Stochastic Processes
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Module - 1
Probability Theory Refresher
Lecture - 4
Problems in Sequences of Random Variables

So, this is stochastic processes module one probability theory refresher lecture four problems in sequence of random variables. As a illustrative examples we are going to discussed four problems in this lecture.

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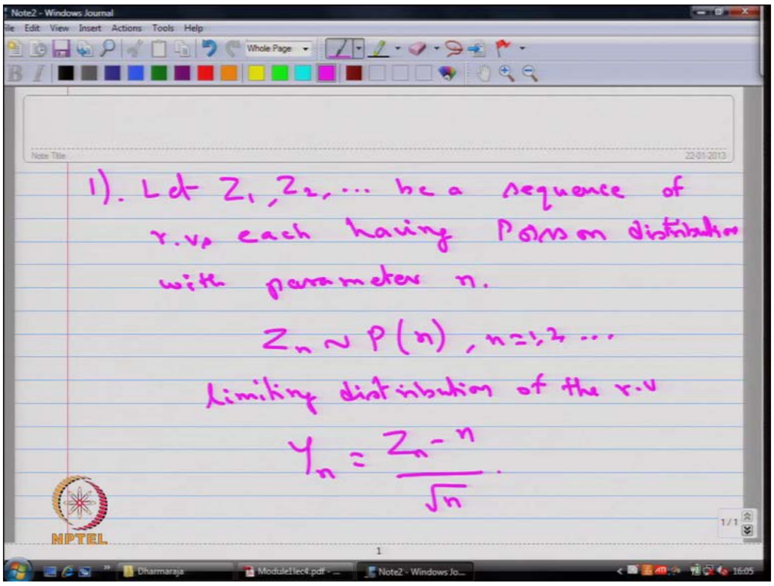

The first problem, let z 1, z 2, and so on be a sequence of random variables each having Poisson distribution with parameter n, that is z n is Poisson distribution with the parameter n, for n is equal to 1, 2, 3 and so on. Our interest is to find the limiting distribution of the random variable that is defined as y suffix n that is z n minus n divided by square root of n.

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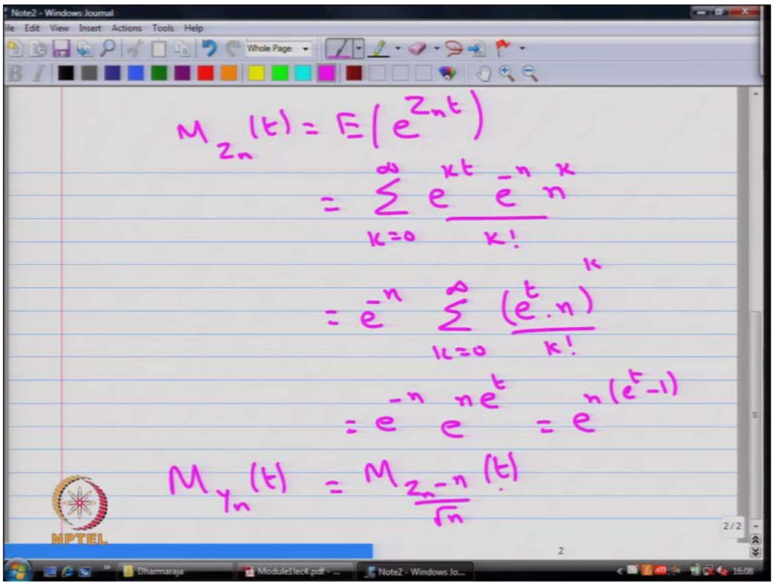

So given z n is Poisson distribution with the parameter n. We can find out the M z f of z n M z f of z n is nothing but expectation of e power z n of t. That is same as summation k is equal to 0 to infinite e power k times t, e power minus n n power k by k factorial, because it is the expectation of e power z n of t, where z n is Poisson distribution with the parameter lambda. Therefore, this is going to be m is k is equal to 0 to infinity this one.

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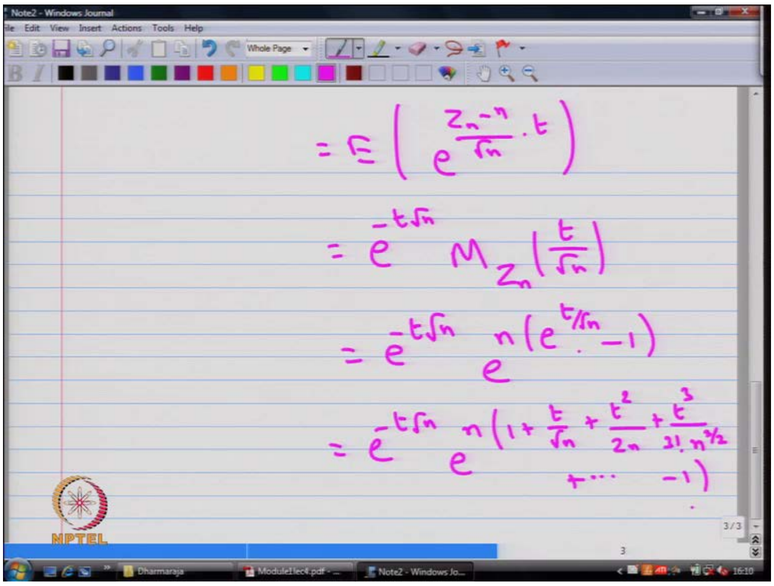

So, you can take e power minus n outside. So, the remaining term becomes, k is equal to zero to infinity e power t multiplied by n to whole thing power k by k factorial. That is same as e power minus n, e power n times, e power t that can be rewritten as e power n times e power t minus 1. Now, we will find out the M z f of the random variable y n were y n is the z n minus n divided by square of n.

Therefore, the M z f of the random variable y n as a function of t that becomes M z f of z n minus n divided by square root of n function of t, that is same as expectation of e power z n minus n divided by root n multiplied by t, you know the rules of a moment generating function the constant is out, so you can use that logic. So, it become e power minus t times root n because n t by root n, therefore it becomes a t times root n, then M z f of the random variable z n use the another will of a moment generating function instead of t if the comes it t divided by square root of n. So, that is same as e power minus t times root n just now we found what is the moment generating function of a z n so use the same thing, but replace t by t divided by square root of n therefore, this becomes e power n times wherever the t you replace by t by square root of n so t by square of n minus 1.

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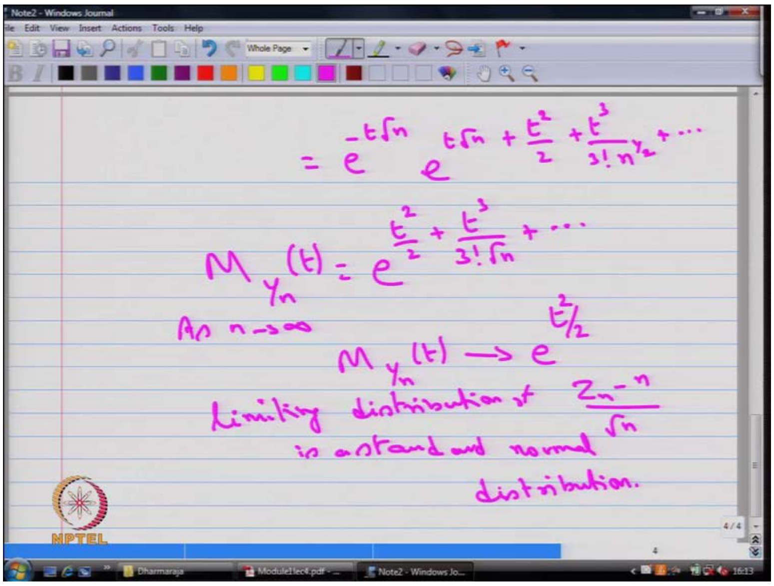

Therefore, we can further simplify by expanding e power t by n; that means, you keep this e power n you expand only e power t by square root of n; that is 1 plus t divided by square root of n then the next term will be t square by 2 times n, and the next term will be t cube divided by 3 factorial n power 3 by 2 and so on. And the last term is. So, this is a expansion of e power t by square root of n minus 1.

So, close the bracket that is same as e power t times square root of n multiplied by. So, this 1 and plus 1 and minus 1 will be cancelled so you will get e power n times t by square root of n that becomes t of square root of n, and the next term becomes t square by 2 then it becomes t cube by 3 factorial n power 1 by 2 and so on. Therefore, this becomes e power t square by 2 plus t cube by 3 factorial square root of n and so on.

Our interest is to find out the limiting distribution of y n so this is the moment generating function of a y n for n. So, as n tends to infinity, because our interest is to find out the limiting distribution as n tends to infinity the moment generating function of y n becomes e power t square by 2. If you recall the moment generating function for standard distributions one can conclude this is the M z f of a standard normal distribution. Therefore we conclude the limiting distribution of y n is standard normal distribution; that is a the limiting distribution z n minus n divided by square root of n is a standard normal distribution. So, this problem is very important in the renewal process therefore we discuss this example has the how to find the limiting distribution of some standard random variables.

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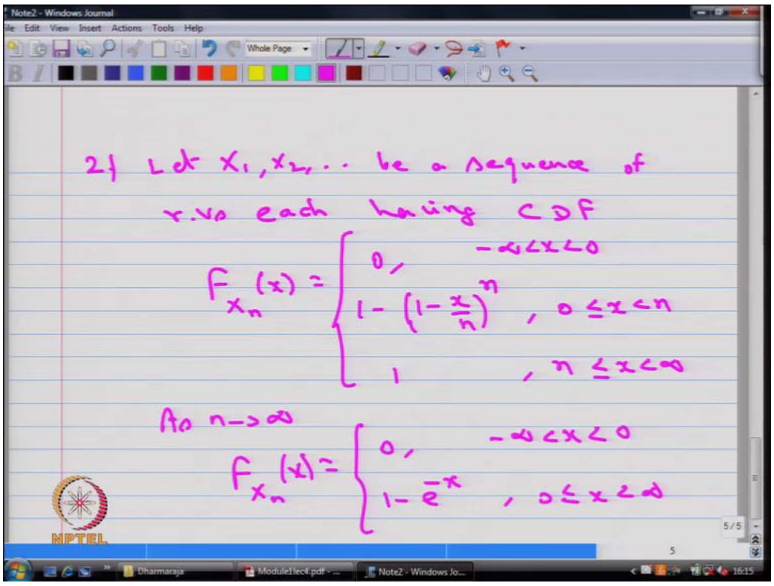

Next example; let x1, x 2 and So, on be a sequence of random variables, each having c d F cumulative distribution function F suffix x n of x 0 from minus infinity to 0 detects the value 1 minus 1 minus x by n power n. For x is lies between 0 to n from n onwards till infinity the value is 1 So, this is the cumulative distribution function for the random variables exercise. It is the function of n therefore I have made it F suffix x suffix n; that means, this is the c d F for the random variable n for every n you have this, as n tends to infinity we get F suffix x n of x that becomes 0 from minus infinity to 0, and it takes the value 1 minus e power minus x from 0 to infinity, as n tends to infinity this c d F of the random variables x n becomes 0 between the intervals minus infinity to 0, the value becomes 1 minus e power minus lambda x, where x is lies between 0 to infinity.

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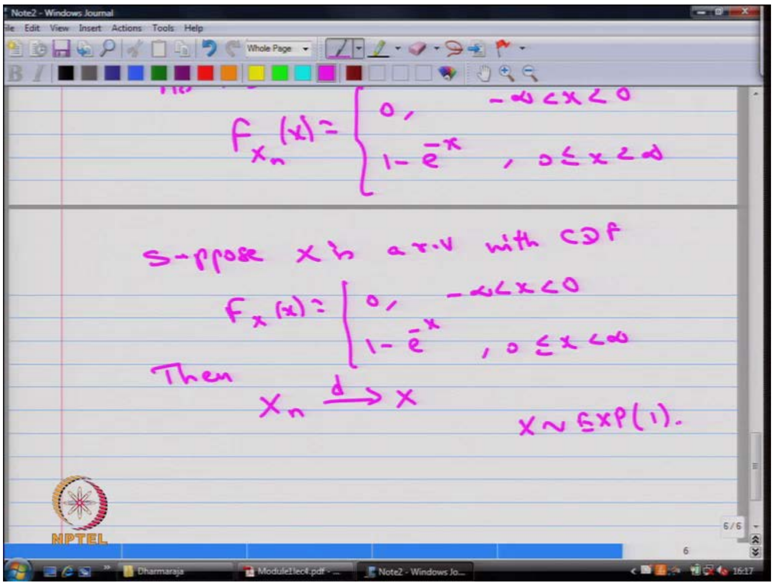

Suppose, x is a random variable with the c d F that is F x of x that is 0 between the interval minus infinity to 0 and 1 minus e power minus lambda x, where x is lies between zero to infinity. Then one can conclude x n converges to x in distribution, since the sequence of F x suffix of n of x tends to F of x, for x is greater or equal to 0 and the value is 1 minus e power minus x. Hence one can conclude the sequence of random variable of x n converges to random variable in distribution.

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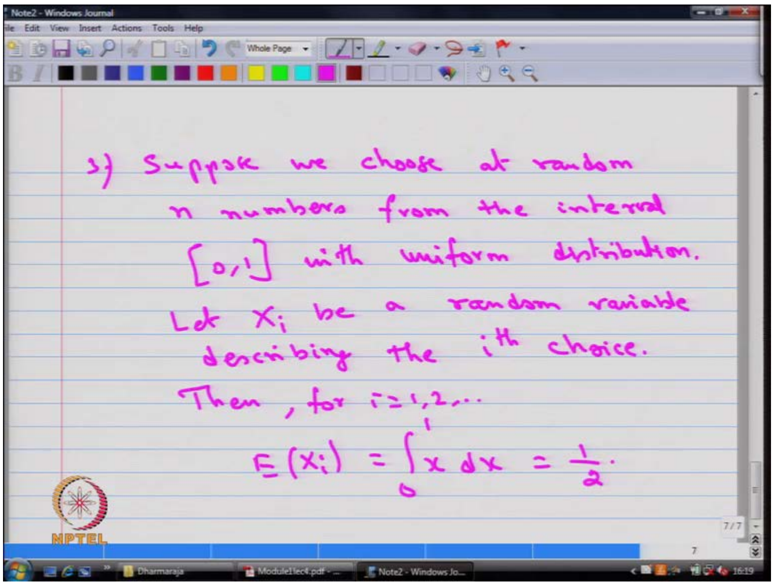

Here, the x is a exponential distribution with the parameter 1. So, this is the one example of a all the sequence of random variable converges to a random variable in distribution. Next I will move in to the third example; suppose, we choose at random n numbers from the interval 0 to 1 with uniform distribution. Let capital x i be a random variable describing the i th choice. Then for i is equal 1,2 and so on you can find out what is the expectation of x is?

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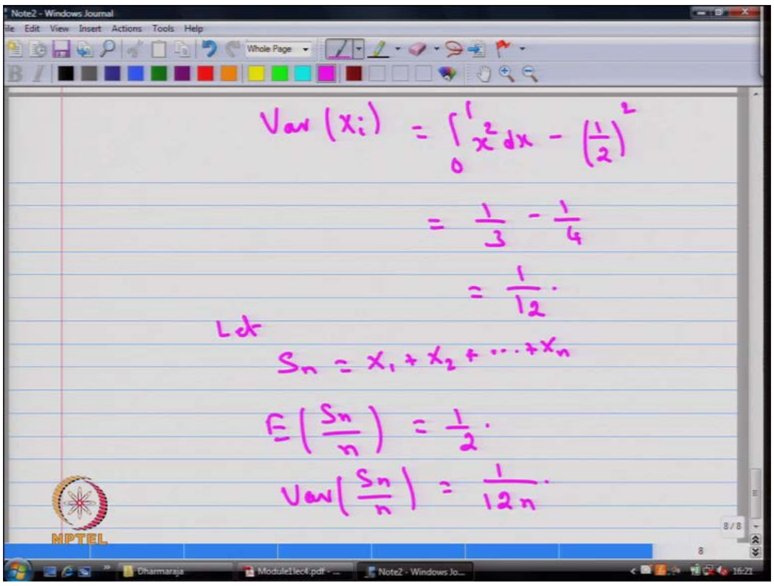

That is nothing but the integration from 0 to 1 x times the probability density function, the probability density function for uniform distribution with the interval 0 to 1 that is 1. Therefore, x into d x if we compute the expectation x i is going to be 1 by 2. Similarly, one can evaluate the variance of x i is that is nothing but 0 to 1 x square d x minus the mean square, expectation of x square minus expectation of x the whole square. So, the expectation of x square is 0 to 1 x square d x. So, if you evaluate this quantity that is 1 by 3 minus 1 by 4 so if you simplify you will get 1 by 12.

If you remember the formulae of variance of uniformly distributed random variable between the interval a to b then the variance of x i x is nothing but you can get it that by substituting the value of a is equal to 0, and b is equal to 1 you will get 1 by 12. Let S suffix n b x 1 plus x 2 and so on till x, one can find lenient variance of x, because you know the lenient variance of x is using that you can find out what is the mean of S n? But our interested is not the finding the mean of S n oriented is to find out the mean of S n by n, that is basically suppose x i is the samples then S n divided by n is nothing, but the sample mean.

So, expectation of x n divided by n that becomes the 1 by 2. Similarly, if you calculate variance of S n by n that becomes 1 divided by 12 times n, because the variance of x is 1 by 12 so the variance of S n is the summation of x i is from 1 to n therefore variance of S n by n becomes 1 divided by 12 time here.

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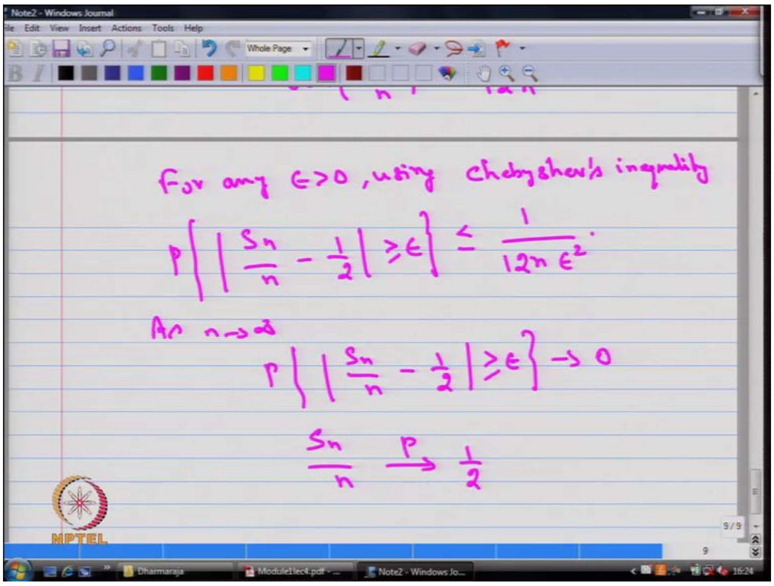

For any epsilon greater than 0 using chebyshev's inequality one can conclude the probability of absolute of S n by n minus 1 by 2 greater than or equal to epsilon that is less than or equal to 1 divided by 12 times n epsilon square. I am using the chebyshev's inequality by knowing mean of S n by n is 1 by 2 and variance of S n by n is 1 divided by 12 n, I get this inequality. Now, as n tends to infinity the probability of absolute of S n by n minus 1 by 2 which is greater than or equal to epsilon will tends to 0, because epsilon is in the n is the denominator, because n is in the denominator as n tends to infinity this probability tends to 0, that is nothing but S n by n tends to the value 1 by 2, and this convergence takes place in probability the sequence of random variables S n by n converges to 1 by 2 in probability.

Therefore, we say the sequence of random variable x n for n is equal to 1, 2 and so on, obeys the weak law of large numbers, because the S n by n converges to 1 by 2 in probability therefore, we say the sequence of random variables x n is obeys the weak law of large numbers. So, that is the intention of giving this example.

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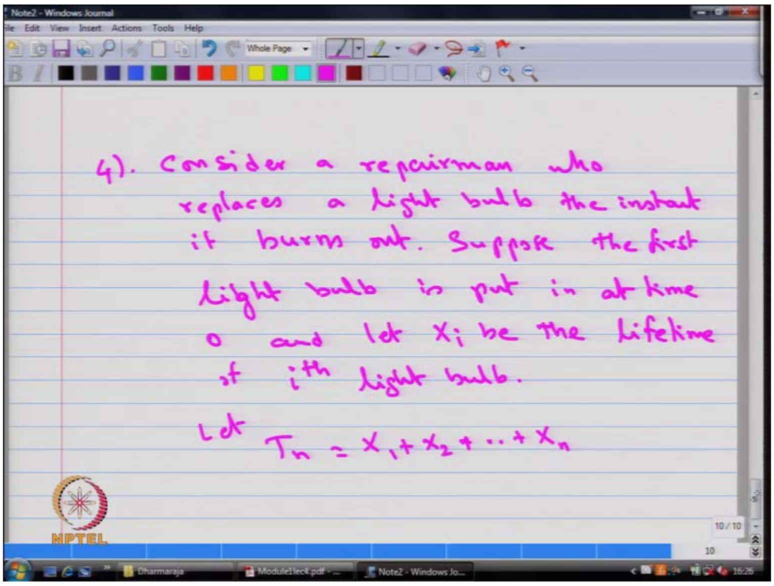

Now, I move in to the fourth example, consider a repair man who replaces a light bulb the instant it burns out. Suppose, the first light bulb is put in at time 0 and let x suffix i be the life time of i th light bulb. You defined the random variables T n is the sum of n x i's.

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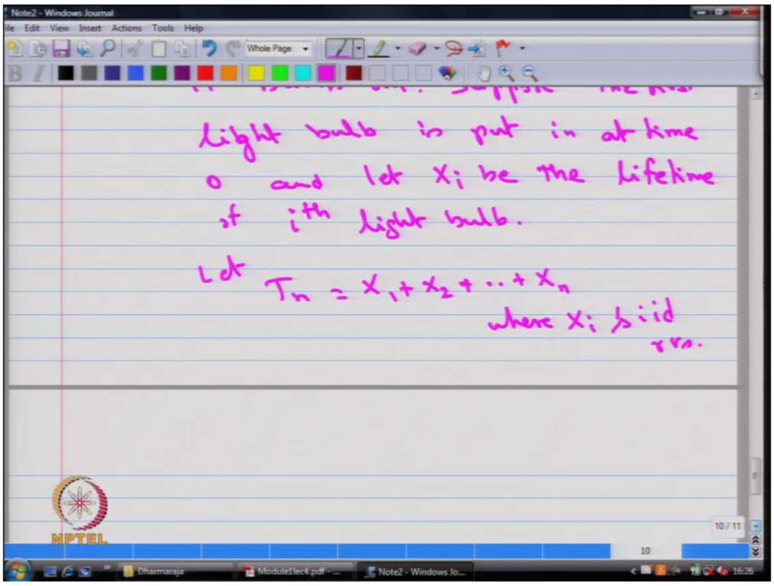

where x i's is are i i d random variables, x i be the life time of i th light bulb, and when x i are i i d random variables you are defining T n is the x 1, x 2 and x n and so on.

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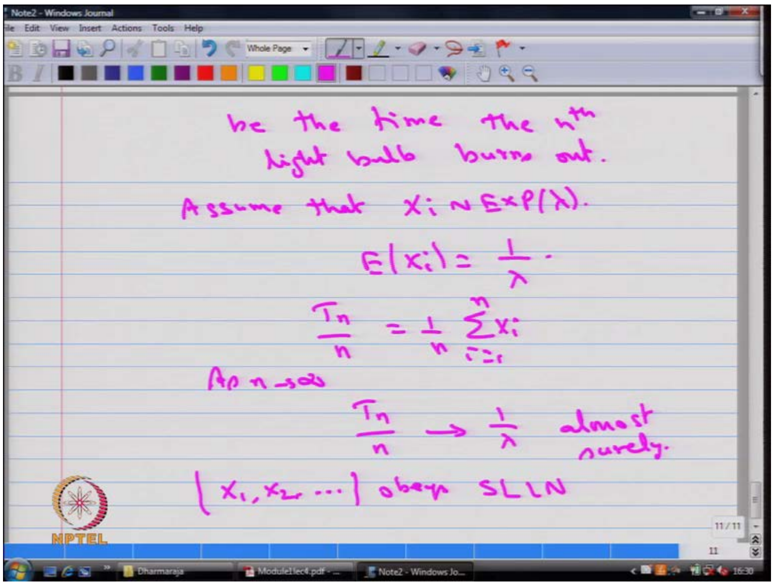

So, the T n be the time of time the n th light bulb burns out, because the T n is the x 1 plus x 2 and so on, till x n therefore, T n be the time the n th light bulb burns out. Assume that x i is exponential distribution with the parameter lambda, you know that already x i are i i d lambda from variable now, I am making the further assumption x i is follows exponential distribution with the parameter lambda; that means, you know what is the mean of this random variable. Since it is exponential distribution with the parameter lambda this becomes 1 divided by lambda, also one can use the result T n by n that is nothing but 1 divided by n summation of x i is, where i is running from 1 to n.

As n tends to infinity as n tends to infinity one can prove T n by n tends to 1 divided by lambda; that is the mean of random variable x i almost surely, I am not proving here the way do the sequence of random variable converges to another random variable converges takes place in probability or in distribution or in (( )) mean or almost surely one can prove this the T n by n converges to 1 by lambda almost surely. That means, you can conclude the random variable x 1, x 2 and so on, obeys strong law of large numbers, because that T n by n is nothing but the 1 by n summation of x i's that converges to the value 1 by lambda almost surely we can conclude the sequence of random variable x i is obeys the strong law of large numbers. Even though, in this problem I made the assumption x i is follows the exponential distribution with the parameter lambda in general the lifetime can be any distribution. So, this problem will be discussed in detail in renewal process.

So, as such here, we are making the assumption of distribution of x i is exponential distribution, therefore I made it converges takes place almost surely to the value 1 by lambda this can be generalized. There are many more problems of the similar kind, but we are discussing only few problems therefore we can use the similar logic of a finding the moment generating function then concluding the distribution and finding the limiting distribution or you verify whether the sequence of random variable converges takes place in mean converges takes place in probability or converges takes place in distribution or converges in the orth mean or converges almost surely this can be used in any problem of the same way what I have done it here.

And I have not discussed any problem in the central limit theorem but that will be used many times, therefore I have not given any problems for the center limit theorem.

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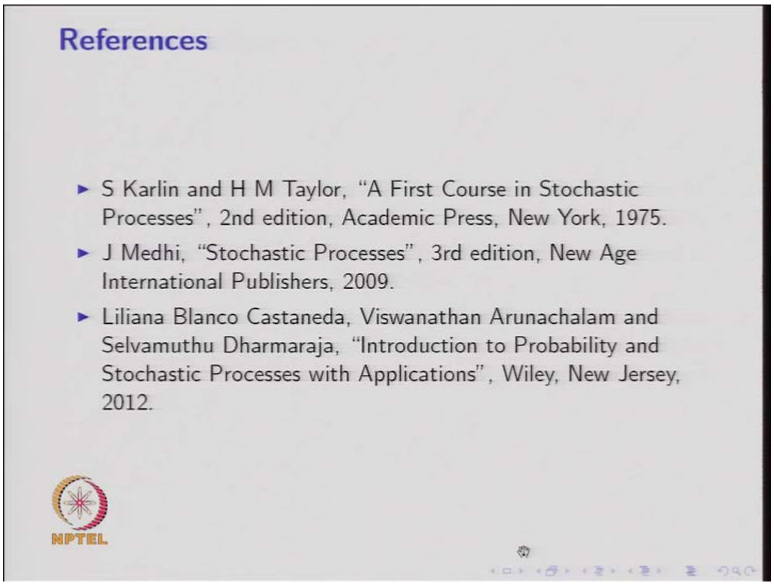

So, if these are all the references reference books we have used for the lecture 3 as well as the lecture 4. It is not end.

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So, what to do then we may go to some more problems than…

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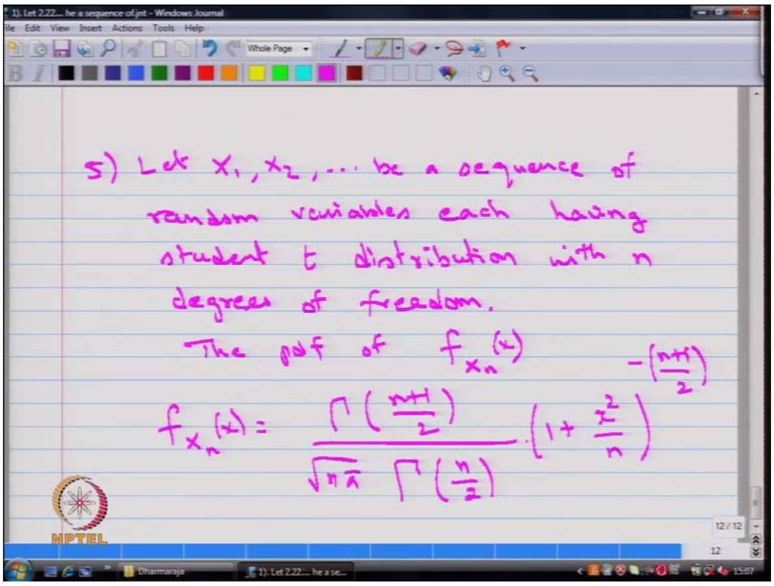

I did not realize that I may lambda for. Let x 1, x 2 so on, be a sequence of random variables each having student t distribution with n degrees of freedom. Our interest is to find out the limiting distribution of the student t distribution. We know that the probability density function of f of x for the random variable x n is given by gamma of n plus 1 by 2 divided by square root of n times phi multiplied by gamma of n by 2 multiplied by 1 plus x square by n power minus n plus 1 by 2. So, this is the probability density function of a random variable x n.

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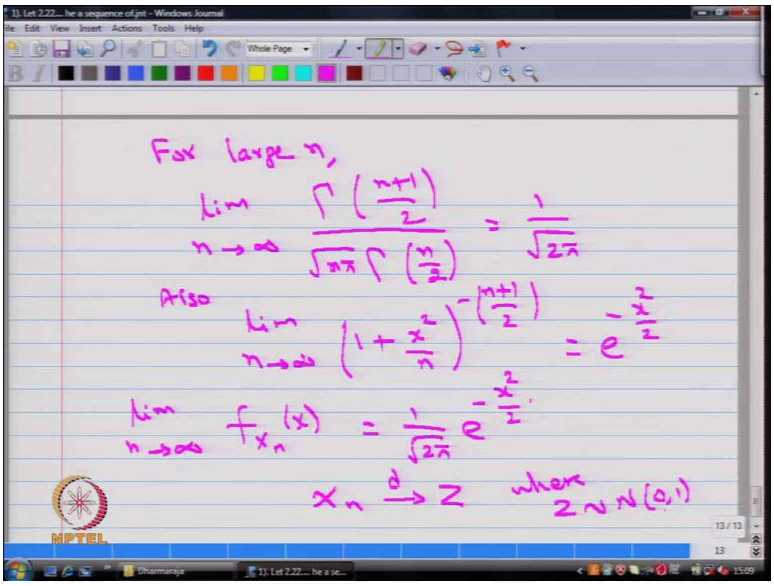

Our interest is to find out the limiting distribution of random variable x n. For larger for large n, we have the results limit n tends to infinity of gamma of n plus 1 by 2 divided by square root of n phi of gamma of n by 2 is 1 divided by square root of 2 phi using stirling approximation, and also limit n tends to infinity of 1 plus x square by 2 the whole power minus n plus 1 by 2, that we know that is e power minus x square by 2. Hence a limit n tends to infinity of the probability density function of the random variable x n becomes 1 divided by square root of 2 phi e power minus x square by 2.

Since, right hand side is the probability density function of a standard normal distribution. We conclude for a larger n the sequence of random variable x 1, x 2, x n and so on, that tends to the random variable z this convergence takes place in distribution, where z is standard normal distribution.

So, this is a simple example of the sequence of random variables, each having a student t distribution. The limiting distribution converges to standard normal in they converge in distribution.

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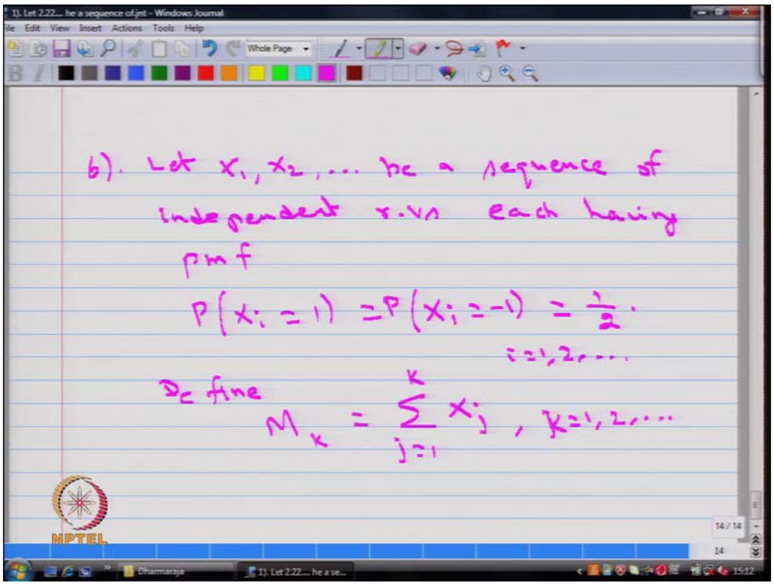

Now, move into next example, example 6, let x 1, x 2 so on, be a sequence of independent random variables each having probability mass function, probability of x i is equal to 1, that is same as probability of x i takes the value minus 1 probability is 1 by 2. This is valid for that means, it is a sequence of i i d random variable, and they are discrete type. Define M suffix k, thus the sum of first k x i random variables. So, this running index is k is equal to 1, 2 and so on. So, we are defining k sequence of a random variable m k by summing first k x i random variables.

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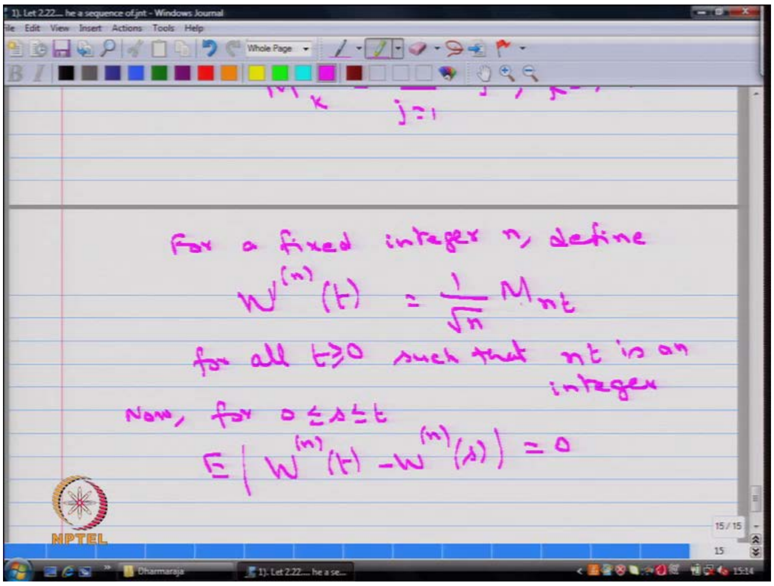

For a fixed integer n, we define another sequence of random variable that is denoted by W superscript n of t, that is nothing but 1 divided by square root of n M suffix n times t. This is for all t greater than equal to 0, such that n time's t is an integer. So, we are defining another sequence of random variable W superscript n of t, that is 1 divided by square root of n times m n of t, where n of t is a integer, so this is valid for all t greater than or equal to 0. To find out the mean and variance for the different of the random variable of a n of t minus W n of S for 0 less than or equal to s, less than or equal to t this quantity will be 0; that means W n of t is the 1 divided by square root of n, M n of t and the way we define the M n of t that is the summation of x i.

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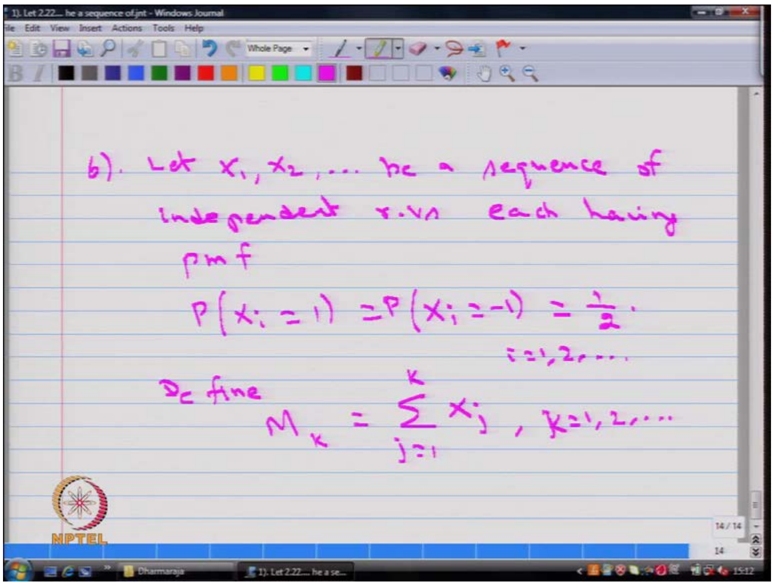

And the probability of x i is equal to 1, and probability of x i is equal to minus 1 minus 1 is 1 by 2, therefore, the mean of x i are going to be 0.

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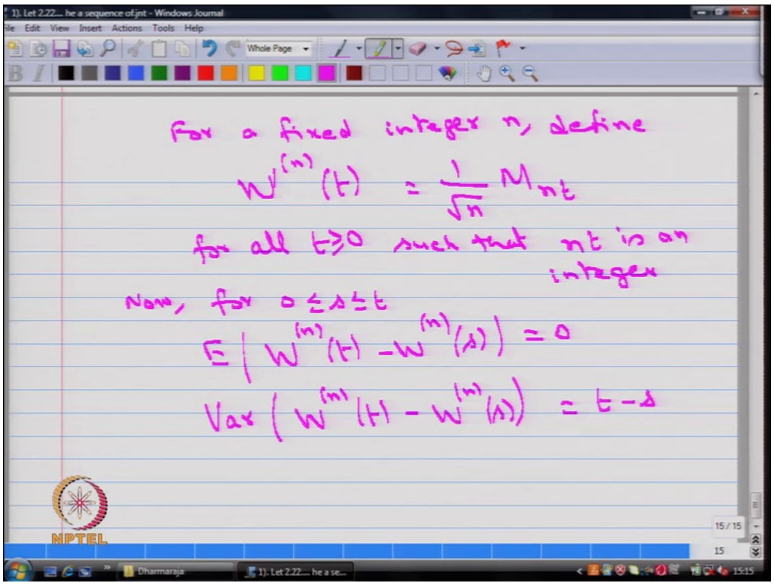

Because, of that the expectation of or mean of W n of t minus W n of s, that is equal to 0. Also, if you evaluate the variance of W n of t minus W n of S by finding first variance of x i's using that you are find out the variance of M n of t, then find out the W n of t minus W n of s, that is going to be t minus s.

Its need calculation of expectation of x i square then using the expectation x i square and expectation of x i's you can find out the variance of x i's, using variance of x i's you can find out the variance of W n of t, then you find out the variance of W n n of t minus W n of s.

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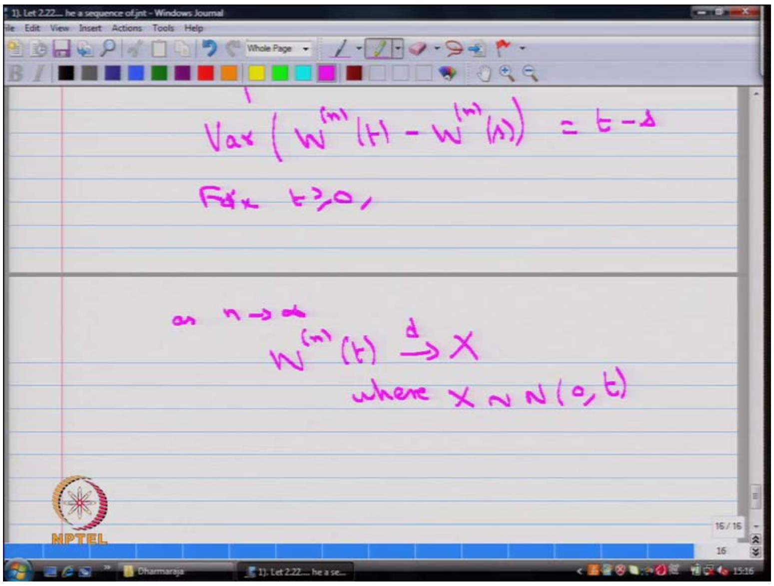

By using mean and variance, for fixing fix t greater than or equal to 0, as n tends to infinity you can conclude W n of t tends to a random variable x, and this converges takes place in distribution using c l t one can control W n of t converges to the random variable x, the converge in distribution, where x is normal distribution with the mean 0 and variance t.

Using a central limit theorem one can prove W n of t converges to x in distribution, where x is a normal distribution if the mean is 0 and the variance t. This result is very useful in Brownian motion, and this same problem will be discussed in detail, when we are discussing the module of a Brownian motion.