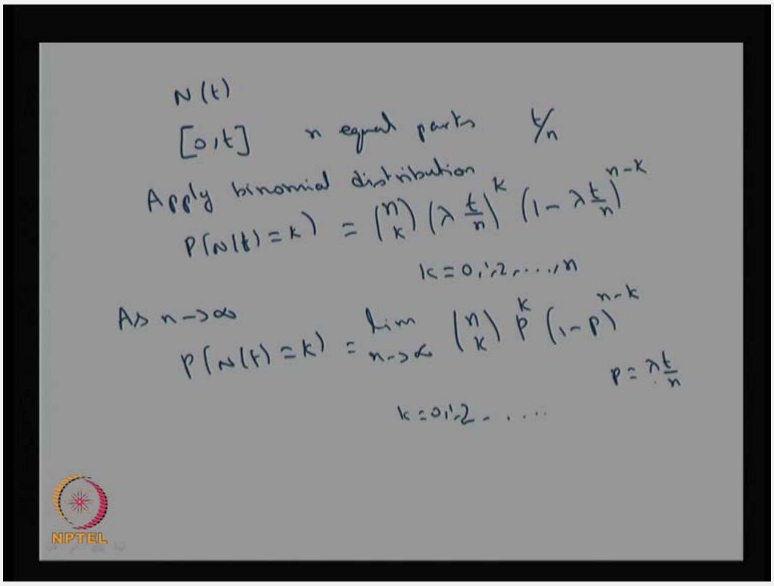

So, with these three assumptions, we are going to develop the Poisson process. So, what I am going to do, since I started with the random variable n of t is a number of arrivals in the interval 0 to t, I am going to partition the interval 0 to t into n equal parts. I am going to partition the interval 0 to t into n equal parts, since I made the interval 0 to t into n equal parts, then each will be of the length t by n.

Since, I made the assumption the non-overlapping intervals are independent and the probability of one arrival is lambda times delta t and the probability of more than one arrival is order of delta t and so on. Therefore, I can apply binomial distribution the way I have partitioned the interval 0 to t into n pieces, therefore, this is going to be a of n intervals of interval length t by n. Therefore I can say, what is the probability that I can be able to find out, what is the probability that k arrivals takes place in the n intervals of each length t by n, what is the probability that k arrivals take place? Therefore, the possible values of k is going to be 0 to n and I can be able to find out by using the binomial distribution, what is the probability that n of t takes the value k. Since, non-overlapping intervals are independent and each probability of one arrival is lambda times delta t, where delta t is t by n, so, each interval behaves as a Bernoulli trail, whether the arrival occurs or there is no arrival.

Like that, you have n such independent trials. Therefore, the sum of n independent Bernoulli trails land up binomial trials. Therefore, by using the binomial distribution, I can able to get what is a probability that n of t takes a value k, that is what is the possible n c k ways and what is the probability of arrival takes place in one interval, that is lambda times this interval length is at by n.

Therefore, it is a lambda times t by n power t by n power sorry t by n power lambda times t by n power k and what is the probability of no arrival takes place in each interval that is 1 minus lambda times t by n power n minus k. So, this is a way I can able to get what is the probability that k arrival takes place in the interval 0 to t by partitioning n p n intervals. So, this is a probability.

But, the way I made a partition n equal parts, so now, I have to go for what is the result as n tends to infinity. That means, my interest is what could be the result, if n tends to infinity of k of, what is the probability that n t takes a value k as n tends to infinity. Therefore, the running index for k is going to be 0 1 2 and so on. What is a probability of n t takes a value k. That means, in the right hand side, I have to go for finding out as n tends to infinity, what is the result for the right hand side and what is the probability of n t takes a value k. We take n tends to infinity because we need to study the limiting behavior of the stochastic process.

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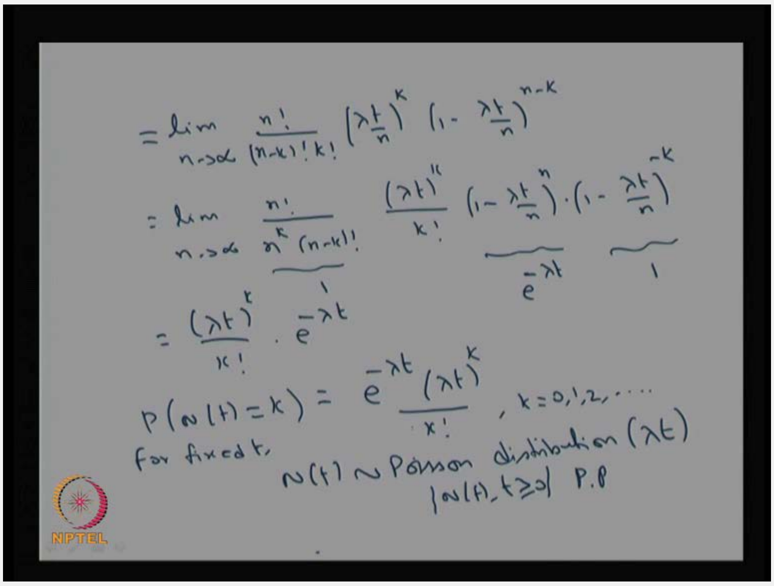

So, that is same as limit n tends to infinity of n c k, I can make it as a p power k, where p is going to be lambda times t by n and 1 minus p power n minus k. Now, I have to find out what is the result for limit n tends to infinity of this expression n c k p power k 1 minus p power n minus k, where p is going to be lambda times t by n. If I do the simple calculation, let me explain. So, limit n tends to infinity that is same as limit n tends to infinity of n c k, I can make it as a n factorial n minus k factorial and k factorial and that is lambda t by n power k. That is 1 minus lambda t by n power n minus k and that is same as the limit n tends to infinity of n factorial. Here, this n power k, I can take it outside and n minus k factorial and lambda t power k and divided by k factorial.

So, this k factorial, I take it inside and the power 1 minus lambda t by n power n minus k, I split into 1 minus lambda t by n power n into 1 minus lambda t by n power minus k. So now, I can look as n tends to infinity, this is nothing to do with n and therefore, lambda t power k by k factorial will come out. So, this result is going to be lambda t power k by k factorial. This will land up as n tends to infinity, this is going to be E power minus lambda t and this will land up 1 and this is also land up 1 as n tends to infinity. Therefore, I may land up it is E power minus lambda t. Hence, the final answer of what is the probability that k arrival takes place in the interval 0 to t, that is going to be E power minus lambda t and lambda t power k by k factorial and the possible values of k can be 0 1 2 and so on.

For fixed t, if you see, this is same as, for fixed t; it is going to be a random variable. For all possible values of t, it is going to be a stochastic process. So, for fixed t, the n of t is a random variable and that probability mass function is E power minus lambda times t lambda t power k by k factorial. So, lambda is a constant for fixed t; lambda into t that is going to be a constant. Therefore, the right hand side looks like the probability mass function of the Poisson distribution. Therefore, for fixed t, the n of t is Poisson distribution. The random variable n of t for fixed t, it is going to be a Poisson distribution with the parameter lambda times t.

Lambda is a constant and for fixed t, t is a constant. So, lambda multiplied by the t again this is going to be a constant therefore, for fixed t it is going to be a Poisson distribution with the parameter lambda multiplied t. Therefore, for possible values of t, the n of t is going to form a stochastic process. Since for fixed t, it is going to be a Poisson distribution, the collection of a random variable and each random variable is a Poisson distribution. Therefore, this is going to be called as the Poisson process.

The way I have we have explained earlier, each random variable is a Bernoulli distributed random variable and the collection of random variable is a Bernoulli process. Similarly, each s n is going to be a binomial distribution and therefore, the collection is going to be a binomial process. The same way for fixed t, it is going to be a Poisson distribution. Therefore, that collection is going to be called as Poisson process. So now, we have developed n of t is going to be a Poisson process, because for fixed t, it is going to be a Poisson distribution. Therefore, this collection of random variable is going to be called as a Poisson process.

Here, the lambda is a constant and there is another name for the default Poisson process is called a homogenous Poisson process, because there is another one called non-homogeneous Poisson process, in which, the lambda need not be a constant. It can be a function of time t also. Therefore, the one we have derived now, it is a homogenous Poisson process in which the lambda is a constant, which is greater than 0. When lambda is going to be a function of t, the corresponding Poisson process is called non-homogeneous Poisson process. So, this is the one particular and very important continuous time or continuous parameter discrete state stochastic process, and that is a Poisson process or this is also we can say, this is going to be a very important continuous time arrival process that is a Poisson process.

The way we are counting n of t is going to be the number of arrivals over the interval 0 to t or number of occurrence of the event over the t, the way you are counting over the time. Poisson process is an example of counting process.

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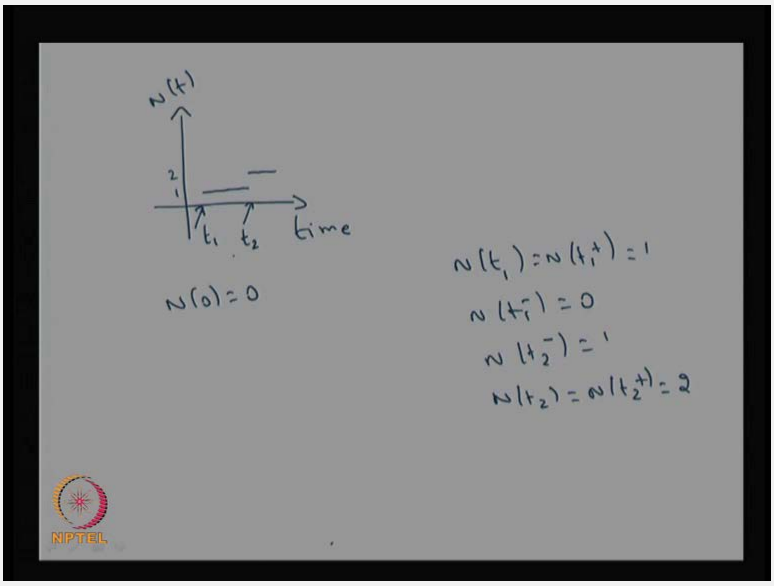

So, the N of t is also called counting process. So, the Poisson process is also call it as the counting process. I can go for giving the sample path of N of t over the time. What is the different values of N of t is going to take? Obviously, N of 0 is equal to 0. Whenever some arrival occurs in some time, then the arrival is going to occur. Therefore, suppose the arrival occurs at this time, I make it as the up arrow, then the value of N of t is going to be incremented by 1 till the next arrival comes. Suppose the next arrival takes place at this time point, then the N of t values is going to be 1 till the time and it is going to be a right continuous function; that means, the time point in which the first arrival occurs, suppose you to make it as a t 1, so, the N of t 1 minus is going to be 0 and the t 1 and the n of t 1 plus t 1, as well as n of t 1 plus, that is going to be 1, whereas, the left limit n of t 1 minus, that is going to be 0.

Suppose the second arrival occurs at some time point t 2, then the n of t 2 minus, that is the left limit at the time point t 2, that is going to be 1 and the n of t 2, that is same as n of t 2 plus, that is going to be 2. So therefore, it is incremented by 1. So, the value is going to be 2. So, this is the random time in which the arrival is going to occur and the way we have made the assumption in a very small interval, only one maximum only one arrival can occur. Therefore, the n of t is going to be a non-decreasing, right continuous and increased by jump of size 1 at the time of epoch of arrival. So, whenever you see the sample path of the Poisson process, it is always going to be a non-decreasing, right continuous and increase by a jumps of size 1 at the time of at the time epoch of arrivals.

Now, I am going to relate another random variable, which involves in the Poisson process or I am going to discuss another stochastic process, which involved in the Poisson process.

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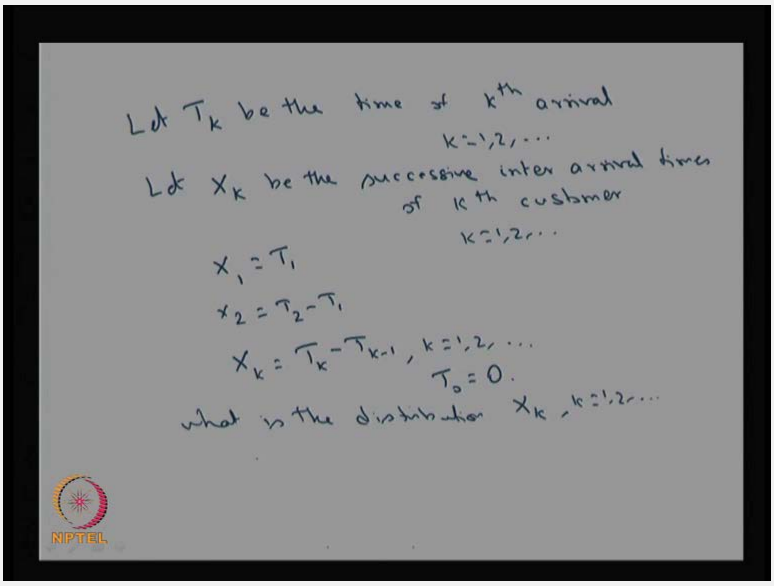

So, for that, I am going to define the new random variable as, let t suffix k be the time of k th arrival. So, k can take the value 1 or 2 and so on. So therefore, the t be the random t be the random variable takes what is the time Point in which the k th arrival occurs. That means, the way I have given the sample path in the previous slide, the t 1 and t 2, the small t 1 and t 2 are the different values of the capital T k. I am going to define another random variable x suffix k be the successive inter arrival times of k th customer. So now, the k can take the value 1 2 and so on.

So, the T k be the time point, whereas the x k be the inter arrival time. That means, the x 1 is nothing but, t 1 minus t 0 and obviously, t 0 is 0. Therefore, x 1 is same as t 1 and x 2 is nothing but, t 2 minus t 1. That means, what is the inter arrival time for the second arrival, that inter arrival time is what time the first arrival occurs. That is, a t 1 and what time the second arrival occurs, that difference is going to be the inter arrival of the second customer. So, this is the way I can define x k is going to be t suffix k minus t suffix k minus 1. So now, the running index for k can take the value 1 and so on; obviously, t 0 is going to be 0. Our interest is to find out what is the distribution of x k for all k 1 2 and so on. Is it feasible to find out the distribution of x k? It is possible. First we can start with k equal to 1. What could be the distribution of x 1? Then, once we get the x 1 distribution, the same analysis can be repeated to get the distribution of x 2 and x 3 and so on, because the scenario which we are going to take it for finding out the distribution of x 1, that is the same as for the x 2 and so on.

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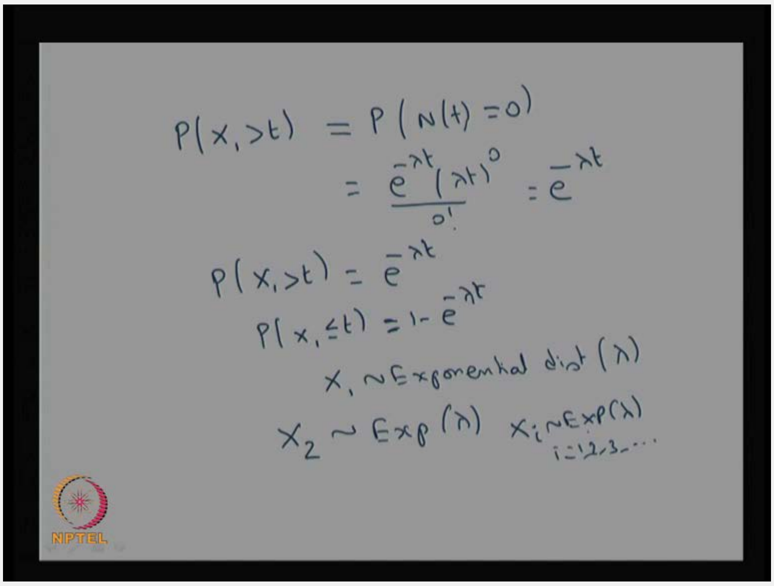

So now, our interest to find out, what is the distribution of x 1? First, we will try to find out that x 1. Now, we will find out the distribution of x 1. Since x 1 is a continuous random variable, we can go for finding out what is the compliment c d f of x 1. So, this is a compliment c d f of x 1 that is nothing but, what is the probability that the first arrival occurs after time t. That is same as what is the probability that till time t no customer enter into the system. The left hand side is the unknown, whereas, the right hand side is the known one. So, we are relating two different random variables.

So here, this is the, what is a probability that the first arrival occurs after time t; that is same as what is the probability that no arrival takes place during the interval 0 to small t. But, we know what is the probability of n t is equal to 0, because just now we have made it for each t, this is going to be a Poisson distribution with the parameter lambda times t. Therefore, the probability of n t equal to 0, that is same as E power minus lambda t and lambda t power 0 by 0 factorial and this is same as E power minus lambda t. So, the left hand side is the unknown. The unknown is what is the probability that x 1 takes the value greater than t, and that is same as E power minus lambda t.

Therefore, we can get what is the probability of x 1 less than or equal to t, that is same as 1 minus E power minus lambda t. So, this is going to be a, what is a c d f for the random variable x 1 and the c d f of x 1 is same as the c d f of exponential distribution with the parameter lambda times t. Therefore, we can come to the conclusion x 1 is going to be exponentially distributed.

The x 1 is exponentially distributed with the parameter lambda. So, the unknown distribution x 1, first we are trying to find out what is the compliment c d f of x 1 and that land up to be E power minus lambda t and therefore, the c d f of x 1 is going to be 1 minus E power minus lambda t. From this, we conclude the x 1 is going to be exponentially distributed with the parameter lambda, where lambda is a greater than 0.

The way we have compute, the way we get the distribution of x 1, similarly, one can show x 2, that is the inter arrival time of the second customer enter into the system, that is also can be proved it is exponential distribution with the parameter lambda. Not only x 2, we can go for the further all the x i's. So, we can able to prove all the x i's are going to be exponential distribution with the parameter lambda for i takes the value 1 2 and so on. Not only that, we can able to prove all the x i's are independent random variable also and identical with each one is exponential distribution with the parameter lambda.

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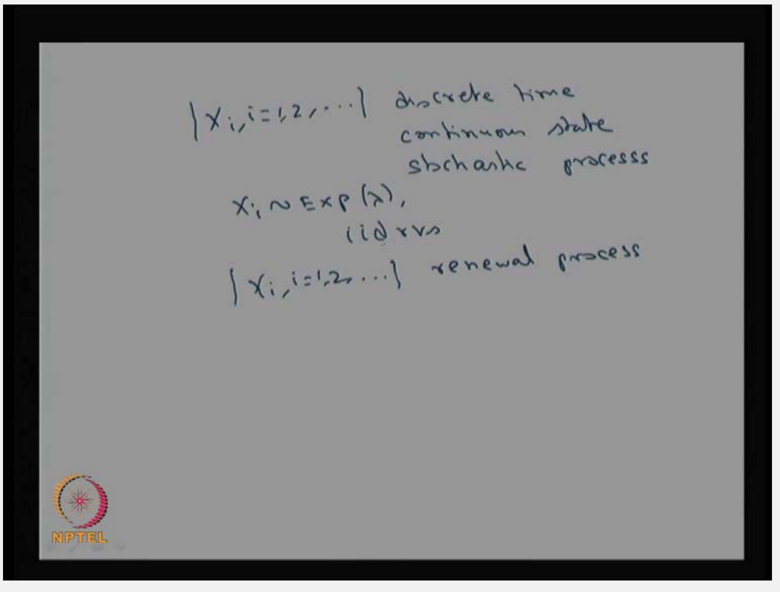

Therefore, the way we land up relating Poisson process with the inter arrival time, so, this x i's will form a discrete time or discrete parameter continuous state stochastic process, in which each random variable x i is going to be a exponential distribution with the parameter lambda and all the x i’s are i i d random variable also. This each x i’s are nothing but, inter renewal time. Therefore, this is going to be called it as renewal process. We are going to discuss the renewal process in detail later of this course, but here, I am just explaining how will you create the renewal process from the Poisson process and the n of t is a Poisson process for different values of t, whereas, the inter arrival time, that is the time in which the renewal takes place or the arrival takes place.

Therefore, the renewals will form a stochastic process and that corresponding process is called a renewal process. Therefore, this is going to be one particular type of renewal process in which the renewal takes place of exponentially distributed time intervals and all the times are i i d random variables also. Now, I am going to explain how we can create the sample path of the Poisson process using the matlab code.

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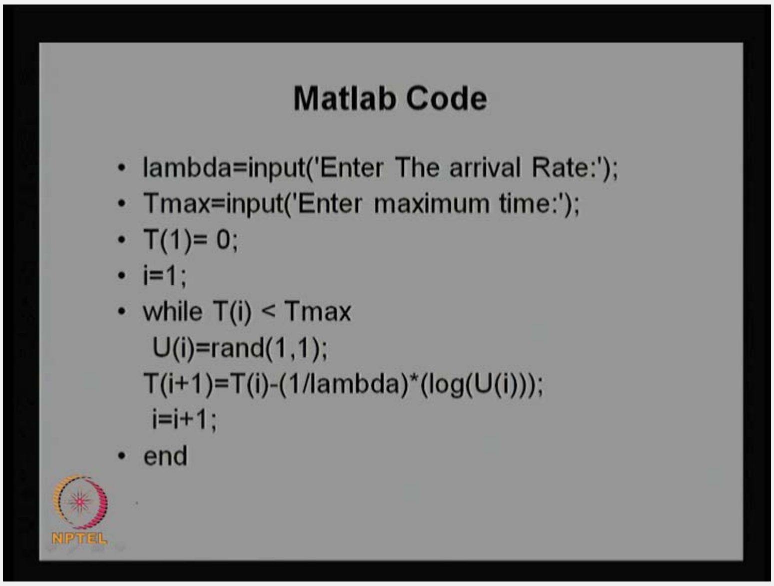

So, since I said the Poisson process is related with the inter arrival times are exponential distribution, so, I can start with the time 0. There is no customer in the system and I can go for what is a maximum time I need the sample path, and then I can keep on create the random variables. From the random variable, I can generate the exponentially distributed the time event, then I can shift the time event by t of i plus 1 by adding the next exponentially distributed time event and then, I can go for plotting the sample path.

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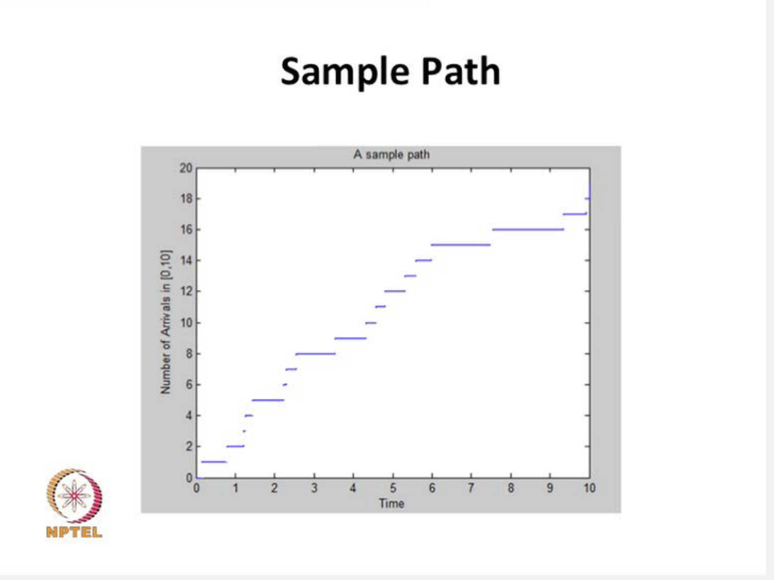

So, this is the one sample path, in which over the time from 0 to 10, the number of arrivals occurs in the interval 0 to time, 0 to 10 in the form of; that means, there is 1 arrival occurs at this time. Therefore, the n of t values is incremented by 1 and it is taking the same value and when at the second arrival occurs, and then the increment is taken by 2 and so on.

If you see carefully the sample path, you can find out the increment is always by one over the time and there is no two arrival or more than one arrival in a very small interval of time. You can come, you can able to see the inter arrival time that is going to be exponentially distributed with a parameter lambda, whatever the lambda I have chosen in this sample path. So, this is the way the sample path of the Poisson process look like. Now, we are going to discuss the third type of stochastic process that is a simple random walk.

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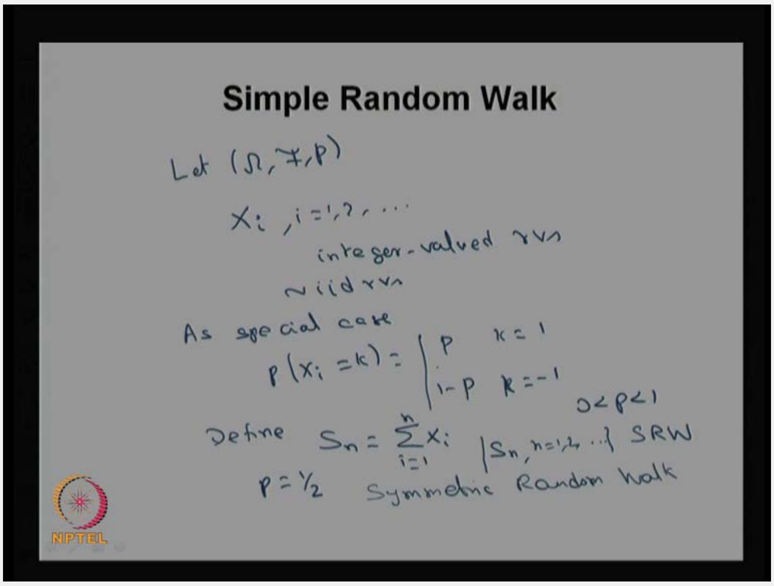

So, how we can create a simple random walk, let me explain. You have a probability space. From the given probability space, you define a sequence of random variable x i's and those random variables are integer valued random variables. Each x i's are integer valued random variable. Not only that, all the x i's are i i d random variables also. All the x i's are i i d random variables and each one is integer valued discrete type random variable. As a special case, I can go for the random variable x i takes a value 1 or minus 1 with the probability p and 1 minus p. This is a special type of random walk. In general I am going to define the in general random walk also. As a special case, I will go for the random variable x i's takes the value 1 with the probability p and x i takes the value minus 1 with the probability 1 minus p, where the p can take the value 0 to 1.

Now, I am going to define the random variable s n that is nothing but, sum of x i's. Sum of first n x i's, that is going to form the random variable s n and the stochastic process s n or the stochastic sequence s n for different values of n, this will form a simple random walk. The s n is going to form a simple random walk. Why it is simple because, it is going to take an integer value random variable and each values are going to take, each random variable is going to take the value 1 or minus 1. Therefore, this is going to be called it as a simple random walk.

In general, the k can take any integers. Accordingly, you land up having s n's are going to be a random walk. I am going to give another special case, when p is equal to half;

that means, each xi random variable takes a value 1 with the probability of or minus 1 with the probability of, then that random walk is going to be called it as a symmetric random walk. Why it is symmetric because, with the probability of it takes a forward one step or with the probability of it takes the backward one step, therefore, that type of random walk is called a symmetric random walk. In general, if it takes a value 1 or minus 1, then it is called a simple random walk. If k can take any integers, then it is going to be called it as a generalized random walk.

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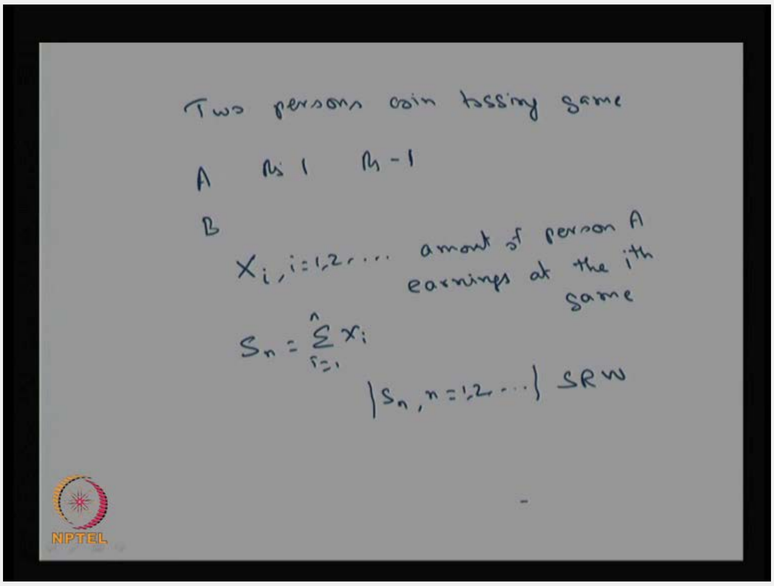

So, this is, this random walk can be created in a simple example of two persons coin tossing game also. This simple random walk can be explained by the example two persons coin tossing example, in which you have a person A and B. If at the end of the coin tossing, if he is going to head, then he is going to win rupees 1 or if he is at the end of the n th coin tossing, if it is going to get the tail, then he is going to lose in this game. If A wins, then B gives rupees 1 to A and if A loses, then A gives rupees 1 to B.

So accordingly, I can go for creating a random variable x n or xi for i is equal to 1 2 and so on. Therefore, xi denotes what is the amount of the person A earning at the i th game. Similarly, we can construct a stochastic process for player B and calculate the measures of interest. I can go for creating a random variable s n is nothing but, summation of xi’s, where I is equal to 1 to n. Therefore, the s n denotes what is the amount earned by the person A at the end of nth game and that is the total amount. So, the x i denotes how much he is going to earn at the end of each game, whereas, the s n is going to be the total amount earned by the person A at the end of first n games. Therefore, this s n is going to form a simple random walk, where x i's are going to take a integer valued with the value 1 and minus 1 with the probability p, it is going to take the value 1 or it is going to take the value minus 1 with the probability 1 minus p. So, I am just relating the simple random walk with the simple scenario of two persons coin tossing game.

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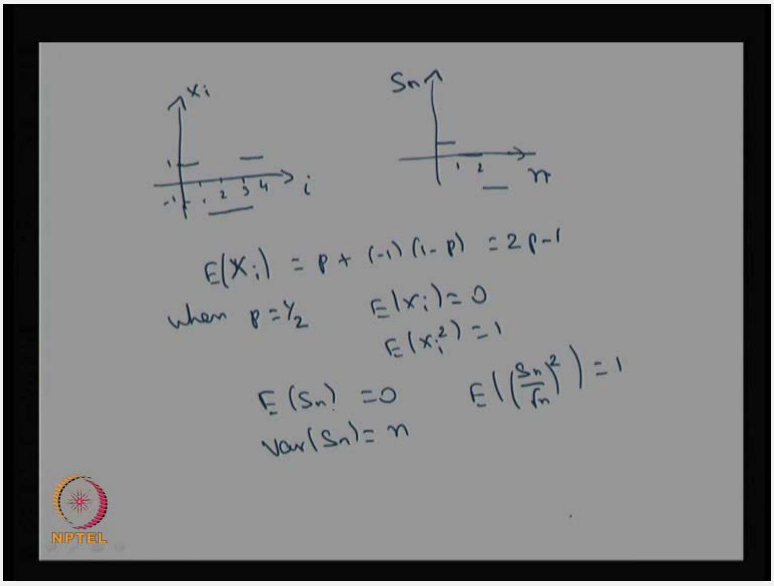

If you see the sample path of the s n, first I can go for what is the sample path of each x i's. Each x i's can take the value 1 or minus 1. Therefore, it is going to take the value 1 or minus 1. Therefore, if x 1 takes the value 1, it is 1. If x 2 takes the value minus 1, it is like this. If x 3 takes the value minus 1, then it is here. If x 4 takes the value 1, then it is like this. So, this is a sample path of xi over the i. The way I have given the x i's, writing what is the possible values of n and what is the possible values of s n. So, since x 1 is equal to 1, therefore, s 1 is going to be 1 and x 2 is going to be minus 1. Therefore, it takes a value 1 plus minus 1 therefore, it is going to be 0.

x 2 is going to be minus 1, therefore, s 2 is x 3 and x 3 is going to be minus 1 and x 4 is going to be 1. Therefore, it is going to be again 0. So, this is the way the sample path goes over the n. So, this is the one sample path for the possible values of x i takes the value 1 and minus 1. Accordingly, I have drawn the sample path of s n over the n.

Since xi's are going to take the value 1 and minus 1 and with the probability p and with the probability 1 minus p takes the value minus 1, I can go for finding out what is the expectation of xi. That is nothing but, x i is equal to p plus minus 1 times 1 minus p. Therefore, this is nothing but, 2 p minus 1. So, when I go for discussing the symmetric random walk, when the p is equal to half, then the expectation of each xi is going to be 0 and also I can able to find out, what is E of x i squares and that is going to be 1. Not only that, when p is equal to half, I can able to find out what is the expectation of s n, that is going to be 0. The variance of s n is going to be n and I can go for writing what is the expectation of s n by root n power n power 1, and that is going to be 1.

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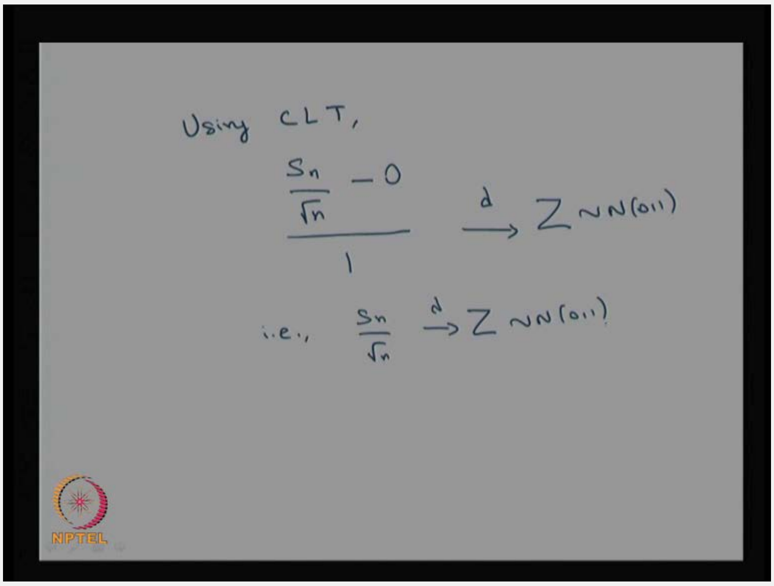

So, the way I have got the result for expectation of s, expectation of xi's and the expectation of s n, I can go for what is the limiting distribution of s n. So, using central limit theorem, I know what is the mean for each s n and I know what is the variance of each s n also and therefore, using a CLT, I can be able to conclude s n divided by square root of n minus the mean of this random variable is 0 divided by the standard deviation is going to be 1. This, as n tends to infinity, this will be a standard normal distribution, where z is going to be a standard normal distribution as n tends to infinity and this convergence is via distribution. That means, I can able to conclude the distribution of s n by squared root of n as n tends to infinity in distribution and this sequence of random variable will convergences to the standard normal in distribution.

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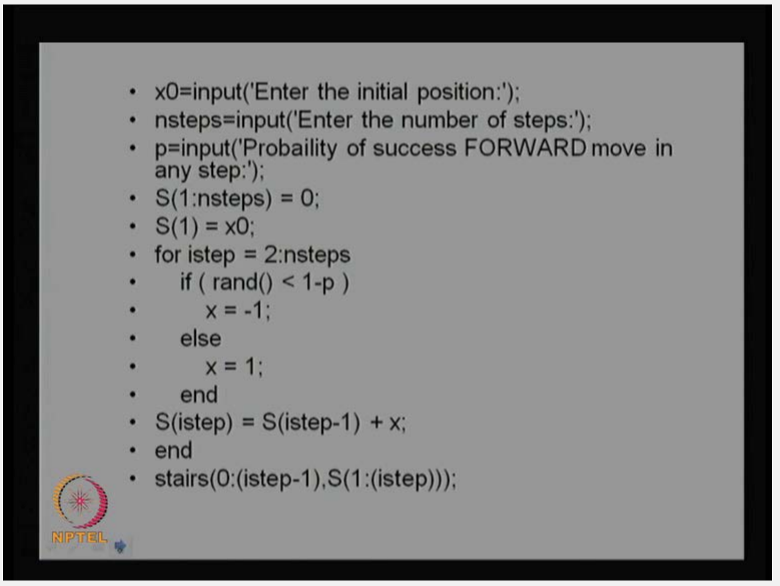

I can go for creating what is a sample path of the simple random walk by using the matlab code. So for that, I have to fix what is the initial position and what is a maximum number of steps I would like to go for finding the sample path and what is the probability of success in each for and what is a forward move probability.

Accordingly, it is going to take the value 1 with the probability p and it is going to take the value minus 1 with the probability 1 minus p. So, I am giving the value of p only and then, I am just going for the possible values of s n by adding the 1 or minus 1. Accordingly, I am just writing the sample path of x i’s.

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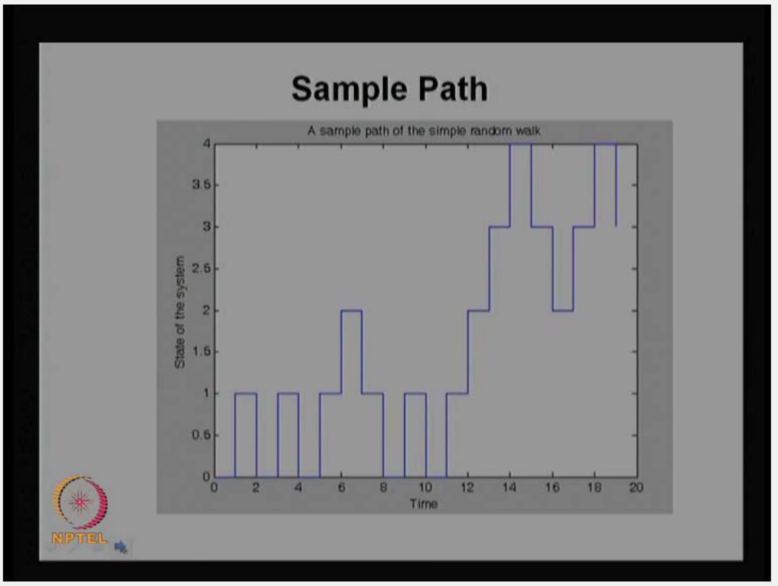

So, if you see the sample path over the time 0 to 10 and each x i's are going to take the value 1 or minus 1, accordingly the s n is going to take the same value or incremented by 1 or decremented by minus 1, according to the values of x i's. Therefore, this is going to be the one sample path, which is depicted using the matlab code.

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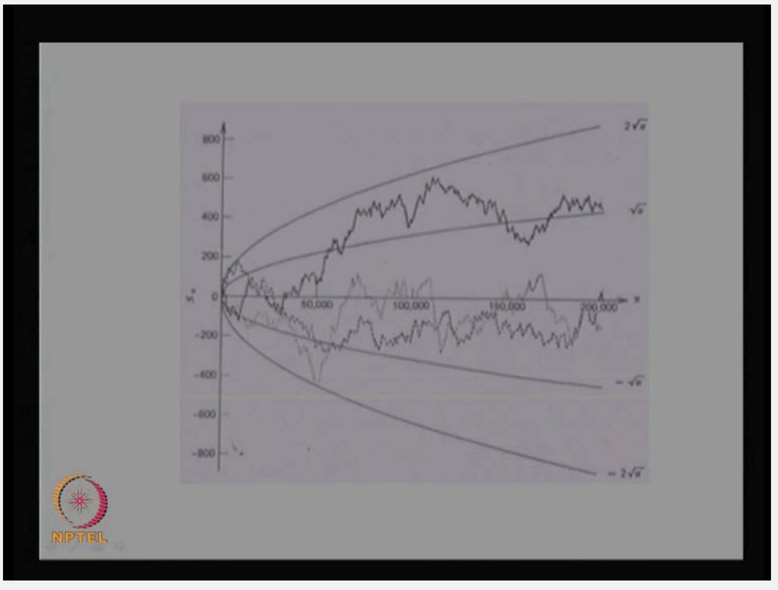

So, this is the earlier I have shown the same graph. This is the s n as n tends to infinity; here you can see the different sample path for as n tends to infinity, you can find out what is the distribution of the s n divided by square root of n as n tends to infinity also.

These figures, it has a 3 different sample path and one can observe, what is the amount of a person A have as n tends to infinity, that depends on whether he is going to take the positive value or he is going to have the negative value, depends on the first few games that can be observed from this diagram.

The first few results, whether he is going to gain by 1 rupee or he is going to lose by 1 rupee, accordingly the possible values of s n will go as n tends to infinity. Now, we are going to discuss the fourth simple stochastic process that comes in the population model.

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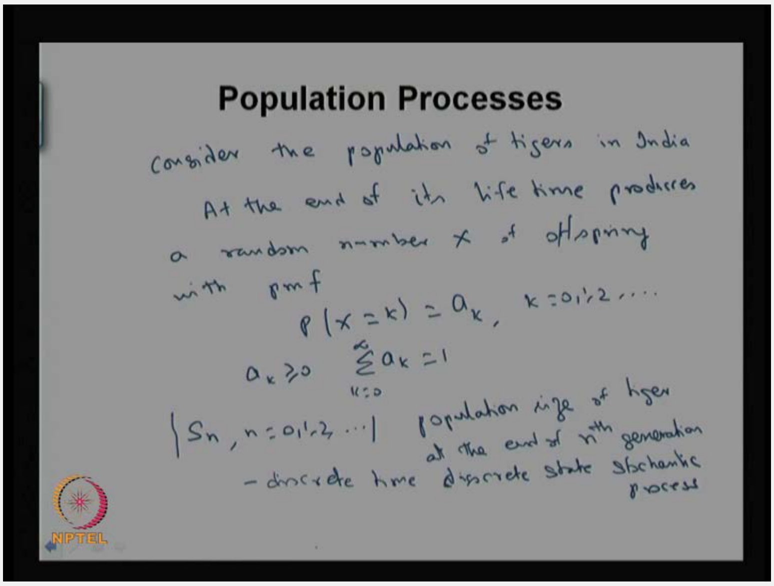

Now, we will see the fourth simple stochastic process arises in the population model. You consider a population of Tigers in India. So, that is going to be a, sorry, over the time, this is going to perform a stochastic process. So, I am going to make the assumption at the end of its life time, it produces a random amount random number x of offspring with the probability mass function, that is a probability of x takes the value k, that is a k where it satisfies, a k's are going to be greater than equal to 0 and the summation is going to be 1. Also, I am making the assumption all the off springs act independently of each other and at the end of their lifetime, individually can have a pregnancy accordance with the probability mass function, the same probability of x i's takes the value k.

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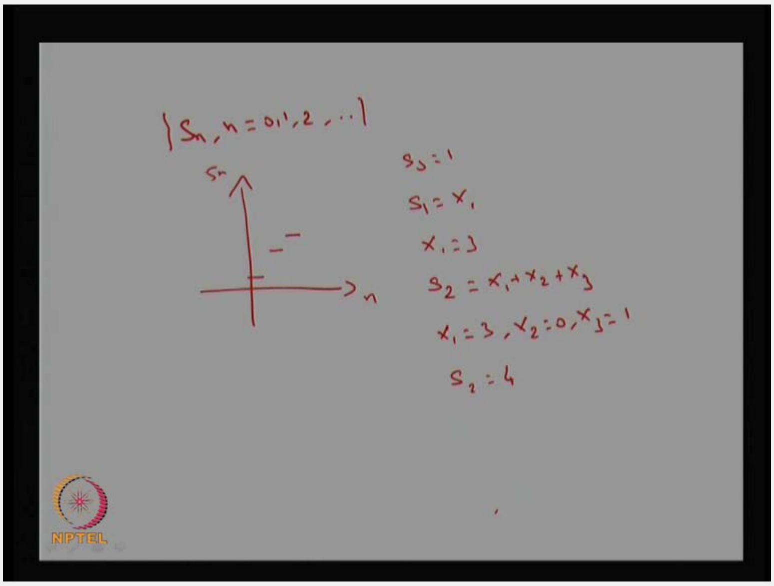

With this, s n will form a discrete time and discrete state stochastic process, where s n is the population size of a Tiger at the end of n th generation. If you see the sample path of s n over the different generation, suppose you make it s naught is equal to 0 and suppose you make it s 1 is equal to x 1 and suppose x 1 takes the value 3 and then, the second generation s 2 is going to be x 1 plus x 2 plus x 3 and suppose you make it x 1 takes the value 3 and x 2 takes the value 0 and x 3 takes the value 1, then we have a s 2 is going to take the value 4.

So, if you see the sample path of s n over the n, it is going to take the value 1, then it is going to take the value 3, then it is going to take the value 4 and so on and this is the sample path of the population size of a Tiger over the n th generation. This is going to form a discrete time discrete state stochastic process. There is a another stochastic process Gaussian process that I will discuss in the later lectures.

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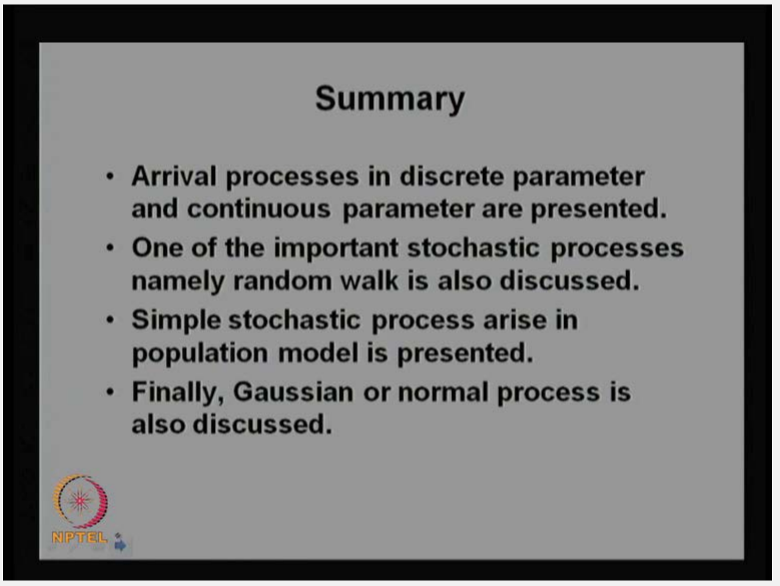

In this lecture, we have covered the arrival process of the two types, one is a discrete time and another is the continuous time arrival process. We have also discussed the random walk and we have discussed a simple stochastic process arises in the population model and the Gaussian process, that I will discuss later.

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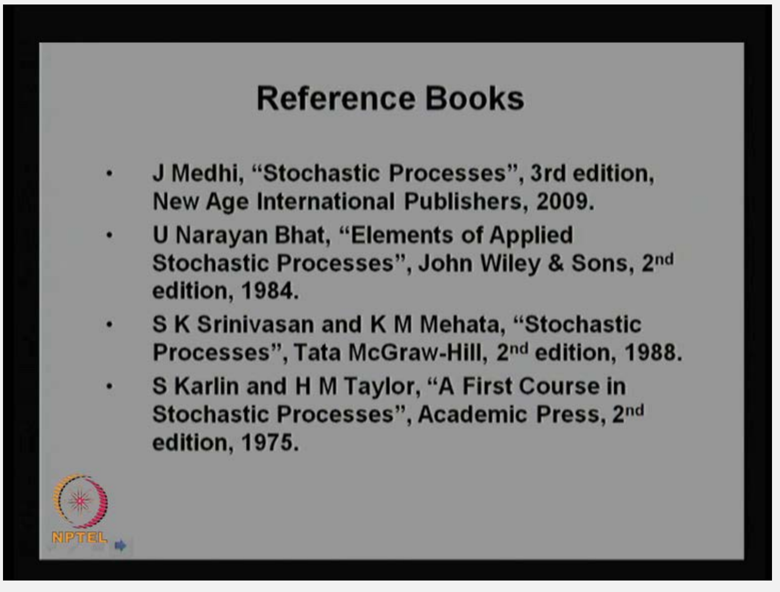

The references books are which, so, with this, I complete the module 2 of definition and the simple stochastic processes.

Thank you.