Stochastic Processes
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Module - 2
Definition and Simple Stochastic Processes
Lecture - 1
Definition, Classification and Examples

This is the module two of stochastic process. In this module, what we are going to discuss is definition, then followed by the simple stochastic process. This module consists of two lectures, and here this is the first lecture in which we are going to describe the stochastic process. Then we are going to discuss the classification of a stochastic process followed by few simple examples, which arises in the real world problem.

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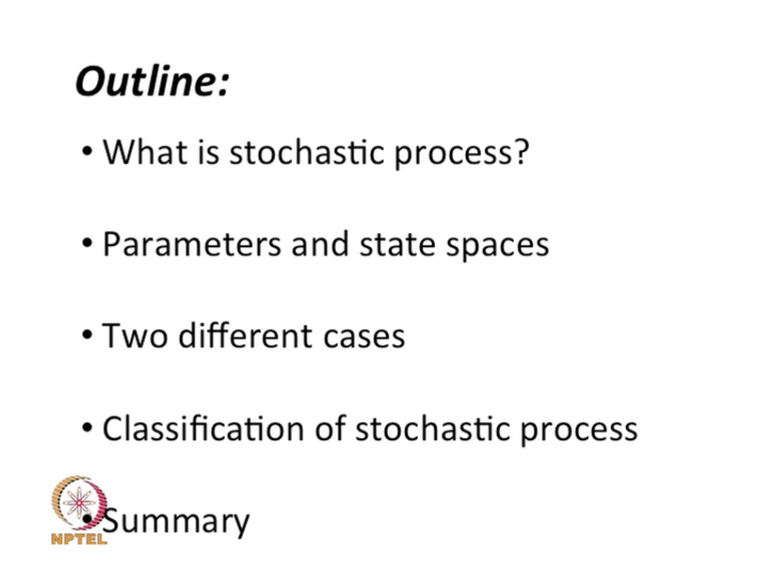

So, the content of this lecture is going to be as I said, let me first give the definition of a stochastic process. Then I will explain how to create or how to develop the stochastic process and how to what is the meaning of a parameter and the state space. Then I am going to give what are all the approaches in which the stochastic process can be described and the classification of a stochastic process based on the parameter and the state space.

Then, at the end of this lecture, we are going to discuss some of the few simple stochastic processes and the summary of the lecture 1. There are few reference books also listed for this course for this preparation.

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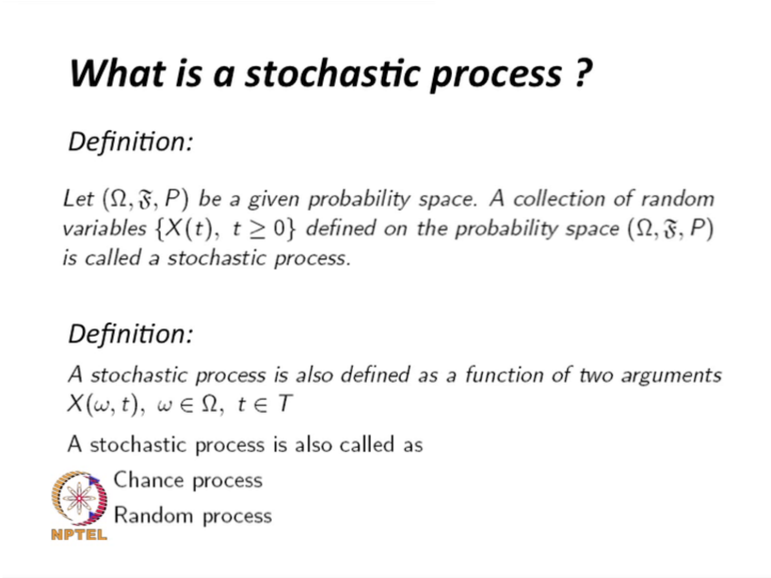

What is stochastic process? Let me give the definition. Let omega f p be a given probability space. That means, you know what is a random experiment. From the random experiment, you know what is the omega and from the collection of possible outcomes, you got the sigma algebra, and that is capital f and you have a probability measure also.

Therefore, this triplet is going to be the probability space and you have a given probability space. From the given probability space, you have the collection of random variables, that is x of T, where T is belonging to capital T defined on the probability space, that is omega f capital P, that is called a stochastic process.

That means, you have a probability space. From the probability space, you have collected random variables, with the T belonging to capital T. This collection is going to be called as a stochastic process. Now, the question is whether we can create only one stochastic process or how to create a stochastic process from this sigma algebra.

That means, suppose you have an omega. From the omega, you can always create a sigma algebra, that is a capital F, and that is a collection of subsets of omega satisfying the condition. If you make a union of few elements, then if you make the elements, if you take a few elements, then the union of elements is also belonging to one of the element. If you take any one of the elements in the f, then the compliment is also belonging to f.

So, if these conditions are going to be satisfied, then that collection of subsets of omega is going to be called as sigma algebra. So, from the omega, we have created a random variable, that is x of T 1. That is nothing but a random variable, that is nothing but a real valued function, which is defined from omega to r, such that, it satisfies the condition x of T 1 of inverse of minus infinity to the closed interval x, that is belonging to f for all x belonging to r.

That means, whatever be the x belonging to r, if the inverse images from minus infinity to some point x, if that is belonging to capital F, then that real valued function is going to be called as a random variable. Like that, if you make a different random variable for different T, where all the T i's are belonging to, so I can go for T I, where I is, so all the t's are belonging to capital T. So, that means, if I have a collection of random variables for the different values of T, then that collection is going to be called as a stochastic process.

Now, the question is, whether we can create only one stochastic process from a given probability space or more than one stochastic process can be created from the same probability space. The answer is yes. You can always create more than one random variable from the same probability space. That means, for a different collection of a capital T, you can have a different stochastic process. More than one stochastic process can be created from one probability space.

Now, the next question, if I change the sigma algebra, what happens? If I change the sigma algebra capital F, then I may land up collecting some other stochastic process, in which those real valued function is going to be a random variable for that particular omega and f and p and for a given probability space, the stochastic process is going to be changed for a different collection of a T belonging to capital T.

That means, once you know the f, then you will have some collection of random variable and that will form a stochastic process. If you change another f, then you may get the different stochastic process and also for a given probability space, you can have more than one stochastic process. By the way you define a collection of random variable, the way you have capital T, accordingly you will have a different stochastic process.

Now, the way I have given the collection of random variable, I can say it in a different way, that is a stochastic process is also defined as a function of two arguments, and that is x of w comma T, where w is belonging to omega and T is belonging to capital T. That means, the same way I can define the collection of random variable as a collection of w comma T, where w is belonging to omega and T belonging to capital T. This is also going to be formed as a stochastic process.

That means, always the w is belonging to omega. That means, the w is belonging to the possible outcomes and the T is belonging to capital T. This is going to set the given probability. This is going to set the stochastic process. The other names for the stochastic process are going to be chance process. There are some authors use the word chance process and there are some authors, they use the notation that is called a random process.

So, either the stochastic process can be called as a chance process or the random process also. Now, what we are going to see, once you have a collection of random variable, so based on the values of x of T and the values of the different values of T, we are going to define what is a parameter space and what is a state space.

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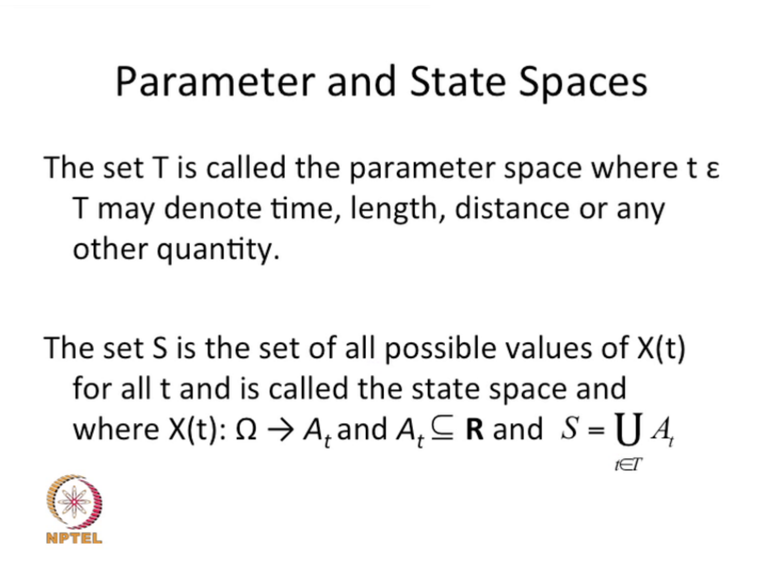

What is the meaning of parameter space? The set, we use the notation capital T, and that is called the parameter space. The set capital T is called the parameter space and it is usually represented as the time, most of the time or it can be represented as the length or it can be represented as a distance and so on.

So, usually we go for T as the time. So, the set T is called the parameter space. Similarly, I can define the state space as the set capital S that is nothing but all possible values of x of T for all t. So, this set is called the state space. x T is a random variable from omega into a suffix T, where a suffix T is a subset of capital R. Then the a t's are going to be the elements of, it is going to be contained in the real line. Then the S is nothing but union of T belonging to capital T. All the a t's that is going to be form a state space.

That means, for a fixed T, you will have a collection of possible values that is going to be the a T and for variable T, you collect all the union and that possible values of x T is going to form a set and that set is called the state space.

Similarly, all possible values of a small t, belonging to capital T and that set is going to be called as a parameter space. So, based on the parameter space and the state space, we can go for classification.

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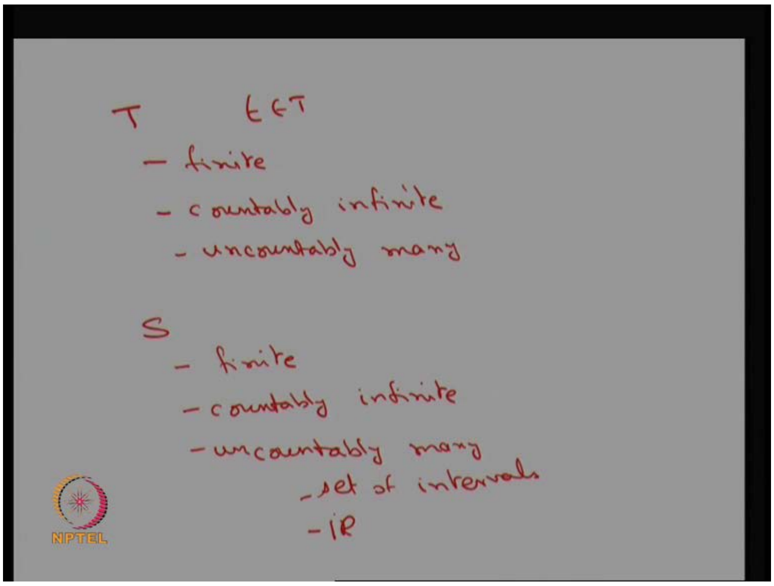

Now, I can explain, what are all the possible values of S can take. So, this T is going to be the collection of capital t. Therefore, this can be a finite. That means, countably finite or it could be countably infinite also or it could be uncountably many elements of a small t. So, that set can be a finite set or it could be countably infinite or it could be uncountably many elements also.

T can also be multi dimensional set. Similarly, the state space capital S, that can be a, same way it could be a finite or it could be a countably infinite or it could be uncountably many elements. So, since the state space are going to be the collection of all possible values of x T and x T is a real valued function and then it is going to be a random variable. Therefore, these elements are going to be always real numbers.

So, either it could be finite elements or it could be countably infinite elements and it is going to be uncountably many elements. That means, it could be a set of intervals on a real line or it could be the whole real line itself.

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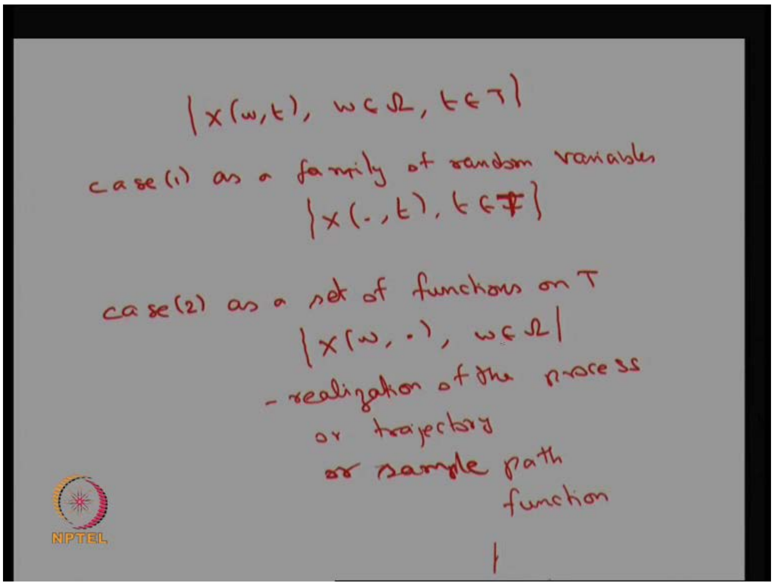

So, based on the values of, the way I have explained the random variable or the stochastic process is going to be x of w comma T, where w is belonging to omega and T is belonging to capital T.

There are two approaches that can define the stochastic process. The first one, that is, we name it as case 1. I can say it as the collection of random variable as a family, family of random variables as x dot T, where T is belonging to capital T. So, this is the way I can create the random variable and this is the easier approach. In the sense, once I know the different T or fixed T, it is going to be one random variable. I have collected family of random variable for different values of small T. Therefore, this is the way we can create the stochastic process and this is the easier approach also.

The next one, that is case 2, that is nothing but as a set of functions on capital T. That is nothing but a collection of x w comma for w is belonging to omega. That means, I have made a function on capital T and once I fix one w, I will have one function and if I fix another w, where w is nothing but possible outcomes. Therefore, if I have different possible outcomes, that is going to create a different stochastic process. Therefore, I can create a stochastic process of x of w comma T either fixing a T or fixing the w.

Accordingly, I can have two different ways of creating a stochastic process. The case 2, the way I have made a collection of random variable by fixing the w, then I made functions on T and therefore, this is going to be the method of realization of the process or it is going to be called as a trajectory or it can be called as a sample path or it can be called as sample function also.

So, these are all the different ways the case 2 can be called. That means, once you know one possible outcome, therefore, you are tracing the stochastic process along the one possible outcome. Therefore, that is going to be called as a realization of the process or the trajectory or the sample path.

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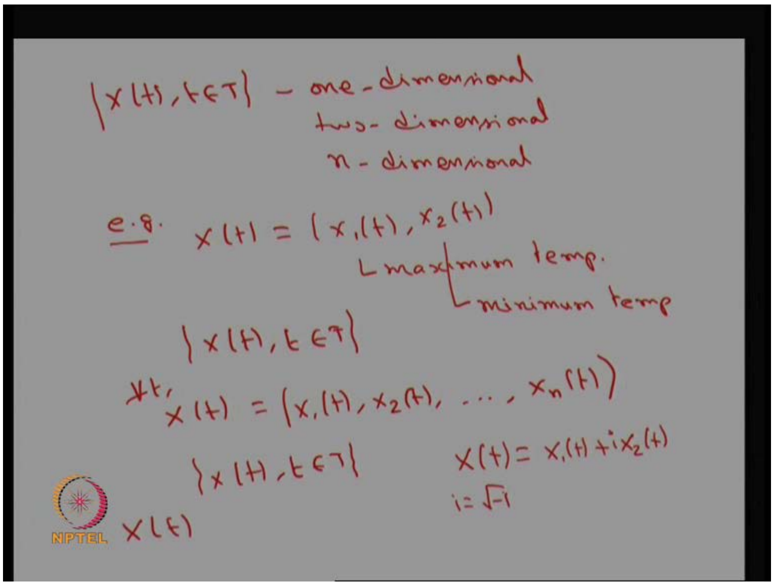

So, the conclusion is, we can always define a stochastic process as a collection of random variable for different values of T or you can go for a collection of functions on T for different values of the possible values of, possible outcomes that is w belonging to omega. So, these are all the two approaches that can create the stochastic process. Not only we can go for making a one dimensional random variable or one dimensional stochastic process, so we can create a stochastic process, it could be one dimensional or it could be two dimensional or it could be n dimensional also. So, first we have discussed what is stochastic process and how to create the stochastic process, whether it exists and so on and then we have given the parameter space and the state space.

Then, we have given what are all the ways we can create the two different approaches you can create the stochastic process. Now, we are discussing what is the dimension of the stochastic process. So, either the default could be one dimensional or it could be two dimensional or it could be a n dimensional. Let me give one simple example, in which it is going to be two dimensional. That means, I have a random variable x of T and that is going to be x 1 of T comma x 2 of T, in which x 1 of T is nothing but the maximum temperature and x 2 of T could be minimum temperature.

The maximum and minimum temperature possible of a place at any time T and this set together is going to be one random variable. That means, this is a random vector which consist of two random variables, x 1 of T and x 2 of t. That means, for x, for fixed T, you have one random vector x of T. Therefore, you have a random vector for over the T and this random vector will form a stochastic process. Therefore, this is going to be 2 dimensional stochastic process.

Therefore, in general, you can define n dimensional stochastic process for fixed, and for every T, you have a random vector x of T and that is going to be x 1 of T comma x 2 of T and so on. It is going to be the n th element is x n of T and that is going to be n triple, in which each one is going to be a random variable for fixed T. This is going to be a random vector for fixed T and this is going to be n dimensional stochastic process, in which each one is going to be a one dimensional random variable for fixed T.

That means, you can go for making a one dimensional random variable. Then you have a collection of random variable from a one dimensional stochastic process or you can have a two dimensional. Like that, you can have n dimensional stochastic process. In the course, what we are going to discuss always is a one dimensional stochastic process. We can always create a complex valued stochastic process also in the form of x of T. Let me define it here. The x of T is going to be x 1 of T plus i times x 2 of T, where i is nothing but the complex quantity square root of minus 1. That means, the x 1 of T is a real valued random variable for fixed T and x 2 of T is also a real valued random variable for fixed T.

The way I have made it, the x of T, this is going to be a complex valued random variable for fixed T. Therefore, the x of T over the T, and that is going to form a complex valued stochastic process. Because for fixed T, x of T is going to be a complex valued random variable. The corresponding stochastic process is called complex valued stochastic process in the one dimensional form.

Like that, you can go for the multi dimensional complex valued stochastic process also. But in this course, what we are discussing is only the real valued one dimensional random variable most of the times. Sometimes, we are discussing real valued two dimensional or n dimensional stochastic process, and that too with the real valued random variable and not the complex valued.

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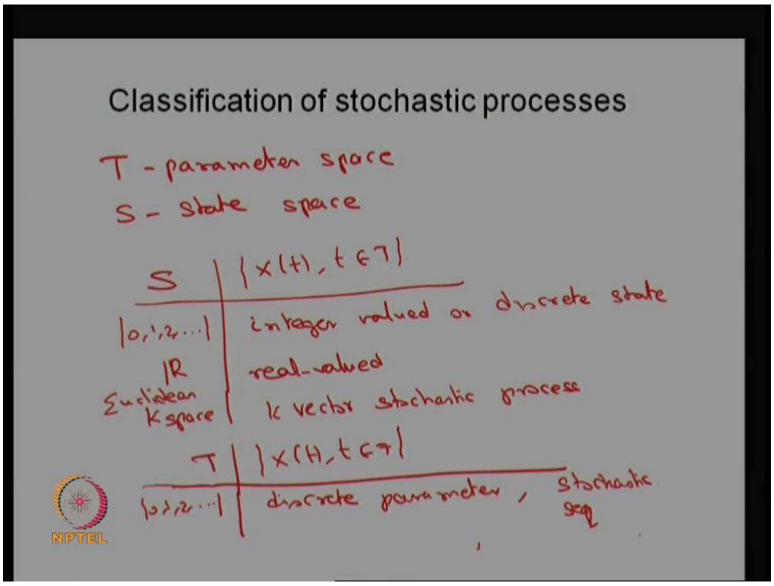

So, now we are going for classification of a stochastic process. The way I have explained the parameter space capital T, the capital T is a parameter space and capital S is going to be the state space. That is nothing but the collection of possible values of x of T and the possible values of small T belonging to capital T and that forms a parameter space. Some books, they use the notation parameter set also and capital S is going to be the state space. Now, based on this, we are going to classify the stochastic process. Suppose, let us start with the capital S.

Suppose, the possible values of S and what is the name of the stochastic process, if S is going to take only countably infinite or countably finite values. Then it is going to be called as, the corresponding stochastic process is going to be called as an integer valued stochastic process or we can call it as a discrete state stochastic process. So, whenever the possible values of S is going to be countably finite or countably infinite, then we say it is an integer valued stochastic process or a discrete state stochastic process.

Suppose, the possible values of S is going to be the real values, then we call it as a real valued stochastic process. Suppose, if we take a Euclidean space with the k dimensional Euclidean, Euclidean k dimensional space, then we call it as a k vector space, k vector stochastic process.

That means, each random variable is going to have a one dimensional random variable and like that, you have k random variables for fixed T. Therefore, you have a k vector stochastic process. Therefore, it is going to be called as a k vector stochastic process, in which, each element is going to be a one dimensional random variable for fixed T. So, the collection the k triple values stochastic process is going to be called as a k vector stochastic process. Similarly, you can go for based on the capital T, what is the name of the stochastic process for different values of T.

That means, if it is going to take the value countably finite or countably infinite or it is going to take only the integer values, then we say it is a discrete parameter stochastic process or there is another name. It is called the stochastic sequence also.

Whenever the possible values of capital T is going to be a countably finite or countably infinite, then we call the corresponding stochastic process as the stochastic sequence or it is a discrete parameter stochastic process. Otherwise, if it takes an uncountably many values in the capital T, then it is going to be called as a continuous parameter and it is or it is going to be called as a stochastic process itself. Therefore, based on the discreticity, uses the word sequence or if it is going to be uncountably many values of capital T, then it is going to be called as a stochastic process.

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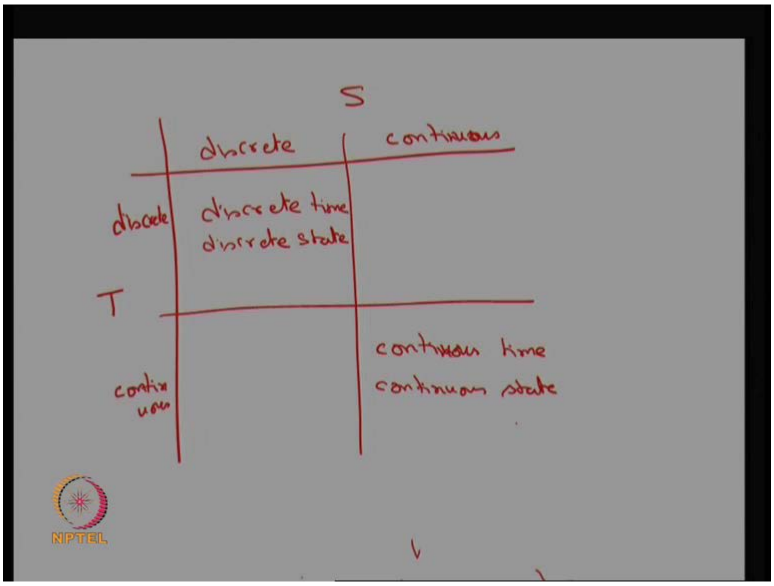

So, based on the classification, I can go for making a one table, in which the possible values of S will take a column and the possible values of capital T will take a row. So, either it could be a countably finite or countably infinite that I uses the word discrete.

If the possible values of T is going to be uncountably many, either it is set of all intervals or it will be a whole real line itself or it is going to be a union of many intervals, in that case, it is going to be called as a continuous parameter. Similarly, if the possible values of S is going to be a countably finite or countably infinite, then the state space is going to be called as a discrete.

Similarly, if it is going to be uncountably many values, then it is going to be called as a continuous. So, accordingly you can classify the stochastic process into four types, in which, if the T is going to be a discrete and as well as S is going to be a discrete, then it is going to be a discrete time or discrete parameter. Both are one and the same. So, discrete time, and discrete state stochastic process. Similarly, if the T is discrete and the state space is continuous, then we can call it as a discrete time continuous state stochastic process. Similarly, this is going to be a continuous time discrete state stochastic process and this is going to be a continuous time continuous state stochastic process. That means, based on the possible values of capital T and possible values of capital S, any stochastic process can be classified into four types, in which, it is going to be a discrete discrete or continuous continuous or discrete continuous, or continuous continuous based on the time and the state space.

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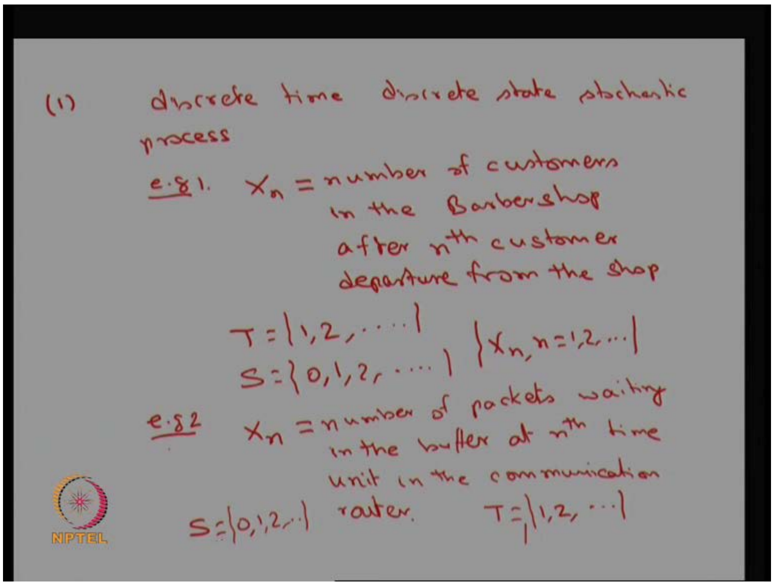

So, let us see some simple examples based on the possible values of capital T and capital s. So, the first one is going to be a discrete time or you can use a discrete parameter also. Discrete time discrete state stochastic process; that means, the possible values of capital S as well as the possible values of capital T has to be either, it has to be of countably finite or countably infinite elements in it.

Let us see one simple example. Let us create a random variable x suffix n as nothing but that is nothing but the number of customers in the barber shop after nth customer departure from the shop. So, here suffix n that will form a parameter space. Therefore, the T can be possible values of n. That means, whenever one customer leaves the system, how many are in the system after he leaves. So, the possible values of T will be the first customer, when he leaves, how many are there he want to find out and so on. Therefore, the possible values of capital T are going to be 1, 2 or 3. Therefore, this is the number of making the number of customers in the system.

Whereas, the possible values of x n for possible values of n, and that is going to be, there is a possibility no customers in the system when someone leaves. So, there is a possibility 0. When someone leaves, only one customer in the system, then it is going to be 1 or 2 and so on. Therefore, there is a possibility it could be finite also. So, the capital

S can be countably finite or in this case, I have made the assumption that it is countably infinite. Therefore, the capital T as well as T is going to be a form of the discrete. Therefore, the corresponding stochastic process x n for possible values of n is going to be 1, 2 and so on and this is going to be a discrete time discrete state stochastic process.

You please note that here, the parameter space, the capital T is not the time. The parameter space, forming 1, 2 and 3, these are all customers, the nth customers. Therefore, n can be 1, 2 and so on. Therefore, usually the capital T is a time, whereas, sometimes it could be a distance or length or the number or whatever the other quantity. So, here is the typical situation in which the parameter space is not considering the time.

Therefore, this is going to be random variable because you never know how many customers are going to be in the system after the nth customer leaves. Therefore, this is going to be a random variable. Obviously, it is a real valued function satisfying all the property of the definition and you can see the probability space for this. From the probability space you have created a random variable and therefore, this random variable is going to be the, this collection of random variable over the n that is going to be the discrete time and discrete state. Therefore, this random variable here, you can create with the help of case 1, by making, for fixed n, what is the random variable and then you make a collection of random variable. So, we can create this stochastic process by using the case 1 or the approach 1, which is the easier one.

I can go for creating one more stochastic process for this discrete time and the discrete state stochastic process that comes under telecommunication problems. x n is going to be number of packets waiting in the buffer at nth time unit in the communication router. That means, there is a communication router in which the packets are coming for transmission. So, after the transmission is over in the buffer, the packets leave the router. So, at any time, you do not know how many packets are waiting in the buffer for the transmission. So, there is a possibility no packets may be there at some time point and there is a possibility there are many more packets may be waiting for the transmission in the buffer.

So, the possible values of capital T, the possible values of capital S that is going to be, there is a possibility no packets in the buffer or 1 and so on and similarly, the possible values of capital T, that is also we are making the nth time unit. Therefore, the time unit could be first time unit or second time unit and so on. Therefore, here the S is going to be the discrete as well as the T is going to be a discrete. Therefore, this collection of random variable x suffix n for possible values of n, that is also going to form a discrete time discrete state stochastic process because the possible, both the values are going to be of a discrete type.

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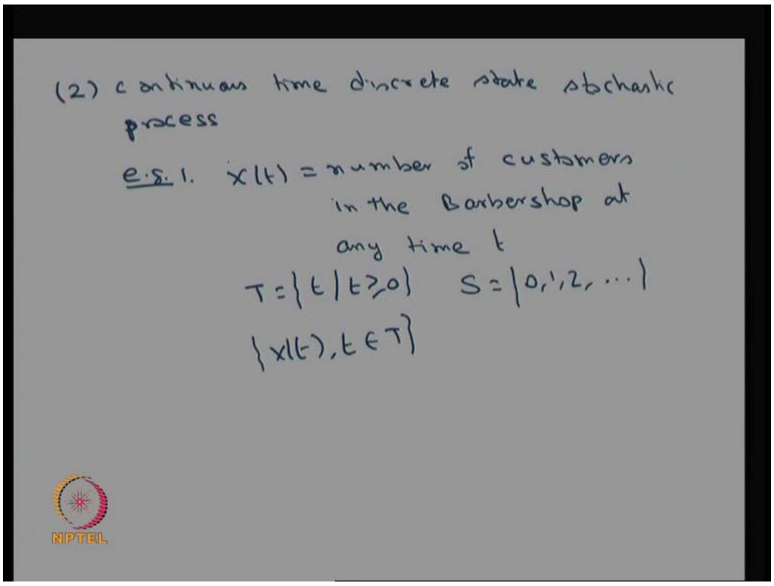

Discussing the simple stochastic process based on the parameter space and state space and we have seen the discrete time discrete state stochastic process, the first one. Now, we are seeing the second one, and that is continuous time discrete state stochastic process. That means, the possible values of parameter space is going to be uncountably many values. Therefore, we get the continuous time and the possible values of the states space, that is going to be countably finite or countably infinite. Therefore, you get the discrete state. So, you will see few simple example of this type. The first example, that is x of T, that is going to be the number of customers in the barber shop at any time T. That is the difference.

In the earlier example, we have seen the number of customers in the barber shop for the nth customer’s departure. Now, we are seeing the number of customers in the barber shop at anytime T. Therefore, we are looking at how many customers at anytime T in the barber shop. Therefore, the possible values of capital T, that is going to be a collection of T, such that the T is greater than or equal to 0. The possible values of S, and that is going to be, still it is the number of customers, therefore, the possible values are 0 or 1 or 2 and can be and there is a possibility it can be countably finite also. So, whether the state space is going to be a countably finite or countably infinite, we classify as a discrete state

Therefore, this is a typical example of continuous time discrete state stochastic process. The collection of random variable is going to be x of T for all possible values of capital T. So, this is going to form a real valued stochastic process, in which for each T, it is going to be a random variable. So, this is going to be a real valued stochastic process of one dimensional type. The T is belonging to the small t, and that is going to be the time. That is a default one and it is going to be uncounatably many. Therefore, it is going to be a continuous parameter. So, it is going to be called as a continuous parameter discrete state stochastic process also.

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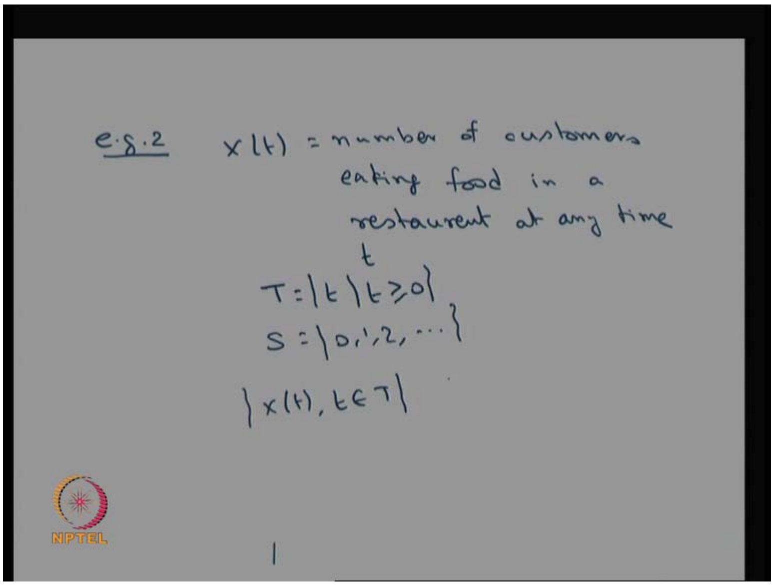

The next example, example 2; in the example 2, let we make a x of T that is going to be number of customers eating food in a restaurant at anytime T. Therefore, you are observing the system. You are observing the restaurant how many customers are taking food. Therefore, the possible values of the parameter space T is going to be T greater than or equal to 0 and the possible values of S, still it is account, therefore, the possible values are going to be countably finite or countably infinite. Therefore, this collection of stochastic, this collection of a random variable over the T, that is going to be a continuous time or continuous parameter discrete state stochastic process. This is a very typical example. So, it could be countably, this S could countably finite or countably infinite also.

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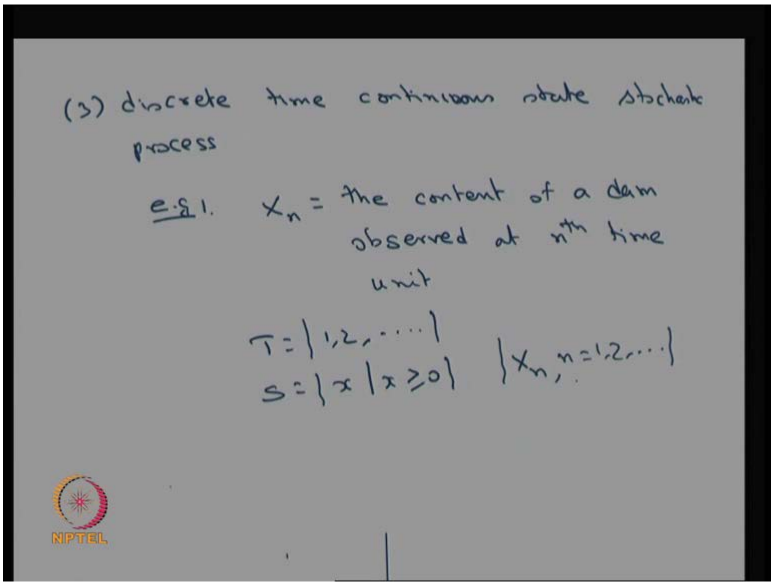

Now, let us see the third type, and that is discrete time continuous state stochastic process. That means, we need the possible values of capital T has to be a countably finite or countably infinite, whereas, the possible values of the state space has to be uncountably many of that type. So, let us create an example for that x suffix n, that is nothing but it is a random variable that denotes the content of a dam or water reservoir observed at nth time unit. So here, the time unit could be every one hour or that could be, because you are seeing what is the content of the dam or water reservoir, it could be every day at fixed time of everyday or it could be fixed time of weekly once. So, that is going to be the time unit. So, at the end of each nth time unit, you are observing what is the content of the dam. So, that is nothing but it is a real quantity. Therefore, the capital T is going to be your observing at only at the time unit. So, either it could be one hour or daily once or weekly once and so on.

Therefore, I can make a one to one correspondence with the countably finite or countably infinite numbers. So, that will form a parameter space and the capital S, this is going to be the possible values of x n for all possible values of n. Therefore, this is the water content of dam that is going to be the real quantity. Therefore, that is going to be some x, where x is always greater than or equal to 0. So that means, you have the parameter space is going to be a discrete, whereas, the state space is going to be a continuous. Therefore, this stochastic process x suffix n for possible values of n is going to be 1, 2, 3 and so on and this is going to form a discrete time continuous state stochastic process.

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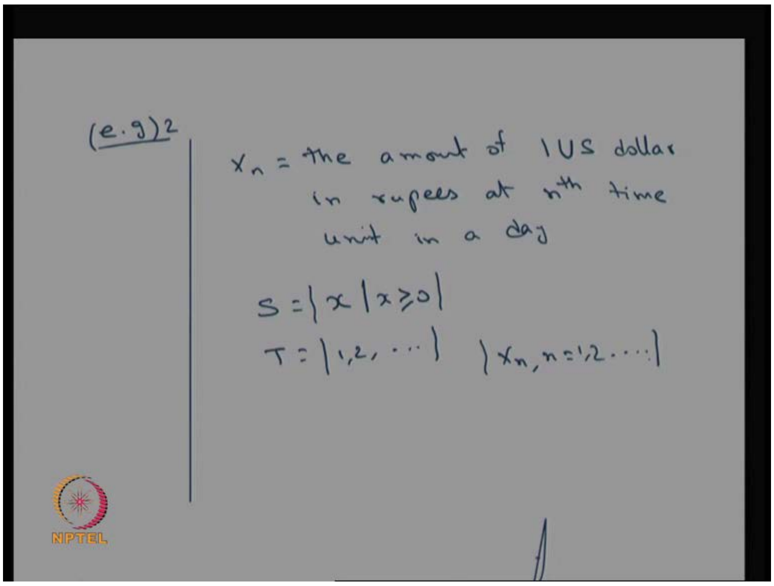

Let me give one more example for the same type, that is example 2, and that is nothing but example 2. x n is nothing but the amount of 1 US dollar in rupees at nth time unit in a day. That means, I am just observing what is the value of 1 US dollar in rupees in a day for the nth time unit. I could be every 5 minutes or it could be a every minute or it could be every hour of any particular day and that is going to form a random variable and that collection is going to form a stochastic process.

In this, the possible values of x is going to be, since it is the amount of 1 US dollar in rupees, it could be some fraction also. Therefore, we do not want take it as the integer number. It can be real numbers. Therefore, it is going to be possible values of x greater than or equal to 0 and the capital T, that is going to be the time unit, either it is every minute or every once in 5 minutes or once in 10 minutes or everyone hour and so on. Therefore, this is going to form a countably finite or countably infinite one and this stochastic process will form a discrete time and continuous state stochastic process.

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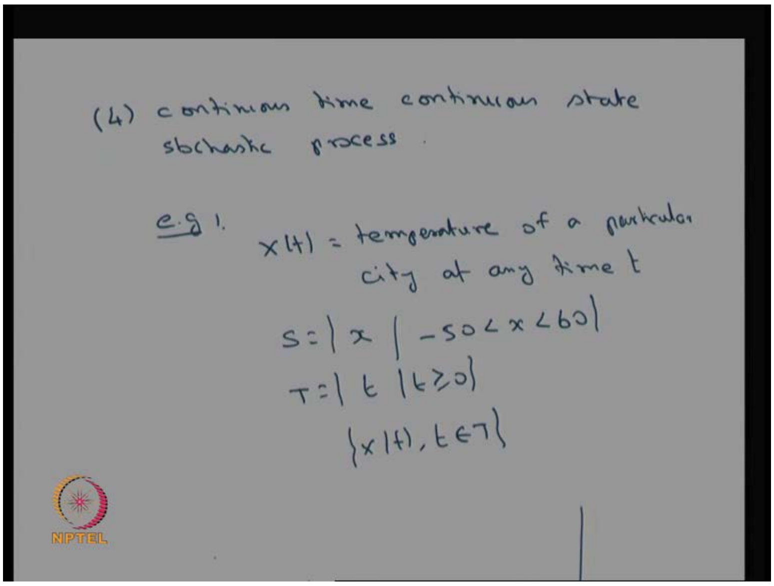

Let me go for the fourth type, that is the fourth classification of a stochastic process, and that is continuous time continuous state stochastic process. That means, the possible values of parameter is going to be uncountably many. Therefore, you get the continuous time or continuous parameter and the possible values of state space, that is going to be uncountably many. Therefore, you get the continuous state stochastic process.

The examples are, the first one, x T is going to be temperature of a particular city at any time t. So, whenever I use any time T, it can take any value. Therefore, the possible values of S is going to be the temperature. So, you can think of the temperature. Suppose some particular city's lies between minus 50 to 60 degree Celsius. So, this quantity, S is going to be the Celsius of minus 50 to positive 60 and the parameter space T is going to be your observing over the time. Therefore, this time is going to be greater than or equal to 0. Therefore, the parameter space is a continuous one and the state space is the continuous one. Therefore, this collection of random variable will form continuous time continuous state stochastic process.

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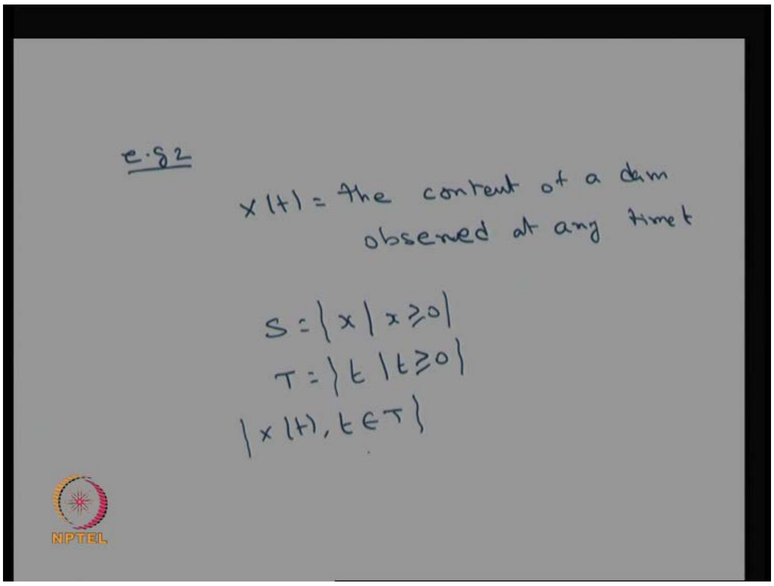

Let me give one more example of the same type, the fourth type, and that is example 2. x of T is going to be the content of a dam observed at any time T. So, the content of a dam or a reservoir, that is going to be the real quantity. Therefore, S is going to be a collection of x, such that x is going to be greater than or equal to 0.

You are observing over the time, therefore, that is also collection of T, such that T is going to be greater than or equal to 0. Therefore, this will form a stochastic process, in which it is going to be the classification, it will be under the classification of a continuous time continuous state stochastic process and this can be created with the help of the first approach. That means, for fixed T, you find out what is the random variable and you collect the random variable over the all possible values of T. Therefore, this is going to be of the continuous time and continuous state stochastic process. So, in this lecture, what we have seen? What is the meaning of a stochastic process or how to create the stochastic process. So, that is nothing but it is going to be a collection of random variable. So, we have defined the stochastic process as well as how to create.

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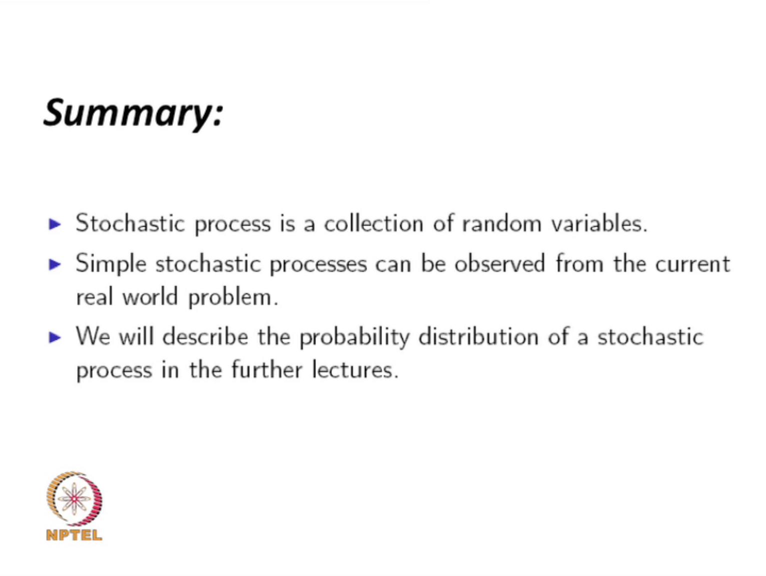

Then later, we have given what is a parameter space and what is a state space and we have given the classification of a stochastic process based on the parameter space and the state space. Also, some of the real world problems from that we can create a stochastic process and that stochastic process are the simple stochastic process and there are many more stochastic process can be created with the help of the definition and so on. So, that will be discussed in the lecture 2.

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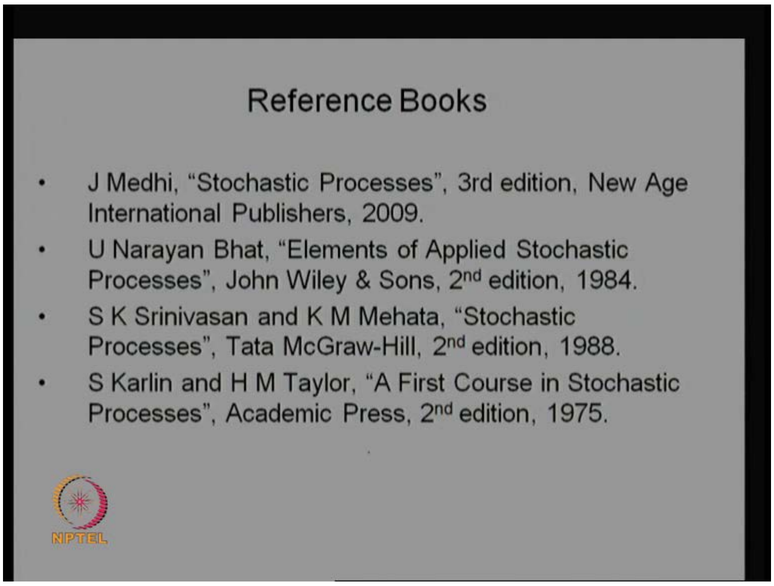

These are all the reference books we have used it for preparing this lecture. One material and I will be continuing the lecture 2 with some more stochastic process, which is of very useful in the later stages.

Thank you.