Stochastic Processes
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Module - 9
Branching Processes
Lecture - 2
Markovian Branching Process

This is a stochastic processes module nine branching processes. In the lecture one, we have discussed the definition and examples of branching processes, important discrete type branching process, Galton-Watson process is discussed in detail. We found mean and variance of Galton-Watson process. Then we have find you will find the probability of extinction for the Galton-Watson branching process. This is a lecture two, in this lecture we are going to discuss Markov branching process.

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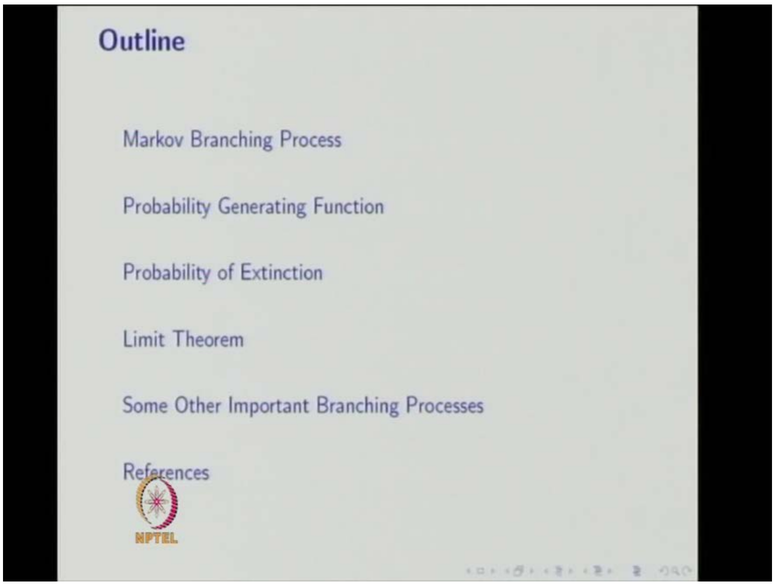

This is a very important branching process of a continuous time. This we are going to start with the probability generating function. Then, we are finding the probability of extinction and we discuss the limit theorem. Finally, we are going to discuss some other important branching processes at the end.

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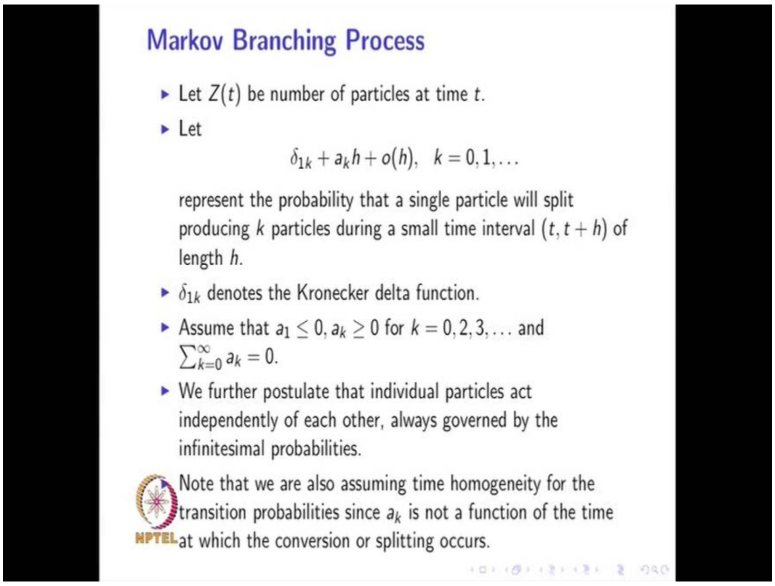

What is a Markov branching process? Let Z (t) be the number of particles at time t. The sequence is Z of t, the collection of random variables Z of t for t greater than or equal to 0 form a Markovian branching process with the following assumptions. Let delta 1 comma k plus a k times h plus order of h for k is equal to, 1, 0, 1, 2 and so on, represents the probability, that a single particle is split producing k particles during a small time interval t to t plus h of length h. Delta of 1 comma k denotes the Kronecker delta function.

Assume, that a 1 is less than or equal to 0 for k is equal to 0, 2, 3 and so on. a k's are greater than or equal to 0 and summation of a k's starting from 0, 1, 2 and so on, that will be 1, that will be 0. We further postulate, that individual particles act independently of each other, always governed by the infinitesimal probabilities. Note that we are also assuming time homogeneity for the transition probabilities, since a k is not a function of time at which the conversation conversion or split occurs; since a k is not a function of the time at which the split occurs. The similar assumptions we have taken care in the discrete type branching process also.

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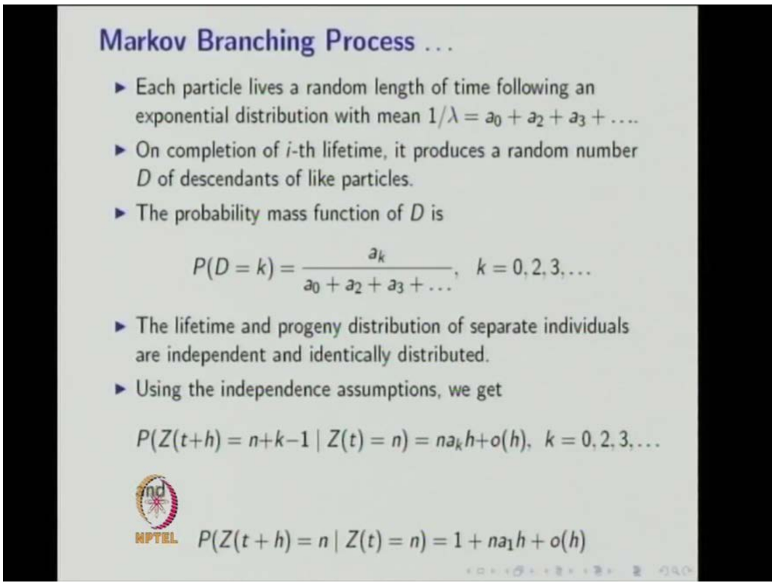

Each particle lives a random length of time following an exponential distribution with the mean 1 by lambda, that is (( )) plus mu a 2 plus a 3 and so on. On the completion of i-th lifetime, it produces a random number D of descendants of like particles. The probability mass function of D is probability, that D is equal to k will be a suffix k divided by a naught plus a 2 plus a 3 and so on. The lifetime and progeny distribution of separate individuals are independent and identically distributed. Using the independent assumptions we get the conditional probability of Z (t) plus h is equal to n plus k minus 1, given Z of t was n that is same as n times a suffix k of h plus order of h.

We know that small order of h means small order of h divided by h tends to 0, as h tends to infinity, as h tends to 0; order of h divided by h tends to 0 as h tends to 0. And for k equal to 1 the probability of Z of t plus h is equal to k given Z of t is equal to n. Probability of Z of t plus h is equal to n given Z of t is equal to n, that will be 1 plus n times a 1 h plus order of h. So, for k 0, 2 and 3 we have a separate expression, for k equal to 1 we have a different expression.

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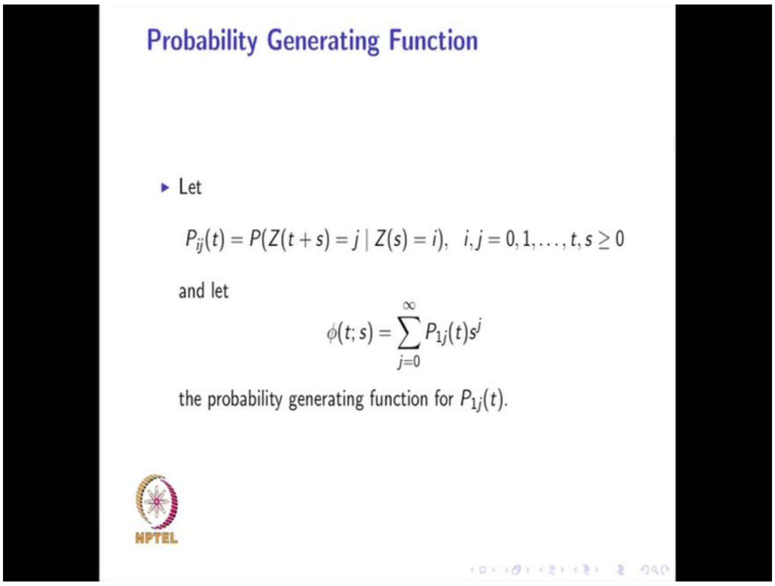

Now, using the condition probability we are going to define the probability generating function. Now, let P i comma j of t is nothing but the conditional probability of, probability of Z (t) plus s is equal to j, given Z of s was i. Using these, we define the probability generating function, that is nothing but psi of t, s, the two variables summation over j P i j of t s power j. So, this will be the probability generating function for P i j of t where P i j of t is transition probabilities.

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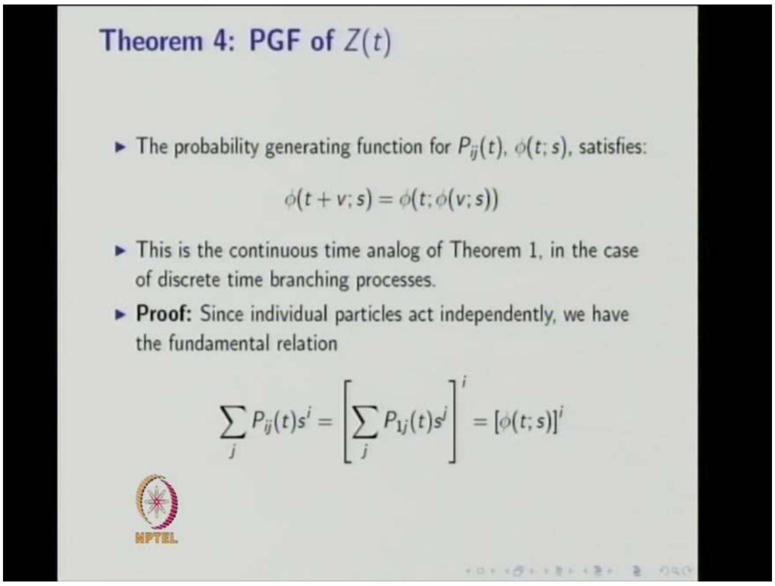

Now, we are going to discuss the probability generating function of Z of t in the theorem 4. Already first three theorems are discussed are the discrete type branching process. So, here we are going to discuss the fourth theorem, the probability generating function for P i j of t. There is psi of t, s satisfies psi of t plus v of psi of t plus (v, s), that is same as psi of t comma psi of v, s. This is a continuous time analog of theorem 1 in the case of discrete time branching processes.

We discuss the proof. Since individual particles act independently, we have the fundamental relation, the probability generating function for P i j of t, that is nothing but summation over j. Instead of the transition probability, i to j is a transition probability of 1 to j of t s power j the whole power i, that is same as the probability generating function power i.

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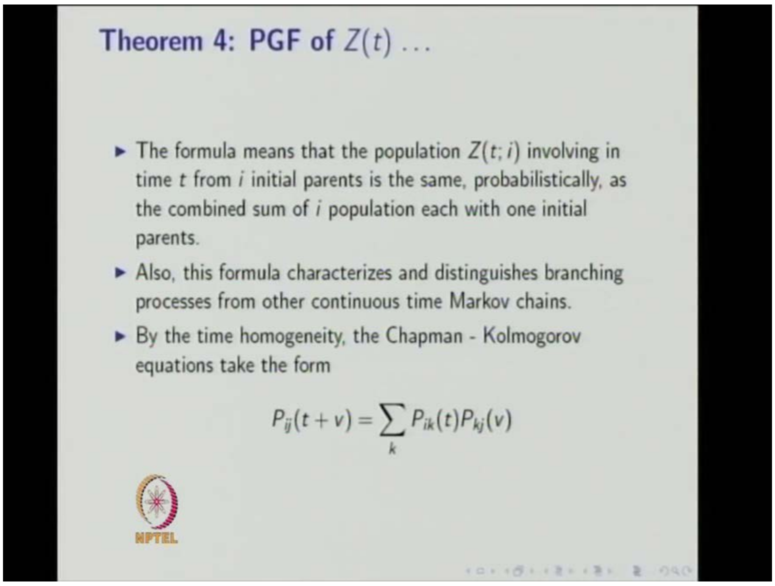

The reason is, the formula means, that the population Z of t, i involving in time t from i initial parents is the same, probabilistically, as the combined sum of i population each with one initial parents. Therefore, the left hand side is the probability generating function of, function for P ij of t, that is same as making summation over j, the probability transition, probability of P (1, j) of t s power j the power I, that is nothing but the probability generating function for P ij power i.

Also, this formula characterizes and distinguishes branching processes from other continuous time branching continuous time Markov chains. By the time homogeneity, the Chapman-Kolmogorov equations take the form, the one step transition probability of P i comma j t plus v can be written in the form of summation k P i to k of t, then P k to j of v. Because it satisfies the time homogeneity one can write the Chapman-Kolmogorov equations because this is the Markov branching process.

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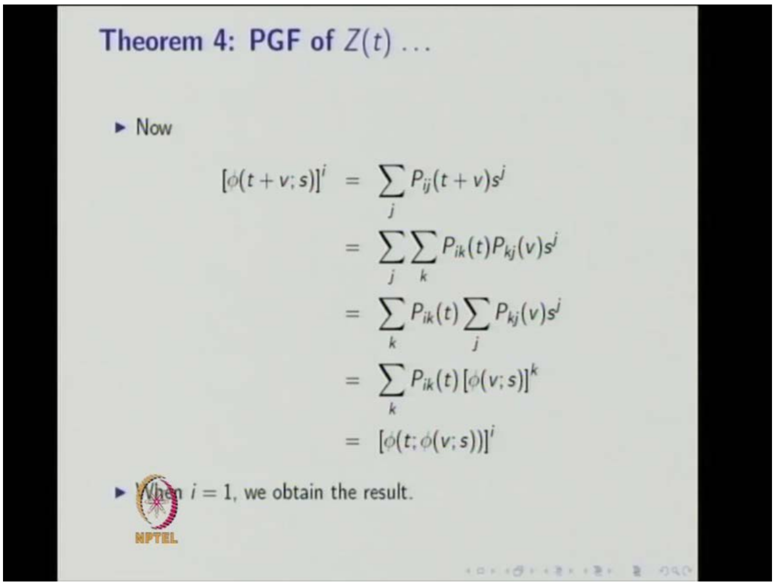

Now, the probability generating function of the time t plus v the whole power i, that is same as the summation j P ij of t plus v power j. So, using Chapman-Kolmogorov equation you can write the P ij of t plus v is summation over k P ik of t P kj of v, that is same as summation over k P ik of t summation over j P kj of v s power j. We know, that this is nothing but the probability generating function for P kj of v, that is same as the probability generating function of v, s power k. This will be written as the probability generating function of psi of t comma psi of v, s the whole power i. When you substitute i is equal to 1 you get the result because psi of t plus (v, s) is same as psi of t comma psi of v, s. When you substitute i is equal to 1 in this equation you will get, psi of t plus (v, s) is same as psi of t comma psi of v, s the whole power i. When i is equal to 1 we will get psi of t plus (v, s) same as psi of t comma psi of v, s.

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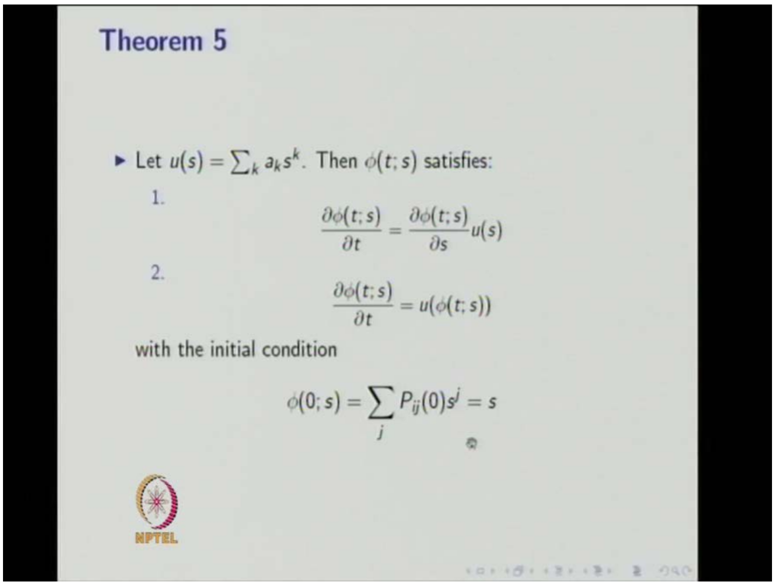

Now, we will move into the theorem 5, which discuss the differential equation corresponding to the probability generating function for P ij of t. Let u of s is equal to summation a k s power k summation over k. Then, the probability generating function for P ij of t satisfies partial derivative of psi of t, s with respect to t is equal to partial derivative of psi of t, s with respect to s multiplied u s. And partial derivative of psi of t, s with respect to t is same as u of psi of t, s with the initial condition psi of 0, s is same as summation over j P ij of 0 s power j, that is nothing but s. So, the theorem 5 gives the partial differential equation and ordinary differential equations satisfied, by the partial, by probability generating function of P ij of t.

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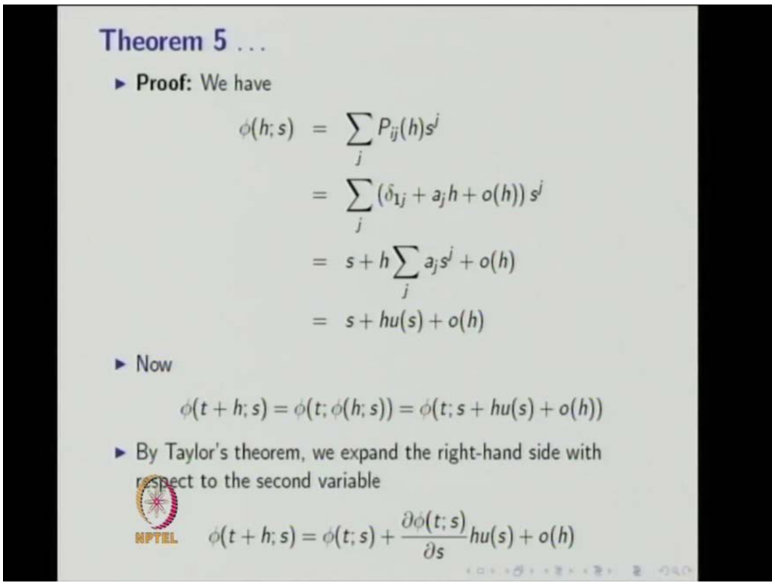

Let see the proof. We start with the psi of h of s, psi of h, s; that is nothing but the summation over j P of i comma j of s P of i comma j of h s power j. Substitute P ij of h and simplify, you will get the first term will be s the second term will be h times summation over j a j s j, the second term will be order of h.

You know, that u of s is same as summation over j, a suffix j of s power j. Therefore, the probability generating function for P ij of h, s, that is nothing but s plus hu of s plus order of h. We know that by the theorem 4, psi of t plus (h, s) will be psi of t comma psi of h of s (h, s). So, substitute psi of (h, s) with s plus h of u s plus o, order of h. Therefore, this will be psi of t, s plus h of u s plus order of h. By Taylor's theorem we expand the right hand side with respect to the second derivative. Therefore, the right hand side will be psi of t, s, the second term will be partial derivative of psi with respect to s times h of u s h times u s plus order of h, all the other term vanishes.

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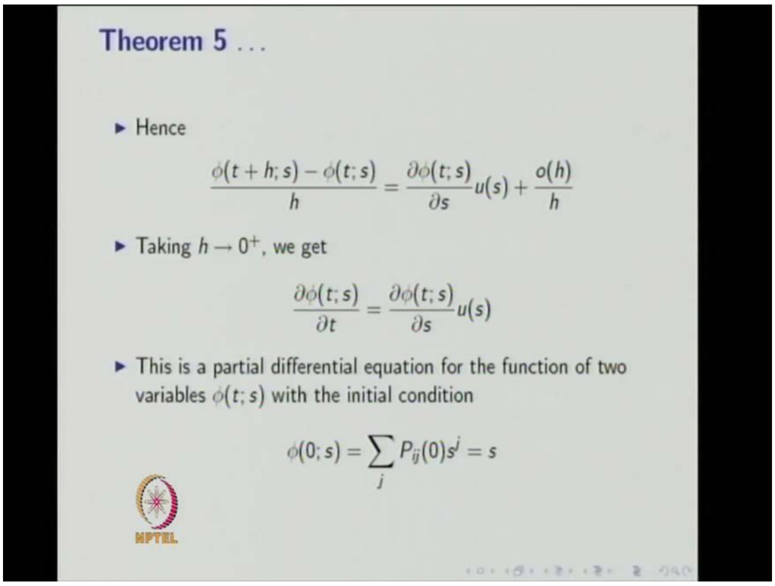

(( )) divide by h and take psi of t, s in the left side, therefore the left hand side becomes psi of t plus (h, s) minus i of t, s divided by h, whereas in the right hand side will be partial derivative of psi with respect to s times u of s order of h divided by h.

Taking h tends to 0 positive, we get the partial differential equation dou psi divided by dou t is equal to dou psi by dou s times u s. This is the partial differential equation for the function of two variables psi (t, s) with the initial condition psi of 0, s is s. So, we have proved the first part of theorem 5. Similarly, one can prove the second part of theorem 5.

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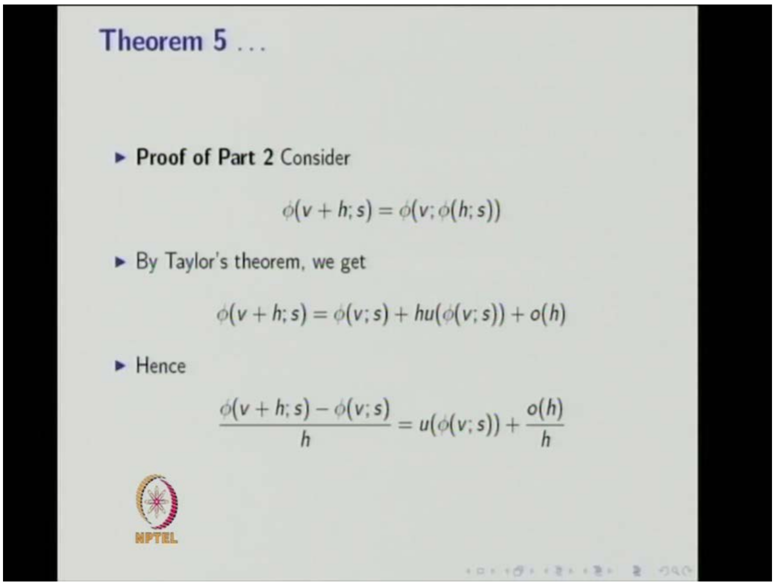

The proof of part two, you start with partial differential, partial, we start with the probability generating function psi of v plus (h, s) is same as psi of v comma psi of h, s. By Taylor's theorem, the right hand side becomes psi of v, s plus h of u psi of v, s plus order of h, then take psi of v, s in the left hand side, divide throughout by h you will get this equation.

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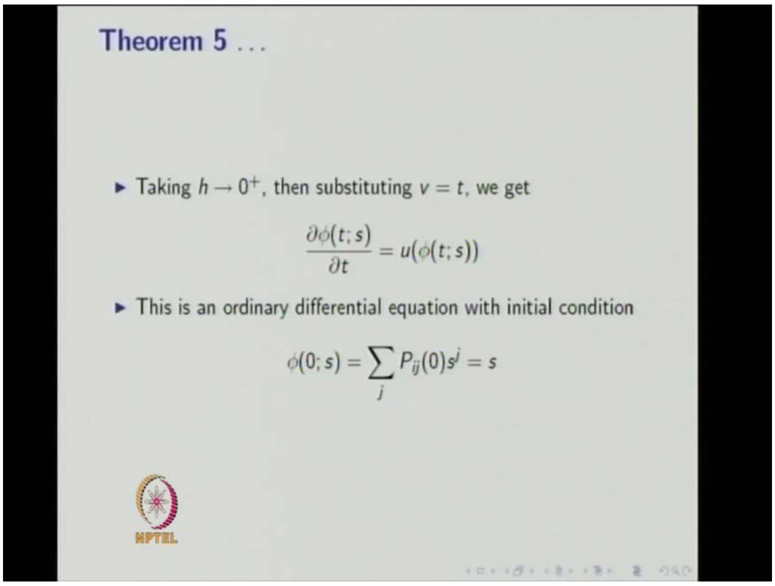

Now, limit h tends to 0 plus and then substitutes v is equal to t. In this equation limit h tends to 0 plus and substitute v is equal to h v is equal to t, we get partial derivative of psi with respect to t is equal to u of psi of t, s. This is the ordinary differential equation with the initial condition psi of 0, s equal to s.

So, in the theorem 5 we conclude, the partial, the probability generating function satisfies the partial differential equation and the initial and ordinary differential equations with the initial conditions psi of 0, s is equal to s.

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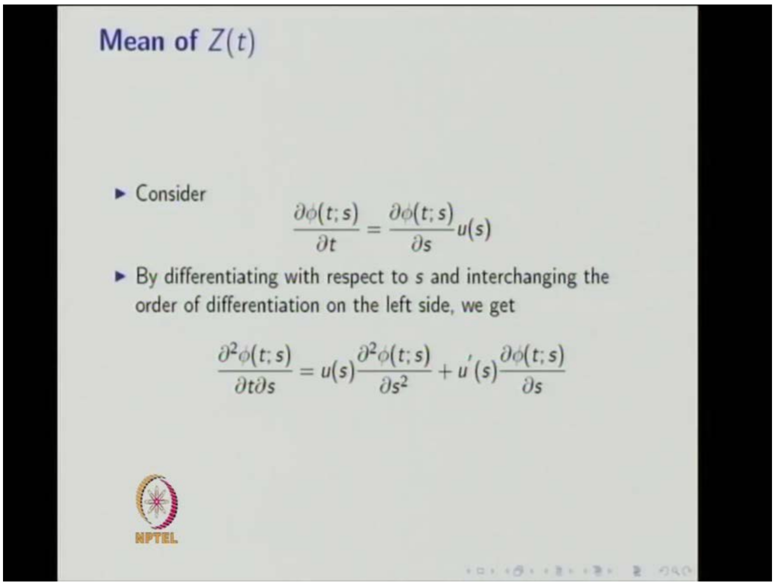

Now, you will find out the mean of Z of t. You start with the partial differential equation satisfied by probability generating function. By differentiating with respect to s and interchanging the order of differentiation on the left hand side we get, the left hand side is the second order partial derivative of psi with respect to t. And with respect to s the right hand side, u of s second order partial derivative of psi with respect to s u dash of s partial derivative of psi with respect to s.

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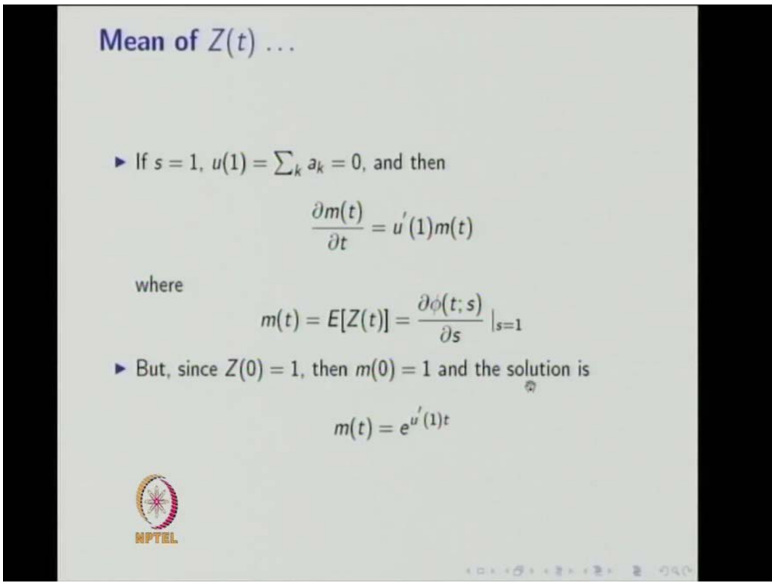

If s equal to 1, you know, that u of 1 will be 0. Suppose the m of t will be the mean of z of t that is nothing but the partial derivative of psi with respect to s. Then, substitute s is equal to 1, therefore this equation becomes partial derivative of m of t with respect to t is equal to u dash of 1 m of t. Since m is with the single variable, so this is the ordinary differential equation. So, d m t by dt is equal to u dash of 1 times m of t, where m of t is a mean of Z of t. But since Z of 0 is equal to 1, m of 0 also 1, therefore you can solve this ordinary differential equation with the initial condition m of 0 is equal to 1. Hence, the solution will be m of t is equal to e power t times u dash of 1.

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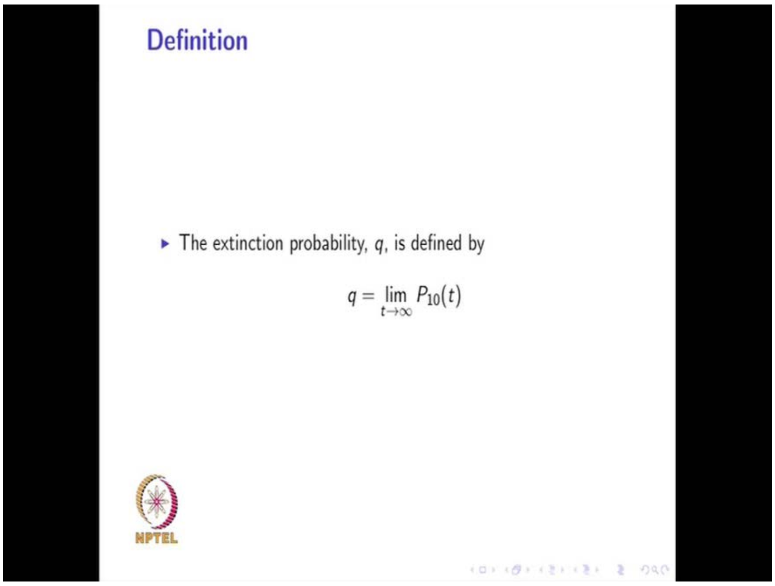

Now, we can discuss the mean of Z of t based on the value of u dash of 1. Before that we discuss the probability of extinction that is defined by q, that is nothing but limit t tends to infinity probability of 1 comma 0 of t. This is called a probability of extinction that is denoted by the letter q.

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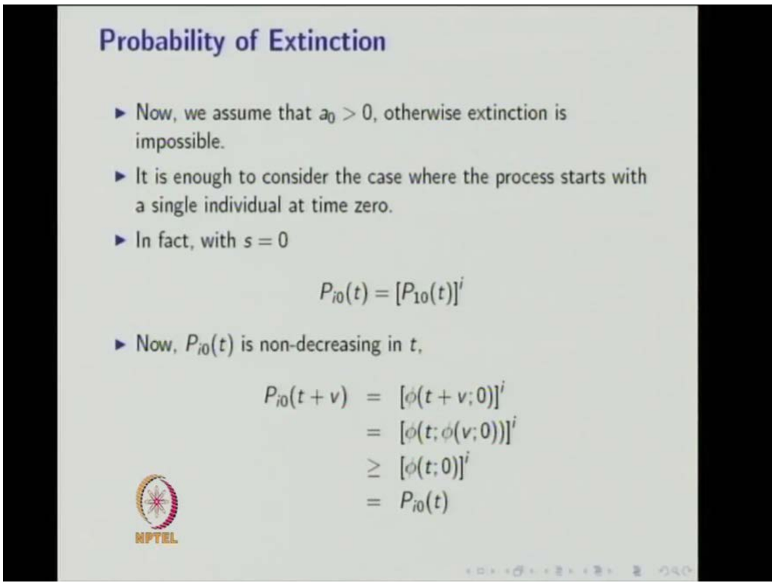

Now, we will try to find out the probability of extinction small q. Assume, that a naught is strictly greater than 0, otherwise extinction is impossible. It is enough to consider the case where the process starts with the single individual at time 0 that means Z of 0 is equal to 1. With s equal to 0 you will get P i of 0 of t, that is nothing but P 10 of t power i in the probability generating function of P ij of t.

Now, we will prove, that pi 0 of t is a non-decreasing in t. You start with P i0 of t plus v, that is nothing but psi of t plus (v, 0) of power i, that is same as a psi of t comma psi of v, 0 power i, that will be greater than or equal to psi of 0, t power i, but that is same as P i0 of t. Hence, we proved P i0 of t is non-decreasing in t.

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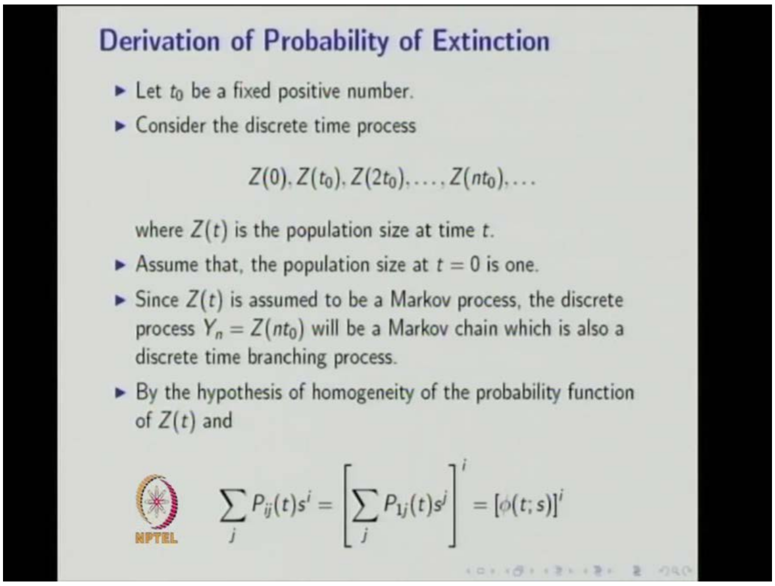

Let t be the fixed positive number. Consider a discrete time branching process Z of 0, Z of t naught, Z of 2 times t naught and so on, Z of n times t naught, where Z of t is a population size at time t. Assume, that the population size at time 0 is 1, Z of 0 is equal to 1. Since Z of t is assumed to be Markov process, the discrete process Y n, Y suffix n, that is nothing but Z of n of t naught will be a discrete time Markov chain, which is also a discrete time branching process because Z of t is a continuous time branching process. Therefore, Y of n will form a discrete time branching process, which is also a discrete time Markov chain.

By the hypothesis of homogeneity of a probability function of Z of t and the probability generating function of P ij of t, that is nothing but probability generating function of P ij of t power P 1 of 1j of t power i. This we have proved it in the earlier; we have proved it in earlier.

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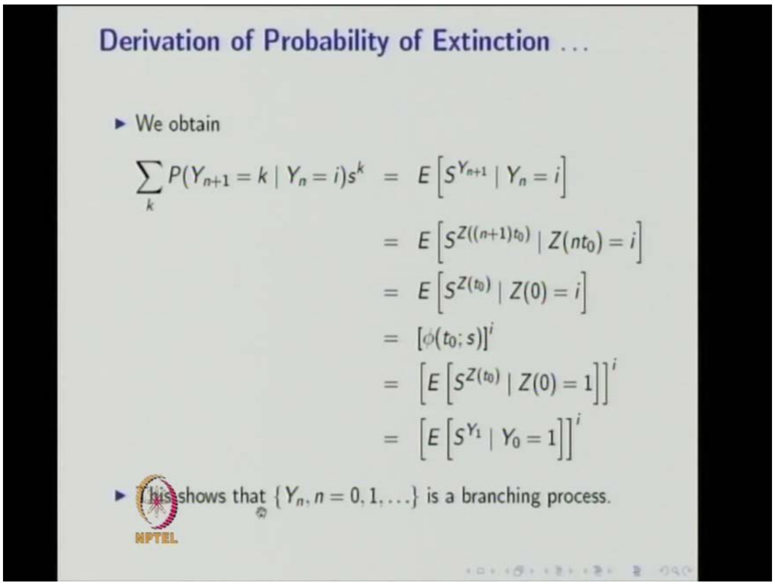

Therefore, using these two we are finding summation over k. The conditional probability of Y n plus 1 is equal to k given Y n is equal to i multiplied by s power k, that is nothing but expectation of S power Y n plus 1 given Y n is equal to i. That is same as, because Y n is nothing but Z of n times t naught, therefore Y n plus 1 is nothing but Z of n plus 1 times t naught.

We replace Y n by Z n n times t naught and Y n plus 1 by Z of n plus 1 times t naught. This is true for all n, therefore that is same as expectation of S power Z of t naught divided given Z of 0 is equal to I, but that is nothing but psi of t naught comma S power i, that can be written as expectation of S power Z of t naught given Z naught is equal to 1 whole power i. That is same as expectation of S power Y 1 given Y naught is equal to 1 the whole power i. This shows that Y n is the branching process; Y n is a discrete time branching process. So, using these we have proved the Y n is a discrete time branching processes.

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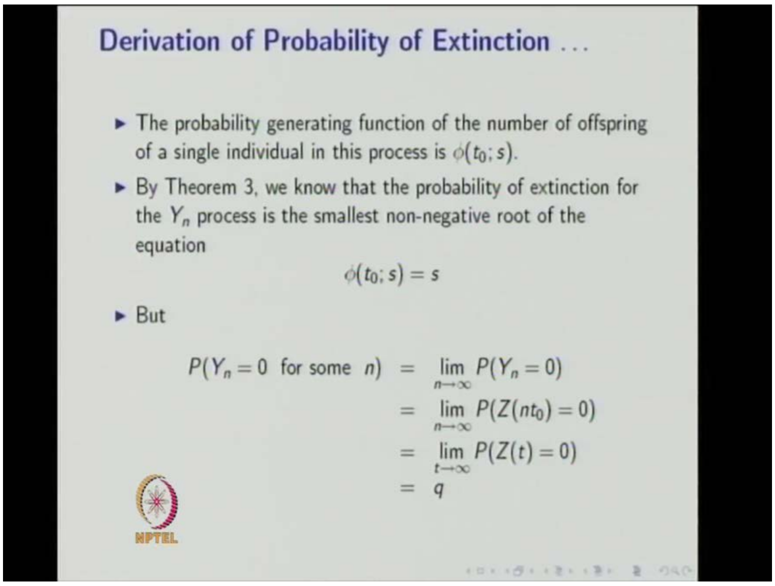

The probability generating function for the number of upstream of a single individual in this process is psi of t naught comma s. By theorem 3 we know, that the probability of extinction for Y n, that is, a discrete time branching process is the smallest non-negative root of the equation psi of t naught comma s equal to s.

So, by using the theorem 3 we conclude, the probability of extinction for the Y n process is the smallest non-negative root of the equation psi of Z naught comma s equal to s. But we know that probability of Y n is equal to 0 for some n, that is same as limit n tends to infinity of probability of Y n is equal to 0; that is same as limit n tends to infinity of probability of Z n times z naught is equal to 0, but that is same as limit t tends to infinity of probability Z (t) is equal to 0. By definition this is nothing, but small q, that is a probability of extinction.

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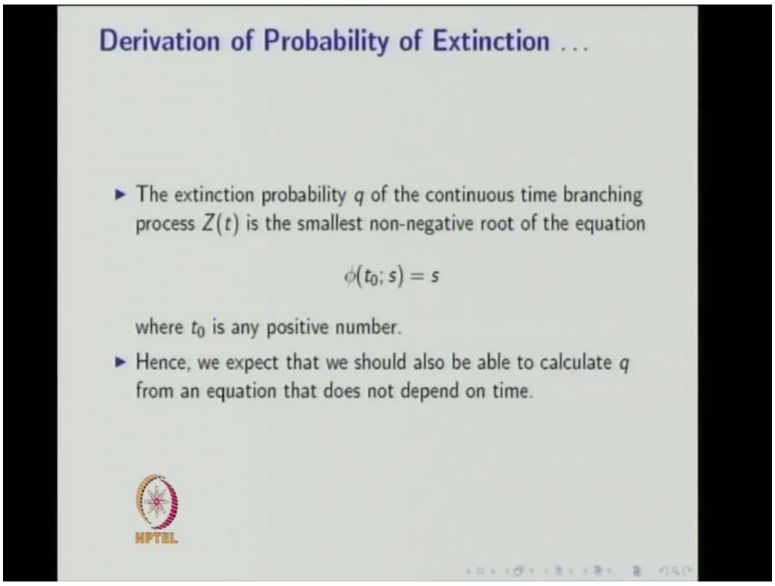

Hence, the probability of extinction q of a continuous time branching process Z of t is the smallest non-negative root of the equation psi of t naught comma s is equal to s. Here, we have to conclude by theorem 3 the probability of extinction for the discrete time branching process. Y n is the smallest non-negative root of the equation psi of t naught comma s equal to s. Because of this we conclude the probability of extinction of the continuous time branching process. Z of t is the smallest non-negative root of the equation psi of t naught comma s equal to s where t naught is any positive number.

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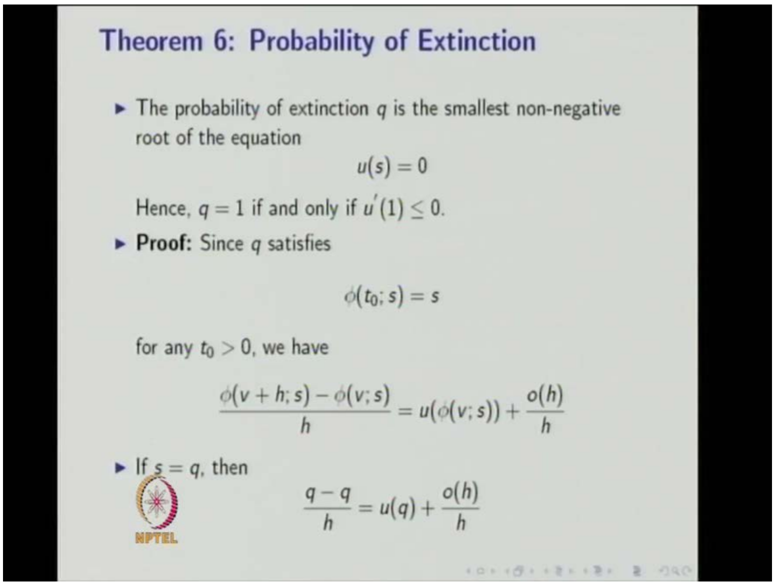

Hence, we expect, that we should be, we should also be able to calculate q from equation, that does not depends on time. From this equation can able to calculate q from the equation, does not depend on time.

Now, we are moving into theorem 6, how to find the probability of extinction. We conclude, q is the probability of extinction for the continuous time branching process. So, here in this theorem we are giving the probability of extinction q is the smallest non-negative root of the equation u of s equal to 0. Hence, q is equal to 1 depend only if q dash is lesser than or equal to 0. So, whenever q dash of 1 is less than or equal to 0, then the probability of extinction will be sure, probability will be 1; extinction event will be sure, the probability of extinction will be 1.

Now, we give the proof of probability of extinction. In the earlier theorem we have concluded, q satisfies psi of t naught comma s is equal to s for any t naught greater than 0. We have, this relation we have in the theorem 4. The theorem 4 discusses, theorem 5 discusses the partial differential equations and ordinary differential equation satisfies by psi of t comma s.

So, we are using these equations to find the probability of extinction. So, here by using theorem 5, psi of v plus (h, s) minus psi of v, s divided by h will be u of psi of v, s plus order of h by h. We know, that this will be tend to 0 has a h tends to 0 if s equal to q. Thus, q is the probability of extinction in the smallest non-negative root of the equation psi of t naught comma s equal to s.

Therefore, if you substitute s is equal to q here, then above equation becomes the, left hand side become 0 when you put s is equal to q here, then the psi of v, q will be q. Therefore, this will be u of q plus order of h divided by h. By substituting s equal to q in the above equation the left hand side becomes 0, the right hand side, first term, psi of v, q will be q. Therefore, it will be u of q plus order of h divided by h.

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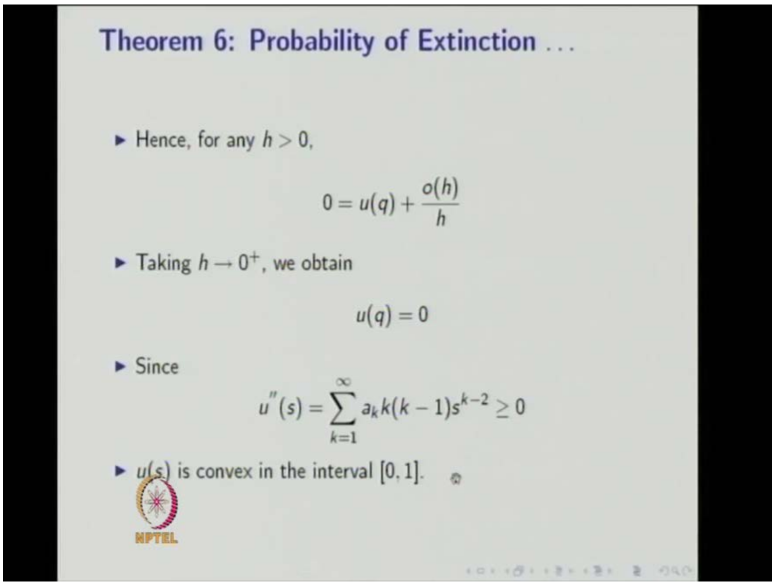

Hence, for any h greater than 0 it will be, 0 is equal to u of q plus order of h divided by h. As h tends to 0 plus you will get, u of q will be 0. Therefore, the earlier theorem we have concluded the probability of extinction will be psi of t naught comma s equal to 0 where q will be the probability of the extinction is the smallest non-negative root of the equation. But here, by using this we concluded u of q is equal to 0. Hence, the probability of extinction q is the smallest non-negative root of the equation, u of s is equal to 0 because we concluded, u of q is equal to 0.

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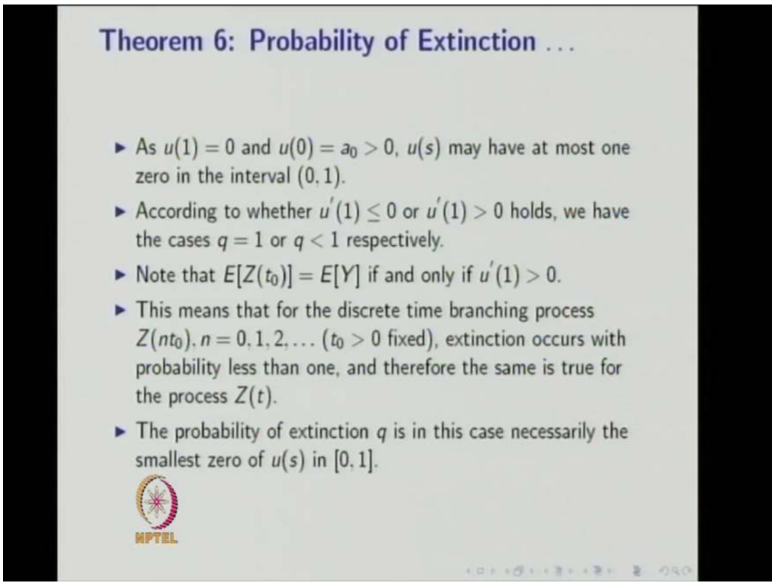

Suppose you find the double derivative of u that will be greater than or equal to 0. Hence, we conclude u of s is a convex function in the interval (0, 1). As u of 1 is equal to 0 and u of 0 is equal to a naught, which is greater than 0, u s may have at most one 0 in the interval (0, 1). The way we defined u of s, u of s is the summation a k s power k, therefore u of 1 will be 0 and u naught will be a naught, which is greater than 0. With that assumption only the probability of extinction is possible.

According to whether u double dash is less than or equal to 0 or greater than 0, we have the case q is equal to 0 or q is less than 1 respectively. That means, when u dash is less than or equal to u dash of 1 is less than or equal to 0, the probability of extinction will be 1. The u dash of 1 is the greater than 0, then the probability of extinction will be less than 1.

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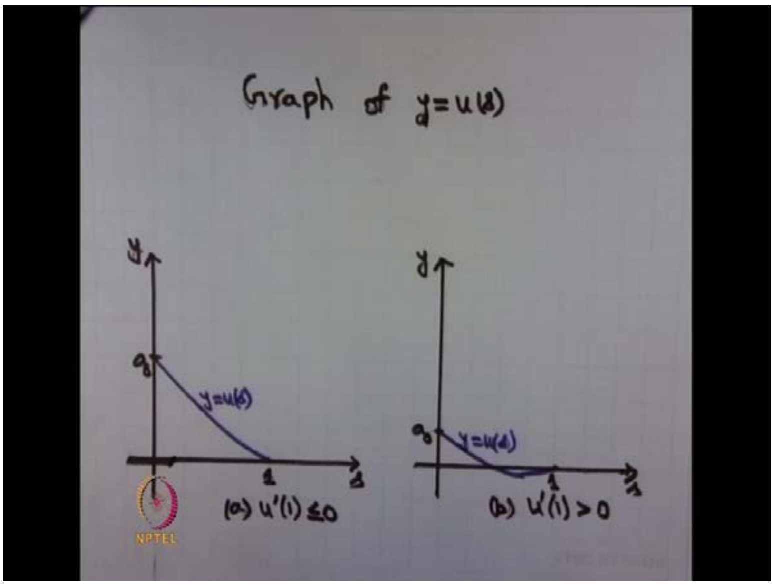

So, graphically one can show, this is a graph of y is equal to u of s. So, here we have two graphs, the graph A related to u dash of 1 is less than or equal to 0. Since u of s is the convex function in the interval 0 to 1 and u naught is a naught u 1 will be 0. So, this is the graphical representation of y is equal to u of s.

Then, the case u dash of 1 is less than or equal to 0. In the case two, then u dash of 1 is greater than 0, the y is equal to the u of s will cut the x-axis at some point, which is less than 1 because u naught is a naught u 1 is equal to 0 and u of s is the convex function. u dash of 1 is greater than 0, the u of s will cut in the x-axis before 1. Hence, the probability of extinction when u dash of 1 is less than or equal to 0, that will be 1 and the probability of extinction, then u dash of 1 is greater than 0, it will be less than 1.

Note, that expectation of z naught, expectation of z of t naught is equal to the expectation of y, depend only u dash of 1 is strictly greater than 0. So, whenever u dash of 1 is strictly greater than 0, the probability of extinction is less than 1. This means, that for discrete time branching process Z of n times t naught extinction occurs with the probability less than 1 and therefore, the same is true for the process Z of t. The probability of extinction q is in the case necessarily the smallest 0 of u of s in 0 to 1.

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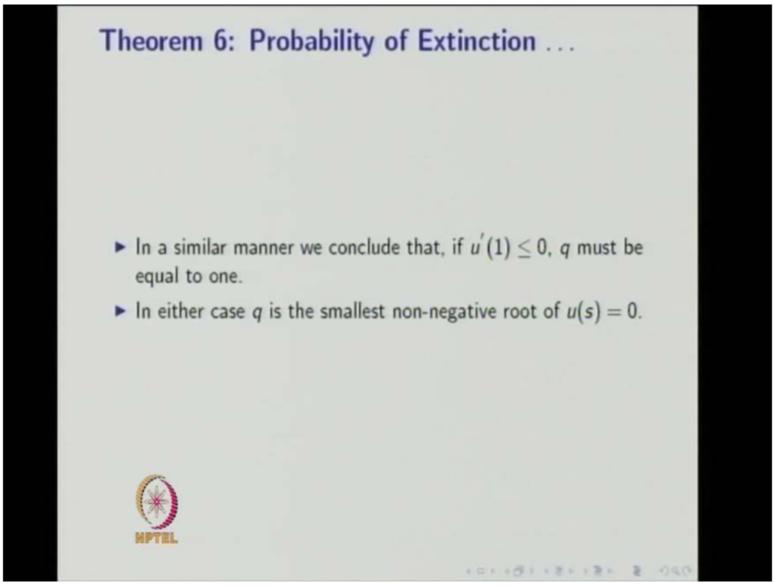

In a similar manner we conclude, that if u dash of 1 is less than or equal to 0, q must be equal to 1. In either case q is the smallest non-negative root of u of s equal to 0. So, hence the probability of extinction q is the smallest non-negative root of the equation u of s equal to 0. When q is equal to, when u dash of 1 is less than or equal to 0 the probability of extinction will be 1.

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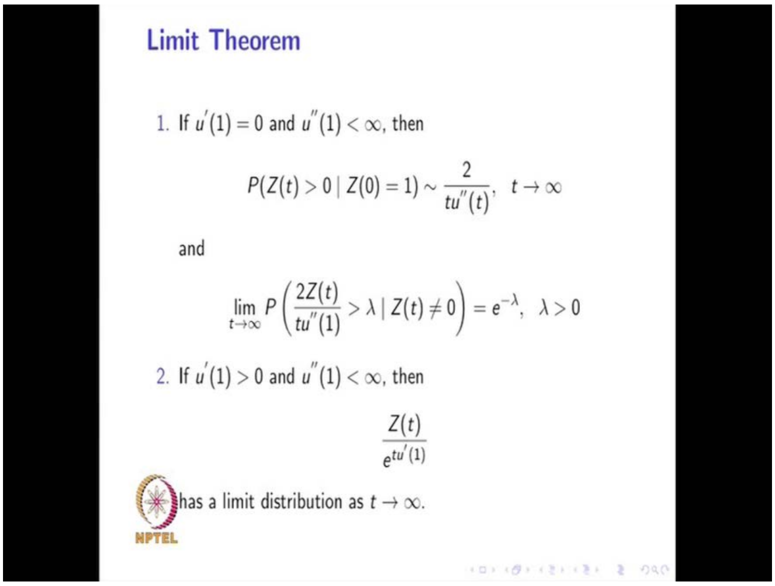

Now, we will consider the limit theorem, if u dash of 1 is equal to 0 and u double dash of 1 is finite. Then, we can show this conditional probability will be approximately 2 divided by t times u double dash of t as t tends to infinity. And also, we can conclude the limit t tends to infinity, probability of this event is e power minus lambda where lambda is strictly greater than 0. When u dash of 1 is strictly greater than 0 and u double dash of 1 is finite, then the Z of t divided by e power t times u dash of 1 has a limit distribution as t tends to infinity. Without proof we are stating this limit theorem.

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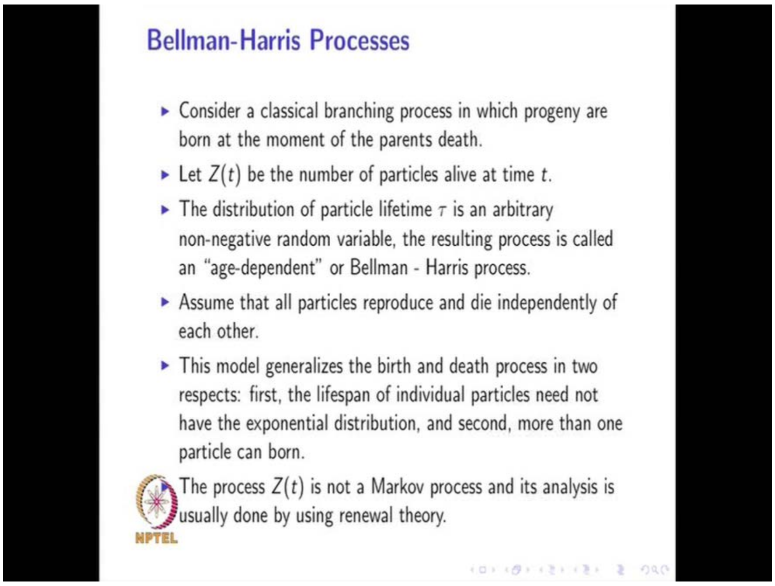

Till now we have discussed the two important branching processes: the first one is Galton-Watson discrete time branching process, the second one is a Markov branching process, which is a continuous type branching process. Now, we are going to discuss some more or some other important branching processes. The first one is Bellman-Harris processes. Consider classical branching processes in which the progeny are born at the moment of parent's death.

Let Z (t) be the number of particles alive at time t. The distribution of a particle lifetime tau is an arbitrary non-negative random variable, the resulting process is called age-dependent or Bellman-Harris processes. So, in the Markov branching process the random variable tau, which is exponential distribution, that here is an arbitrary non-negative random variable, then the resulting process is the age dependent or Bellman-Harris process. So, when tau becomes exponential distribution, then age dependent Bellman-Harris process becomes Markov branching process.

Assume that all particles reproduce and die independently of each other. In the similar assumption, we have taken care in the discrete type, as well as, continuous type branching processes. This model generalizes the birth, death process in two aspects: the first, the lifespan of individual particles need not have the exponential distribution and second, more than one particle can born. Because of these two aspects this model generalizes the birth, death process. The process Z of t is not a Markov process and it is analyzes, usually done by using renewal theory, we have discuss renewal processes in module 8.

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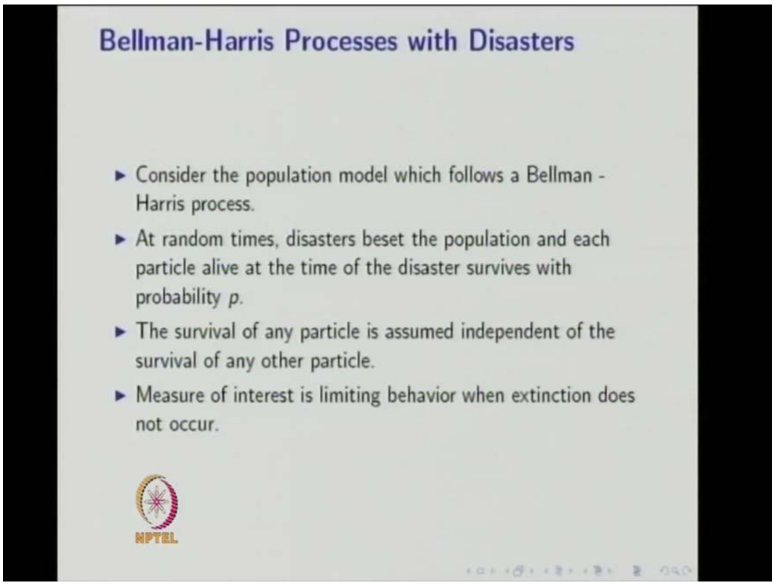

Now, we discuss the Bellman-Harris processes with disasters. Consider the population model, which follows a Bellman-Harris process. At random times, disasters beset the population and each particle are alive at the time of disasters survives with the probability p. The survival of any particle is assumed independent of survival of any other particle. In this model, the measure of interests is limiting behavior when extinction does not occur.

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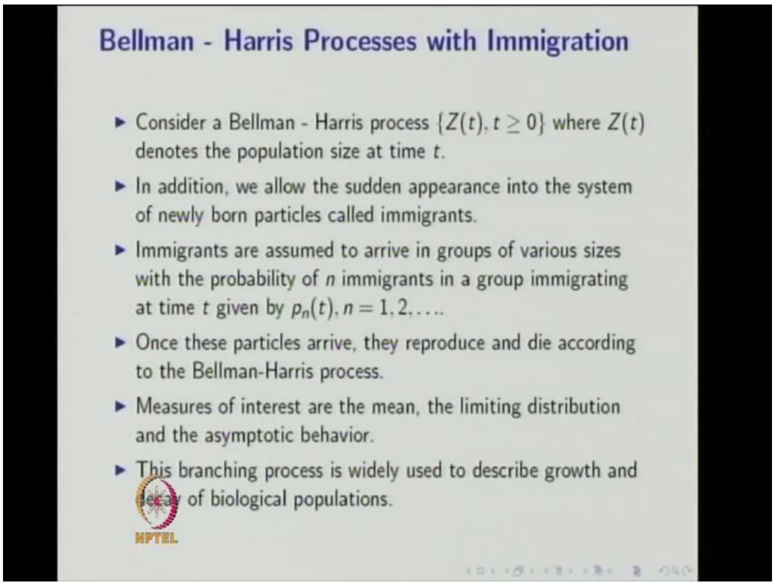

We move into the other important branching process, that is, Bellman-Harris processes with immigrants. In addition to the Bellman-Harris process we allow a sudden appearance into the system of newly born particles called immigrants. Immigrants are assumed to arrive in group of various sizes. The probability of n immigrants in a group immigrating at time t given by p n of t. Once these particles arrive, they reproduce and die according to the Bellman-Harris process. In this model measures of interest are the mean of Z of t, the limiting distribution and asymptotic behavior of Z of t. This process is widely used to describe growth and decay of biological populations.

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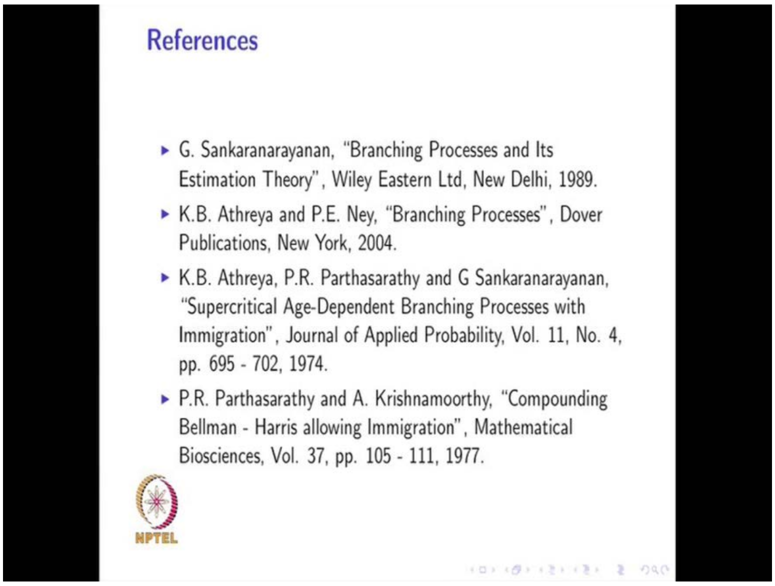

We are not discussing in detail of Bellman-Harris process in this lecture. In this module we have discussed in detail two important branching processes, Galton Watson process and Markov branching process. We have briefed Bellman-Harris process with disasters and with immigration. In the first two branching processes we have discussed mean and variance of Z of t, limiting distribution probability of extinction in both branching processes. Here are the references for this module 9 branching processes.