Stochastic Processes
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Module - 8
Renewal Processes
Lecture - 3
Markov Renewal and Markov Regenerative Processes

This is stochastic processes, module eight renewal processes. In the first two lectures we have discussed renewal functions and its properties, and then we have discussed renewal theorems. There are three importance theorems we have discussed in the lecture two. Today's lecture is a lecture three, Markov renewal and Markov regenerative processes.

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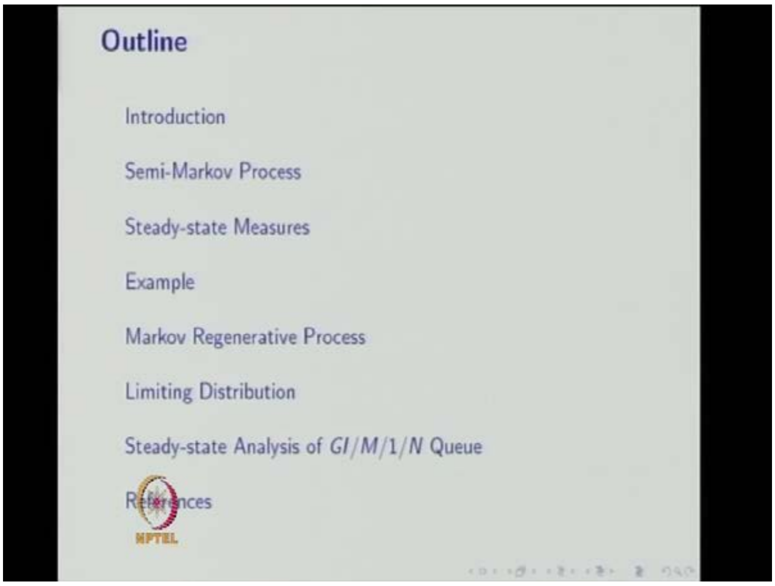

In this lecture, I am going to cover Markov renewal process or semi Markov process, the definition and it is properties, followed by the definition and properties I am going to discuss the steady state measures, and I am going to discuss a one simple example for the semi Markov process. Then the second part of today's lecture, I am going to cover Markov regenerative process; the definition and the properties I am going to discuss, followed by the definition and properties I am going to discuss the limiting distribution. As a example we are going to discuss the steady state analysis of G, M, 1, N queue.

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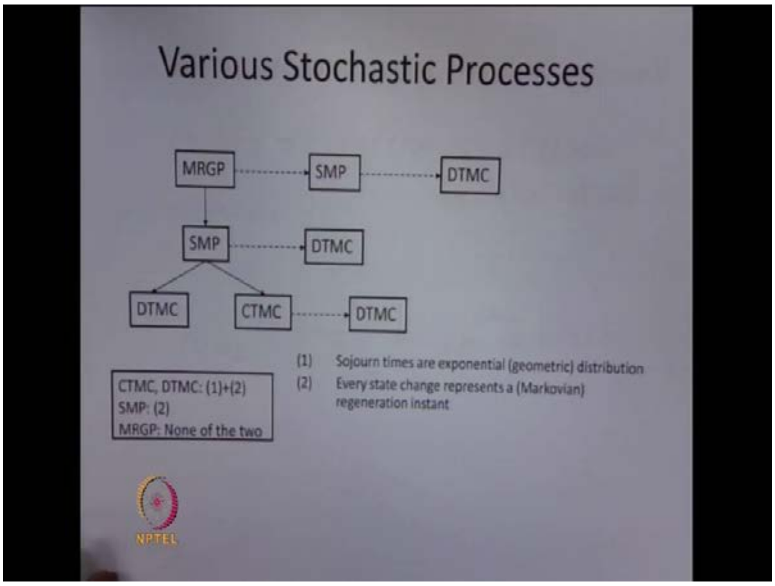

First, we are going to discuss the various stochastic processes which consist of DTMC, CTMC, semi Markov process, and Markov regenerative process. First you consider number one, the sojourn times are exponential distribution; number two, every state change represents a regeneration instant. The epochs or points t 1, t 2 and so on, in which the process probabilistically restarts from scratch that is from epoch t 1, the process is independent of its past; and the stochastic process behavior from epoch t 1 is the same as it had from t naught is equal to 0. These epochs, these epochs or points are called regenerative epochs or regenerative time points or regenerative instance. So, these are all the two important properties, one is sojourn times are exponential distribution, the second one is every state change represents the regeneration instant.

Accordingly, we can classify the stochastic process. The first one is, if both the properties are satisfied, then the corresponding stochastic process are either it is, it is a Markov process. So, based on the time spaces is a discrete are continuous, we have a discrete time Markov chain or continuous time Markov chain. Whenever the two properties are satisfied, the stochastic processes is said to be a Markov process, and the states space is discrete then the Markov process is called a Markov chain, and based on the time space or parameter space is a discrete or continuous, accordingly we have a discrete time Markov chain or continuous time Markov chain.

If any stochastic process satisfies the property number two, only not the property number 1, that means sojourn times are exponential distribution; if that property is not satisfied then that stochastic processes is called a semi Markov process. We are going to discuss the, in detail about the semi Markov process that is the stochastic process satisfying the property number 2 only. If both the properties are not satisfied, but still not every states change represents the regeneration instant, instead of these there are few state change represents we are regeneration instant, then the stochastic processes is called a Markov regenerative process.

So, if two properties are satisfied then it is a CTMC or DTMC; if only the property number 2 satisfies, then it is a SMP; if both the properties are not satisfied, but few state change represent a regeneration instant then the stochastic process is called a Markov regenerative process. So, that we have shown it in the diagram. From the stochastic, from the semi Markov process you can have a embedded DTMC; by making a proper assumptions that he sojourn times are exponential distribution, then it will be a CTMC. From the CTMC you can create the DTMC in the embedded. Similarly, from the SMP you can make a embedded DTMC, by proper assumptions you can get the DTMC or CTMC.

The general case of semi Markov processes is a Markov regenerative process, if you make some assumptions then it will be a semi Markov process. The down arrow means if you, if you make a additional assumptions in the Markov regenerative process then you will get the semi Markov process. If you make additional assumptions in the semi Markov process then you will get a DTMC or CTMC; whereas the, these dotted arrows mean a embedded stochastic process. So, from the DTMC, from the CTMC you can have a embedded DTMC; from the SMP you can have a embedded DTMC; from the MRGP you can have a embedded SMP; and from the SM SMP you can have a embedded DTMC. So, this is a pictorial representation of a various stochastic process based on the, these two properties.

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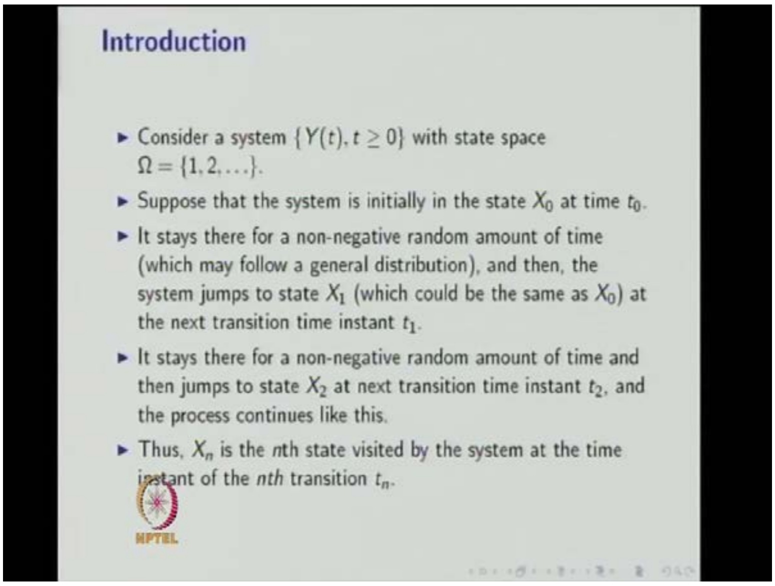

Now, we are moving into the semi Markov process, introduction. Consider a system Y t with the state space omega; that means, the Y t is a discrete state continuous time stochastic process. Suppose that the system is initially in the state X naught at time t naught. It stays there for a non-negative random amount of time, which may follow a general distribution, and then, the system jumps to the state X 1 at the next transition time instant t 1.

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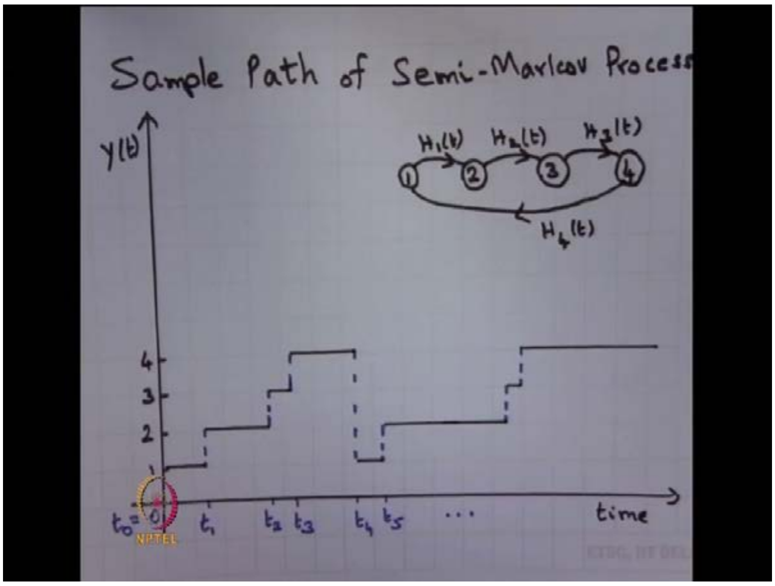

It stays there for non-negative random amount of time and then jumps to the state X 2 at the next transition time instant t 2, and the process continues like this. Thus, X n is the n th state visited by the state system at the time instant of n th transition t n. This can be depicted through the sample path. So, sample path means it is a trace.

Consider, this sample path are semi Markov process. The corresponding Y t is in the y axis. The time is in the x axis, y axis Y of t. The corresponding stochastic process as the 4 states, omega is a 1, 2, 3, 4. So, at time 0, the system is in the state 1, and the system is in the state 1 till time t 1, t 1 time instant. At the time t 1 instant, the system moves to the state 2. So, the system was in the state 0 at time t 0 that is equal to 0; and, at the time t 1 it moved to the state 2. The system is in the state 2 till time instant t 2, and then it moves to the state 3 at time instant t 2. So, from the state, from the state 2, the system moves to the state 3 at the time point t 3 sorry t 2. The system, the system moves to the state 4 at the time point, time instant t 3. Now the system is in the state 4. And, the system was in the state 4 till the time point t 4, then it moves to the state 1, and so on.

And, the t 1, t 2, t 3, t 4, are the time instant. In those time instant, the system is move from one state to the other states. Go back to the previous slide, the system was in the state X naught at time t naught, and at time t 1 the system move to the state X 1, and so on. Thus, X n is the n th state visited by the system and the time instant of n th time transition t n.

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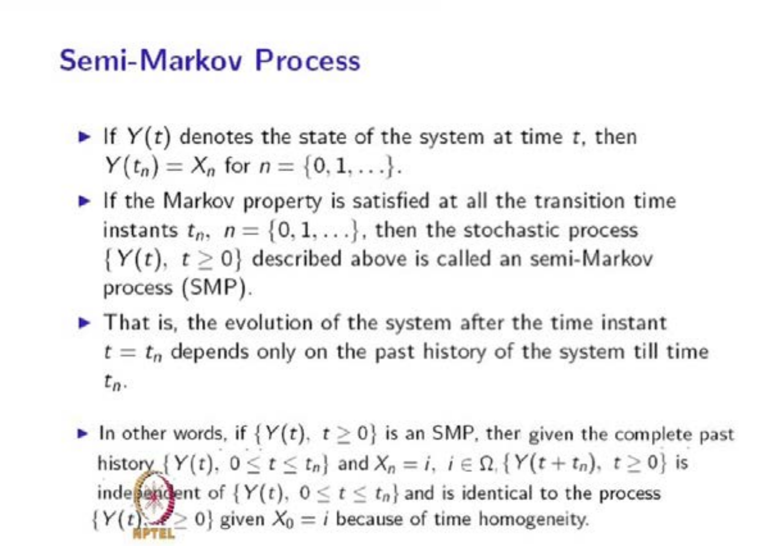

So, in this sample path, t 1, t 2, t 3, t 4, t 5, are the time instance and the system is visiting the states. So, here, it is a 4 state model. The omega is 1, 2, 3, 4. Therefore, the system is keep moving into the, from one state to other states according to the, this state transition diagram, we will come back to the same example again. So, this is a illustration of a sample path of semi Markov process, and the t 1, t 2, are the time instant.

If Y t denotes the system, if Y t denotes the state of the system at time t, then Y of t n is equal to X n, whenever you observe at the time instant in the system that is the possible values of X n. If the Markov properties satisfied at all the time, all the transition time instants t n, then the stochastic process described above is called the semi-Markov process, whenever the Markov process is satisfied at the time instant t naught, t 1, t 2, t 3, t 4, and so on, all the time instance. The time instances are nothing but the system is moving from one state to another state at those time extremes. So, if the Markov properties are satisfied at these time instance, then the stochastic process Y of t is called the semi Markov process, or in other words it is called a Markov renewal process.

That is, the evolution of the system after that time instant t equal to t n depends only on the past history of the system till the time t n. In other words, if Y t is a semi Markov process then the process is Y of t plus t n is independent of a semi Markov processes is thus a stochastic process in which changes of state occur according to the Markov chain, and in which the time interval between two successive transitions is a random variable whose distribution may depend on the state from which the transition takes place, as well as on the state to which the next transition takes place. Y of t, Y of t plus t n is independent of Y of t, given the complete past history Y of t and X n is equal to i, and is identical to the process Y of t, given X naught is equal to i, because of time homogeneity. Not only Y of t plus t n is independent of Y of t, it is also identical to the process Y of t, given X naught is equal to i because of time homogeneity, it is satisfying the time invariant property also. So, it is very important, when the Markov properties satisfied at the time in transition time instance t n then the stochastic processes is called a semi Markov process.

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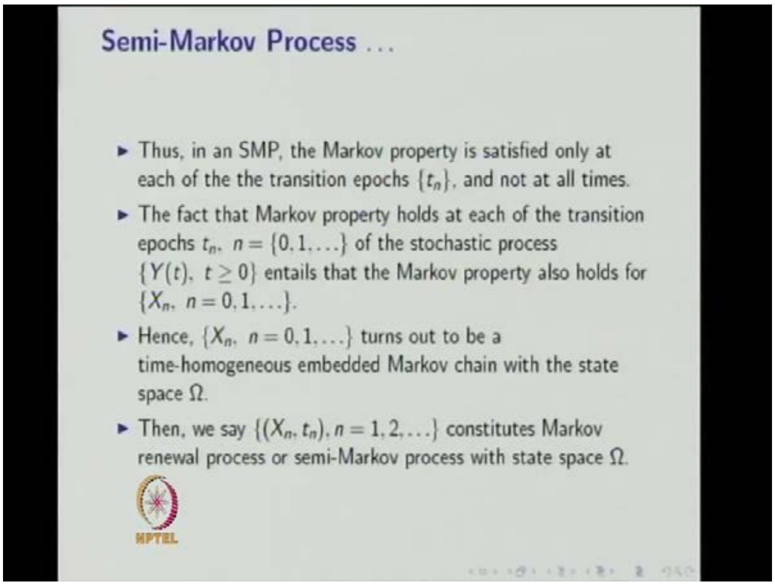

Thus, in a semi Markov process, the Markov property is satisfied only at the each of the transition epochs, and not at all times, this is very important; the Markov property satisfied only at the each of the transition epochs t n, not at all the times. If the Markov properties satisfied at all the time, all times then the stochastic process is called a Markov process. Since, it satisfies only at the each of the time transition instance the Markov process is called a semi Markov process. The fact that the Markov property holds at each of the transition epochs t n of the stochastic process Y of t, entails the Markov property also holds for X n. Since, X n is nothing but the Y of t n, therefore, the Markov properties satisfied for the stochastic process X n, not for the Markov property is satisfied for the Y of t for all times.

Hence, X n turns out to be the time homogeneous embedded Markov chain with the state space omega. Y of t is a stochastic process, and X of n is nothing but the Y of t n, where t n is the transition time instance. And, the Markov properties is satisfied only at all the time, all the transition time instance, therefore, X n form a discrete time Markov chain. Since, X n is Y of t n, this X n stochastic process is called a embedded Markov chain. X n stochastic process is embedded in the stochastic process Y of t, therefore, X n is a time homogeneous embedded Markov chain, because it satisfies the time invariant property as well as Markov property, therefore, X n is the time homogeneous embedded discrete time Markov chain.

Now, we can say X of n, t of n constitutes a Markov renewal process or semi Markov process with the state space omega. X n is a embedded Markov chain, where X n is Y of t n; and the t n is nothing but the transition time instance. Therefore, the together X n, Y n constitutes Markov renewal process, because these are all the time points in which the system is moving into the different states, and the Markov properties satisfied only at those time points, therefore, it is called the semi-Markov process or Markov renewal process, with the state space omega.

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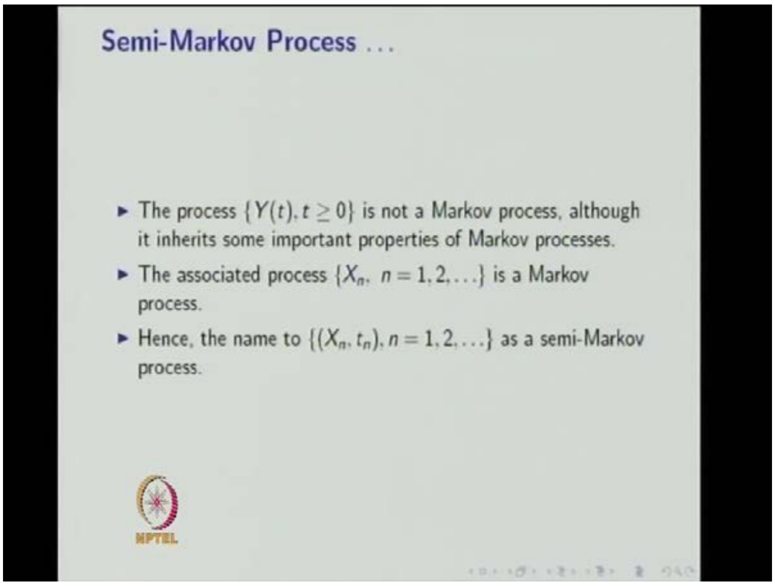

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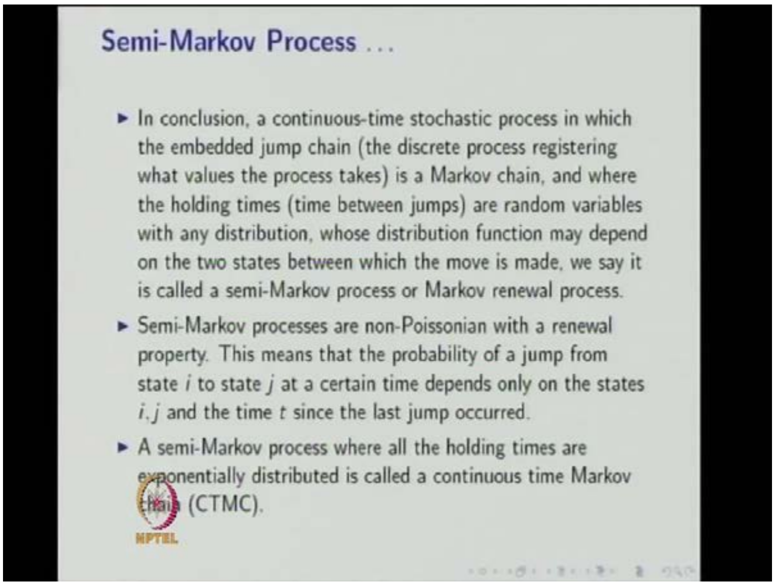

The stochastic process, the stochastic process Y of t is not a Markov process, although it inherits some important properties of Markov processes. The associated process that is X of n is a Markov process. Hence, the name (X n, t n) as a semi-Markov process.

In conclusion, a continuous time stochastic process in which the embedded jump chain that is nothing but the discrete process registered what values the process takes. So, the embedded jump chain, jump chain is a Markov chain. In conclusion, a continuous time stochastic process which embedded has a Markov chain, and where the holding times are random variables with any general distribution, whose distribution function may depend on the two states between which the move is made, we say it is called, we say it is semi-Markov process or Markov renewal process. Whenever these properties are satisfied, we say the stochastic process is a semi Markov process or Markov renewal process.

The semi Markov processes are non-Poissonian with the renewal property. This means that the probability of a jump from a state i to j at a certain time depends only on the states i, j and the time t since the last jump occurred. If you restrict the holding times or exponential distribution, and each time transition instance or the renewals then the special case of semi Markov process is a Poisson process. But in general, semi Markov processes are non-Poissonian. If you make assumptions of holding times are exponential distribution in the same parameter, and each time transitions are nothing but the renewals then the special case of semi Markov process is the Poisson process. A semi Markov process where all the holding times are exponential distribution is called a continuous time Markov chain. So, if I restrict only the holding times are exponential distribution, the each transition need not be the renewals, then a semi Markov process is a continuous time Markov chain.

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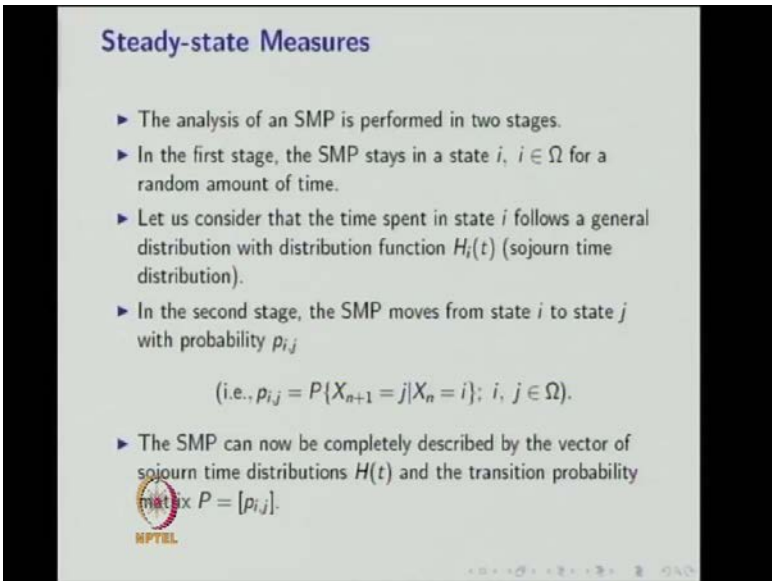

Now, we are going to discuss the steady state measures of semi Markov process. The analysis of a semi Markov process is performed in two stages. In the first stage, the semi Markov process stays in a state i for some random amount of time. For example, in the sample path of semi Markov process, in this model we have a 4 state semi Markov process, so, in the each state, the system stays the random amount of time. Let us consider that the time spent in state i follows a general distribution with the distribution function H i of t; that means, in this example, H 1 of t that is the times spent in this system spent in the state 1, H 2 of t is the distribution of the system staying in the state 2, and so on.

In the second stage, the SMP moves from the state i to j with the probability p i, j, where p i, j is defined, what is the probability that the system was in the state i at the n th time instance. The system will be in the state j at the n plus 1 th time instant, i comma j belonging to omega. So, that is a conditional probability. The condition probability of P X n plus 1 is equal to j, given X n is equal to i, for i, j belonging to omega. So, the SMP can now be completely described by the vector of sojourn time distributions H of t and the transition probability matrix P that is p of i, j. So, the transition probability matrix is the transition probability, therefore, all the row sums are equal to be 1, and the values are lies between 0 to 1. And, you have to supply the sojourn time distribution for each state. So, if these two information are given, then we are known with semi Markov process.

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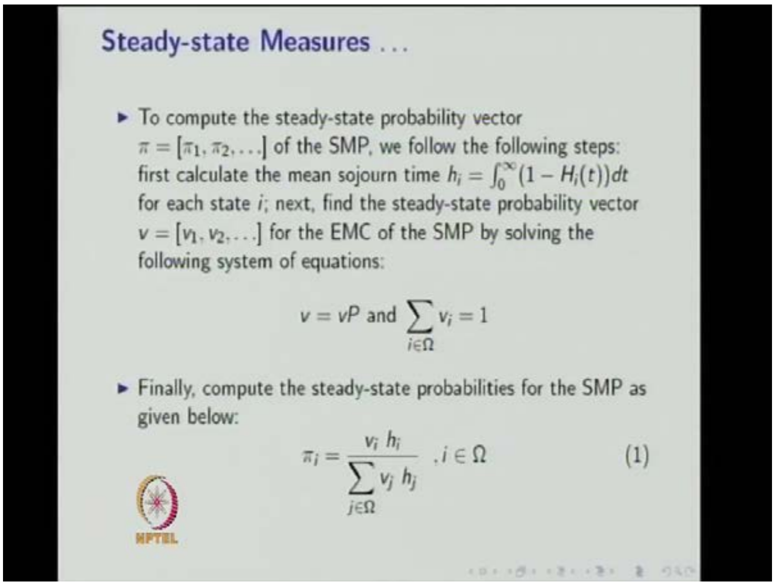

To compute the steady state probability vector, let us assume that pi is the vector, pi 1, pi 2 is the elements of a semi Markov process. First calculate the mean sojourn time that is nothing but the small h i. Since, h i of t is the distribution function, so, 1 minus H i of t the integration between 0 to infinity, will be the mean sojourn time, because each, each random variable is a nonnegative random variable, therefore, the mean will be 0 to infinity 1 minus the c d f integration, for each state i.

Next find the steady-state probability vector v i's, for the embedded Markov chain of the semi Markov process. First we have to find out the steady state probability vector for the embedded Markov chain of the semi-Markov process, and using the steady state probability vector and the mean sojourn time you can get the steady state probability vector of a, steady state probability vector pi.

So, how to find the steady state probability vector of embedded Markov chain? So, you know, P is a transition probability matrix. So, solve v is equal v P, and summation of v i is equal to 1, we will get v i's. The v is equal to v P is the homogeneous equation, and including summation of v i is equal to 1 you will have a nonhomogeneous system of equation. So, you can get the nontrivial solutions, satisfying these two conditions.

So, once you know the v i's, you can compute the steady state probabilities of the semi Markov process, that is nothing but v i's multiplied by h i's divided by summation of all the v j's, h j's, where j is belonging to omega. So, the v i's are nothing but the steady state probability vector of embedded Markov chain, and h i's are nothing but the mean sojourn time, and pi i's you will get, by using this formula.

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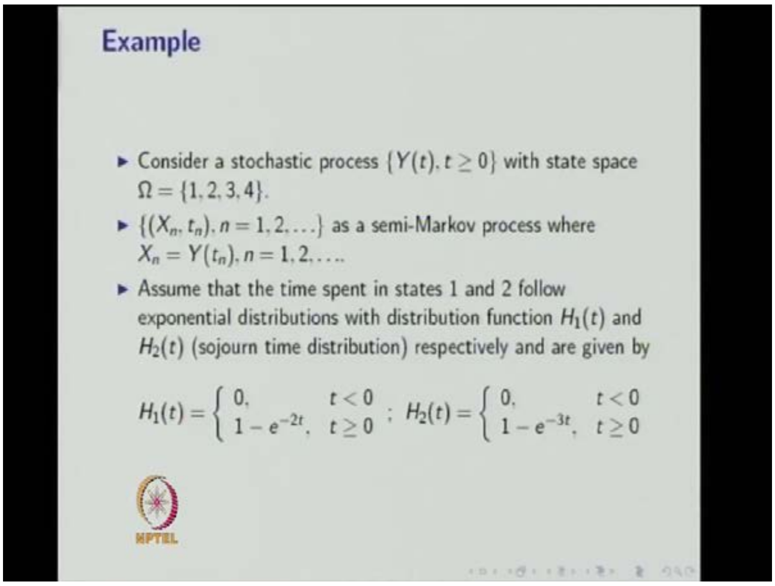

Now, let us consider the simple example, the stochastic process with the state space omega 1, 2, 3, 4. In the previous, previous steady state measures with the assumption that the steady state probability vector, probability, probabilities exists, we are giving the, how to compute the steady state probability measures. So, here the assumption is steady state probabilities are exist. Now, you come to the example. So, this is the 4 state stochastic process, with the states 1, 2, 3, 4; and h i's are nothing but the c d f of sojourn time in each state; and, X n is nothing but Y of t n; and, (X n, t n) will form a Markov renewal process or semi Markov process. Assume that the time spent in the states 1 and 2 follow a exponential distributions with the c d f H 1 and H 2.

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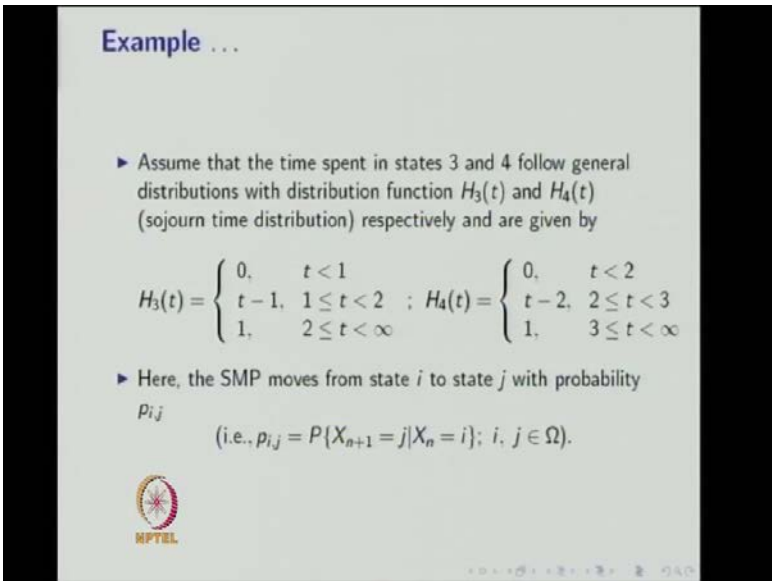

Whereas, the time spent in the states 3 and 4 follow uniform distribution with the c d f H 3 of t and H 4 of t. So, the sojourn time in the state 3 is uniformly distributed between the intervals 1 and 2. And, the sojourn time spent in the state 4 is also uniform distribution between the intervals 2 and 3. Therefore, the c d f's are in this form H 3 of t, and H 4 of t. The semi Markov process moves from the state i to j with the probability p i of j.

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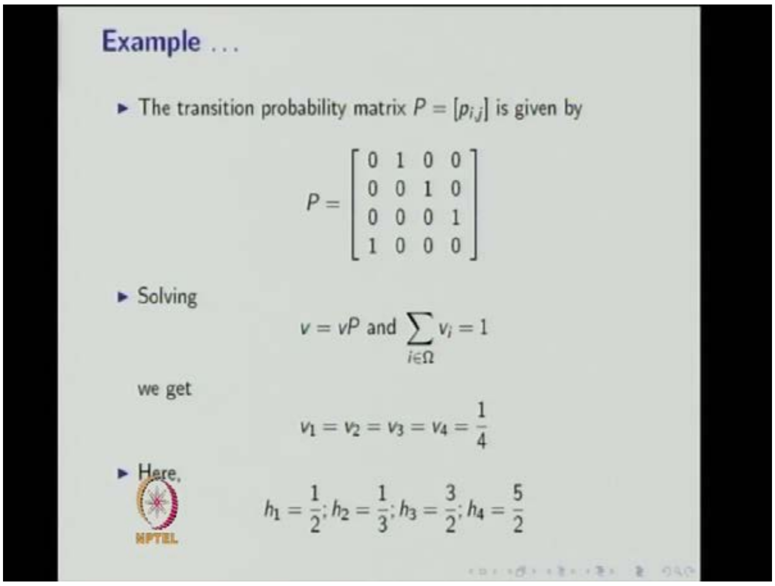

That is nothing but the transition probability matrix can be in the form, the states are 1, 2, 3, 4. Therefore, in the one step, transition probability of the system is moving from 1, 2, the state 2 is assure, therefore, that probability is 1, and all other probabilities are 0's. Similarly, the system move from the state 2 to 3, that probability will be 1, and all other states moving probabilities are 0. Therefore all other transition probabilities from the state 2 to 1, 2 to 4, are 0; whereas, 2 to 3 will be, 2 to 3 will be 1. Similarly, 3 to 4 will be 1, and all other states, all other transition probabilities are 0, then the system moving from the state 4 to 1 is 1, and all other states are 0.

So, you know the transition probability matrix as well as you know the means, you know the sojourn time distribution. So, using that transition probability matrix, by solving you can get the steady state probability vector of embedded Markov chain. If you see the transition probability matrix, since it is a transition probability matrix, the values are lies between 0 to 1 and the rows sum is 1. But, in this particular transition probability matrix, it satisfies the one more additional condition, the columns sum is also 1. Therefore, if you solve the v is equal to v P and the summation of v i is equal to 1, the solution will be 1 divided by the number of states. So, this steady state probabilities are uniformly distributed, uniform distribution.

So, number of states of 4, therefore, the steady state probability of the embedded Markov chain is 1 by 4. So, you know the steady state probabilities of embedded Markov chain, and from the sojourn time distribution we can find out the mean sojourn time. Since, the first two random variables sojourn, first two states sojourn times are exponential distribution, therefore, the mean sojourn time is a 1 by 2, 1 by 3 respectively, for the states 1 and 2. And, the sojourn time in the state 3 is uniform distribution between the interval 1 to 2, therefore, the mean sojourn time is 3 by 2. And, for the state 4, the sojourn time distribution is uniform distribution between the intervals 2 to 3. Therefore, the mean sojourn time is pi by 2.

So, using transition probability, sorry, using the steady state probability, steady state probabilities of embedded Markov chain and the mean sojourn time, using the steady state probabilities of embedded Markov chain and the mean sojourn time, times you can get the steady state probabilities of semi Markov process.

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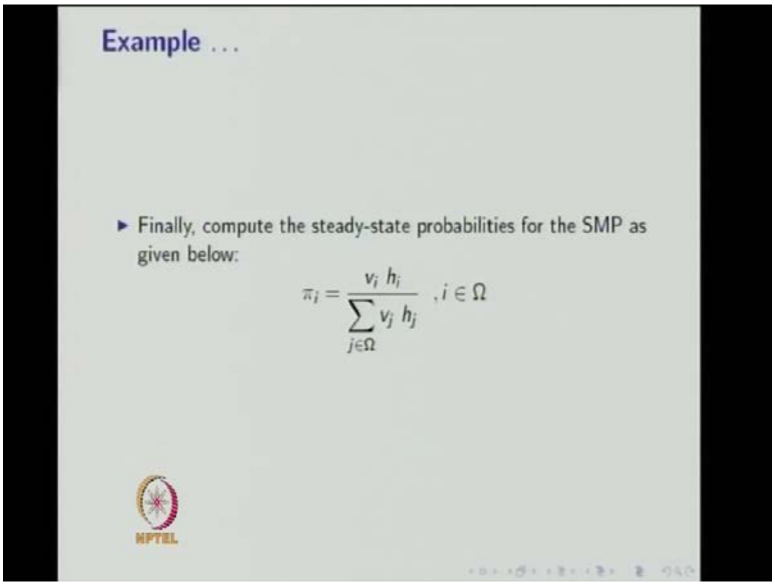

Finally, compute the steady state probabilities are semi Markov process using v i's and h i's; v i's are 1 by 4, and h i's are h is 1 by 2, h 2 is 1 by 3, h 3 is 3 by 2, h 4 is 5 by 2. And, substitute the values in this equation to get the steady state probabilities of semi Markov process.

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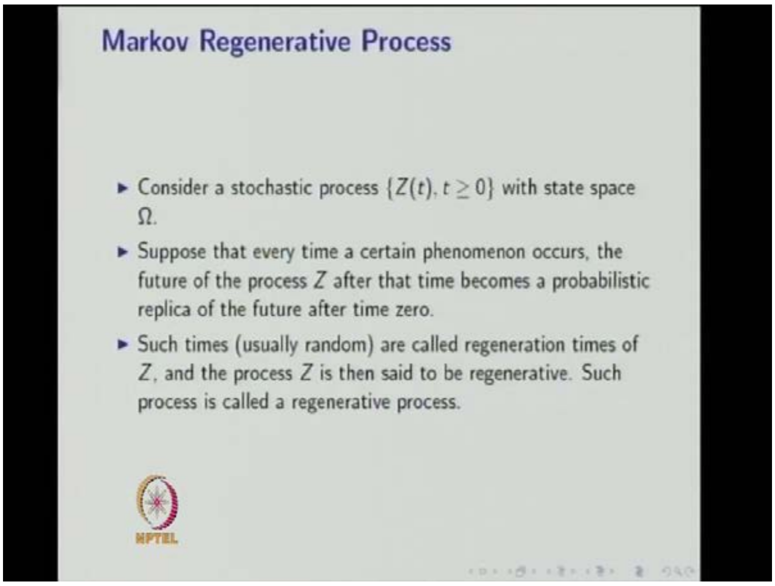

Now, we are moving into the second part of lecture three, that is Markov regenerative process. Consider a stochastic process Z of t with the state space omega. Suppose that the every time a certain phenomenon occurs, the future of the process Z; that means, Z of t after that time becomes a probabilistic replica of the future after time 0. Such times usually a random, such times, usually random, are called regeneration times of the stochastic process Z of t, and the process Z of t is then said to be regenerative. Such a process is called a regenerative process.

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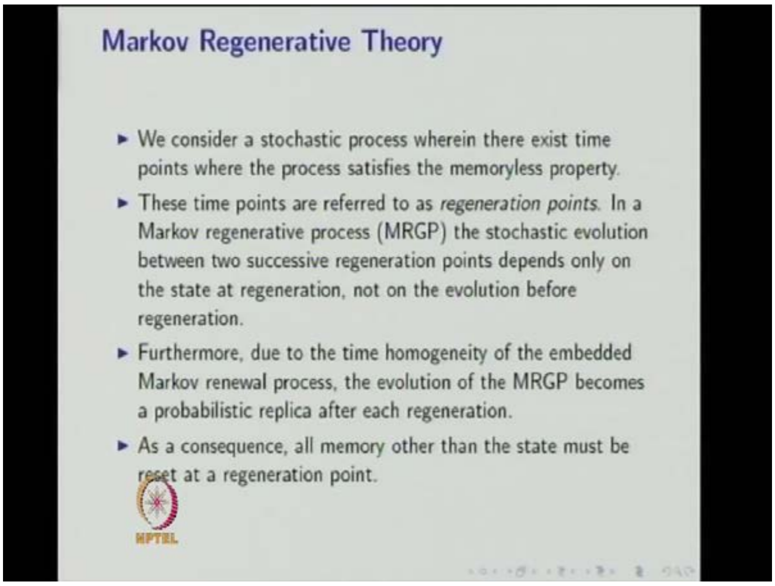

We consider a stochastic process wherein there exists a time points where the process satisfies the memoryless property. These time points are referred to as a regeneration points. In a Markov regenerative process, the stochastic evaluation between two successive regeneration points depends only on the state at regeneration, not on the evolution before regeneration. Furthermore, due to the time homogeneity of the embedded Markov renewal process, the evaluation of the Markov regenerative process becomes a probabilistic replica after each regeneration, because of time homogeneity, time invariant.

As a consequence, all memory other than the state must be reset at the regeneration point. As a consequence, all memory other than the state; that means, the future depends on the state, but not the process; a d property of a inter arrival time is making arrival instance independent and hence, memory less, and hence, memory less has the next arrival does not depend on how long the previous arrival took also, since between any two arrivals, same pure death process operates, hence, the arrival points become regeneration time points, must be reset at a regeneration point.

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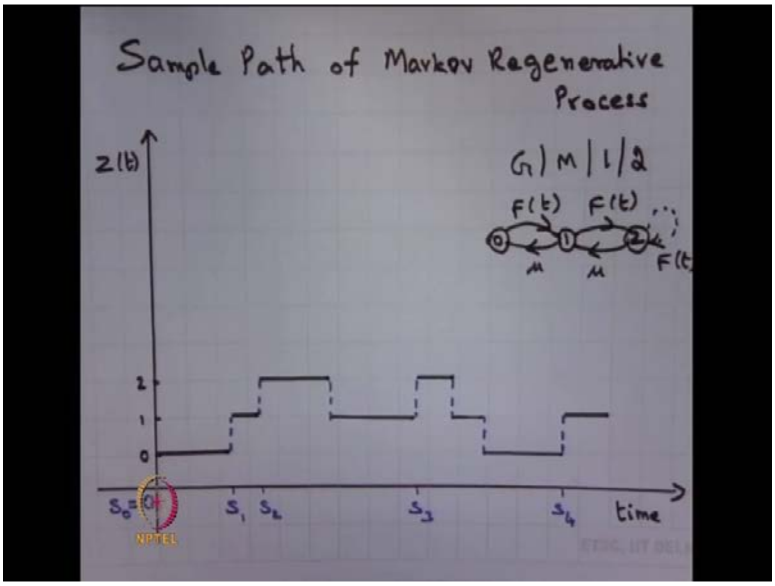

Now, we are going to consider the sample path of Markov regenerative process, through that I am going to explain the regeneration points. Consider sample path of Markov regenerative process. In this stochastic process, the omega is 0, 1 and 2. Consider the simple example of G M 1 2 queuing 1; that means, the arrival is non-Poissonian, non Poisson process; that means, inter arrival distributions are not a exponential distribution; whereas, the service times are exponential distribution. Only one server and the capacity of the system is true. Therefore, the state transition diagram is like this.

This is a, c d f of the system spending in the state 0, before moving into the state 1. The system spending in the state 1, before moving into the state 2, the c d f will be F of 2, F of t. Because, the capacity of the system is 2, there is the possibility, the arrival can come, but the system will be in the state 2 again, therefore, I made self loop with the dotted arcs. Whereas, if the service is completed, if the assumption of service is exponential distribution with the parameter mu then the system can move from the state 2 to 1; 1 to 0, also the rate will be mu .

So, now, you see the sample path. At time 0, the system is in the state 0. The inter arrival time is any distribution, need not be a exponential distribution. At time s 1, at time instant s 1, the first arrival enter into the system, therefore, the system size is now 1, so, system move to the state 1. Now, there are two possibilities, either the service would have been completed before the next arrival, or the arrival occurs, the next arrival occurs first and before that service completion of the first arrival, first server, first customer whose under service. So, suppose you assume that second arrival occurs first, before the completion of the, service completion of the first customer, therefore, the system moved to the state 2, at the time point s 2.

Now, consider a scenario, the first customer who is under service is service completed first, before the third arrival. Therefore, the system size now it will be 1; that means, the earlier the system was in the state 2, since the service is completed, the customers, the system size becomes 1, for sometime. At this duration, the third customer enter into the system, at the time point s 3. Now, the system size is 2, again; that means, initially the systems, system state was 0; from state 0, the system moved to the state 1 because of F of t. Then one more arrival, therefore, the system size 2, first customers service completion, system size is, system state is 1, then the third arrival, now the system state is 3 sorry 2. Then the service completion, therefore, it goes to the 1; one more service completion, therefore, the system goes to the state 0, then the arrival s 4.

Note the time point s 1, s 2, s 3, s 4; these are all the time points in which the arrival occurs that is nothing but the arrival epochs; whereas a, this is the time point in which the service completes, this is a time point in which the service completes, this is the time point in which the service completes, we are not noting down the service completion time points; whereas, we are note down the arrival time epochs. Because, at the time of service completion epochs, still we should know how much elapsed time of the next arrival; because, inter-arrivals are now it is any distribution, not a exponential distribution. If it is exponential distribution then the memory less properties are satisfied, therefore, the residual or the elapsed or the remaining inter-arrival time is also exponential, if inter arrival times are exponential distributions.

But here, therefore, at the time of service completion, we should remember the elapsed or remaining inter-arrival time; therefore, at the service completion time epochs the the memory less property would be satisfied. Whereas, at the time instance s 1, s 2, s 3, s 4, and so on, these are all the time points in which the arrival occurs, at those time points the memory less property satisfied; even though the system is moving into the different states in between the arrival time epochs, for instant between the time epochs s 2 to s 3, the system is moved from the state 2 to 1, at the time 2; at the time point s 2 the system was in the state 2; whereas, at the time point s 3 also the system is in this state 2, but in between the system was in the state 1 for some time.

Similarly, in between the time instance s 3 and s 4, the system was in the different states in between the, these two time epochs; at, in between the, even though the transition occurs at those time points, the memory less property was not satisfied; whereas, at the arrival time epochs, the memoryless properties are satisfied. Therefore, these time points are called, go back to the definition, the stochastic process where there exists time points where the process satisfies the memory less property, the time points are referred as a regeneration time points. So, here, the s 1, s 2, s 3, s 4, are the regeneration time points. In a Markov regenerative process the stochastic evaluation between two successive regeneration time points, that means between s 2 to s 3, or s 3 to s 4, it depends only on the state at the transition and at regeneration, not on the evaluation before regeneration; that means, you should remember where the system was at the time point s 2 as well as where the system was in the state at the time point s 3, and you do not want before the regeneration time.

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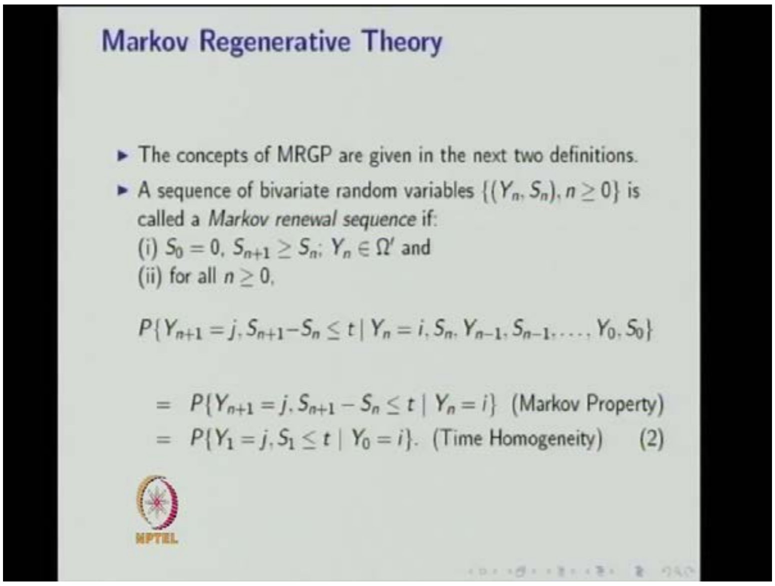

As a consequence, all memory other than the state must be reset at the regeneration point, therefore, this stochastic process is called a Markov regenerative process; and, I have made a sample path for a Markov regenerative process with this example, because we are going to consider the same example later. So, here s 1, s 2, s 3, are the regeneration time points, not the time instance at which the service completion.

The concepts of MRGP for given in next two definitions. The first definition: a sequence of bivariate random variables (Y n, S n) is called a Markov renewal sequence or Markov renewal process; S naught is equal to 0, in this example also you made it S naught is equal to 0, S n plus 1 is greater than or equal to S n; and, Y n is belonging to omega dash, where omega is the state space, the omega dash is the subset of omega. For all n greater than or equal to 0, the Y n, the conditional distribution Y n has to be satisfy this property.

The first line, the probability of Y n plus 1 is equal to j, with the difference of time instance is less than or equal to t, given that the system was in the state, some state at Y naught at the time instance S naught, till the system was in this state i at the time instant S n. This conditional distribution is same as the conditional distribution with the only the latest information, the probability of Y n plus 1 is equal to j, the difference of regeneration time points is less than or equal to t, given only Y n is equal to i; that means, the conditional distribution depends only the current information or latest information, not the complete history. So, that is nothing but the Markov property.

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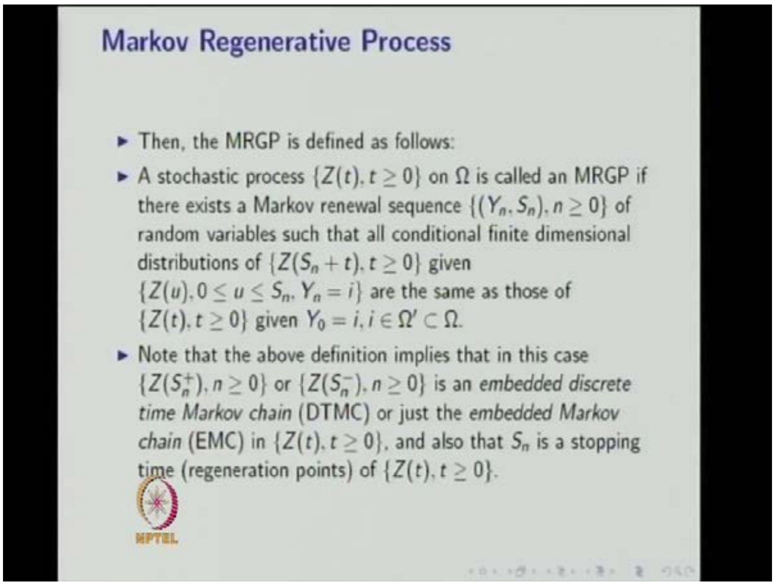

Next, that is same as the conditional distribution of a, instead of Y n to Y n plus 1, you can find out the distribution of Y naught to Y 1, because of, it is time invariant, because of it is a time homogeneity this conditional distribution is same as probability of Y n is equal to j, the first time, the first regeneration time point is less than or equal to t, given that Y naught is equal to i, so that means, the conditional distribution depends the current state, not the past history, including the time homogeneous property, then the, that is a way we define the bivariant random variables that is (Y n, S n) satisfies this property.

Then, the MRGP is defined as follows: a stochastic process Z of t with the states space omega is called a Markov regenerative process, if there exists a Markov renewal sequence for (Y n, S n), such that all conditional finite dimensional distribution of Z of S n plus t, given Z of u, where u is lies between 0 to S n, Y n is equal to i, are the same as those of Z of t given Y naught is equal to i. So, this is the probabilistic replica. The stochastic process Z of t is said to be a Markov regenerative process, if all conditional finite dimensional distribution of Z of, Z S n plus t, given all the past history till S n including Y n is equal i, that is same as the distribution of Z of t, given Y naught is equal to i, that means it includes the time homogeneity as well as the Markov property.

Note that, the above definition implies that Z of S n plus n, plus or Z of S n minus is a embedded discrete time Markov chain, or just embedded Markov chain in Z of t; also, S n is the stopping time or regeneration points, stopping time is nothing but the Markov property is satisfied at the those time points for the given stochastic process. So, in this example before the arrival occurs Z of S n minus Z embedded discrete time Markov chain, just before the arrival occurs will be a embedded Markov chain in these example.

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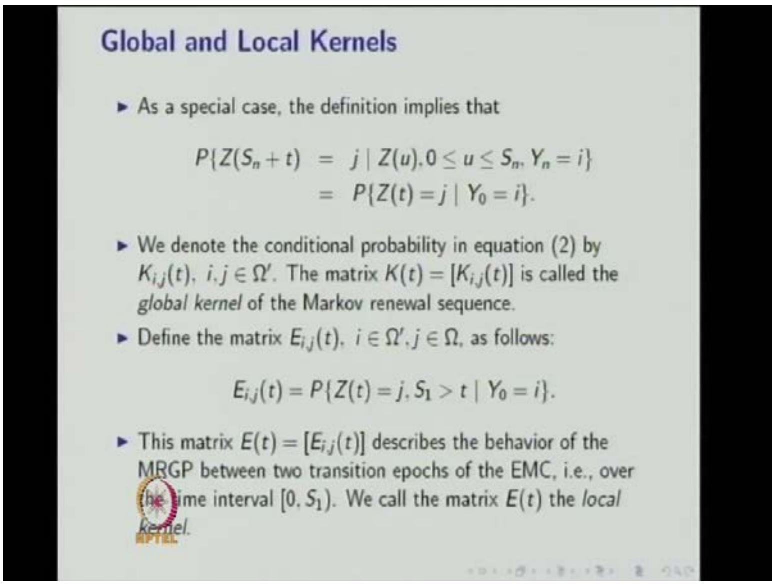

The way we discuss that the semi Markov process with the transition probability matrix and the sojourn time distribution, here we have to explain the global kernel and local kernel. So, that we are going to discuss now. We denote the condition probability in the equation number 2 by K i of j of t, equation number 2 is nothing but the conditional distribution of Y n plus 1 is equal to j, with the difference of timing, time, regeneration time are less than or equal to t, that is same as because of Markov property and the time homogeneous property, this is the probability of Y n is equal to j, S 1 is less than or equal to t, given Y naught is equal to i.

A Markov renewal sequence is also defined in the bivariant as this, and usually this form of definition is frequently used since renewal time and the state of the time, and the state of the system at renewal instant, both are important. So, this condition, this conditional probability becomes the transition probability that is, this conditional probability will form a matrix K of t, and that is called a global kernel of the Markov renewal sequence. For the Markov renewal sequence we can find the global kernel; and, the global kernel is the matrix K of t that consists of K of i, j of t, where each K i, j of t is nothing but probability that, P of Y 1 is equal to j, with S 1 is less than or equal to t, given Y naught is equal to i.

Now, we are going to discuss the local kernel. That is also a matrix that consists of E of i, j of t, where are where i is belonging to omega dash, and j is belonging to omega. Omega dash means the collection of states at which the time transitions of, this system satisfies the Markov property at those time instance, and those collections of states forms a omega dash. And, that is a subset of omega. So, E of i, j of t is nothing but, what is the probability that, the system will be in the state j, with the first regeneration time point is going to be greater than t; that means, the system will be in the state j after the time t, the first regeneration going to occur after time t. The system will be in the state j at that time t, given it was in the state i, at the previous regeneration time point, or at S naught the system was in the state i. So, this will form a glow; this will form a local kernel. So, using global kernel and the local kernel, one can find the steady state and the transition behavior of Markov regenerative process.

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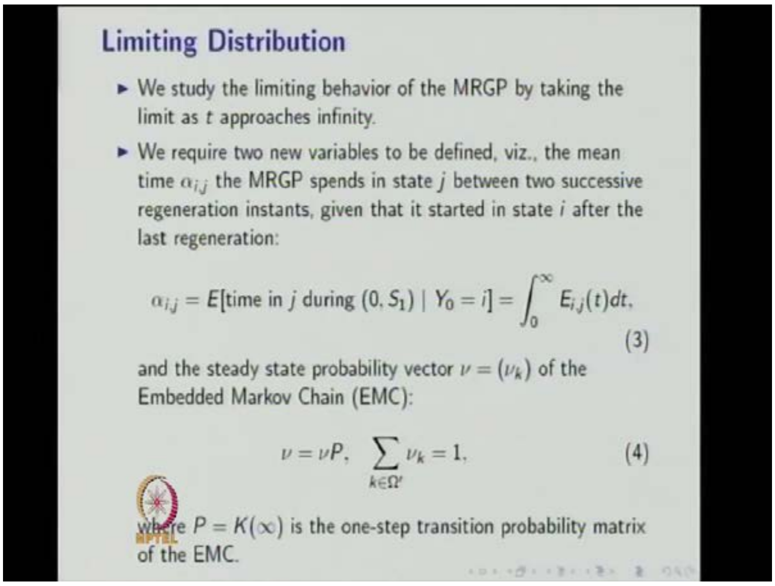

Now, we are going to discuss the limiting distribution or steady state measures. We study the limiting behavior of the MRGP by taking limit as t approaches infinity. We require two new variables to be defined, namely, the mean time alpha i, j of the MRGP spends in the state j between two successive regeneration instants, time instants, given that it started in the state i after the last regeneration.

So, this is nothing but the average spending time in the state j, given that it was in the state i at the last regeneration. So, alpha of i comma j is nothing but the expected, expectation of time in state j, during the interval 0 to S 1, where S 1 is the first regeneration time instant, given that the system was in this state i at the previous or last regeneration time, and the steady state probability vector v of the embedded Markov chain.

That means v is equal to v P, and the summation of v K' s are is equal to 1, where k is belonging to omega dash, and P is the one step transition probability matrix of embedded Markov chain. So, from the global kernel K that is K of t, if you make a t tends to infinity, you will get the one-step transition probability matrix P. So, from using P, you can get the steady state probabilities v, by solving v is equal to v p and the summation of v K is equal to 1. Once you solve the, this, using the alpha i j you can get the limiting distributions.

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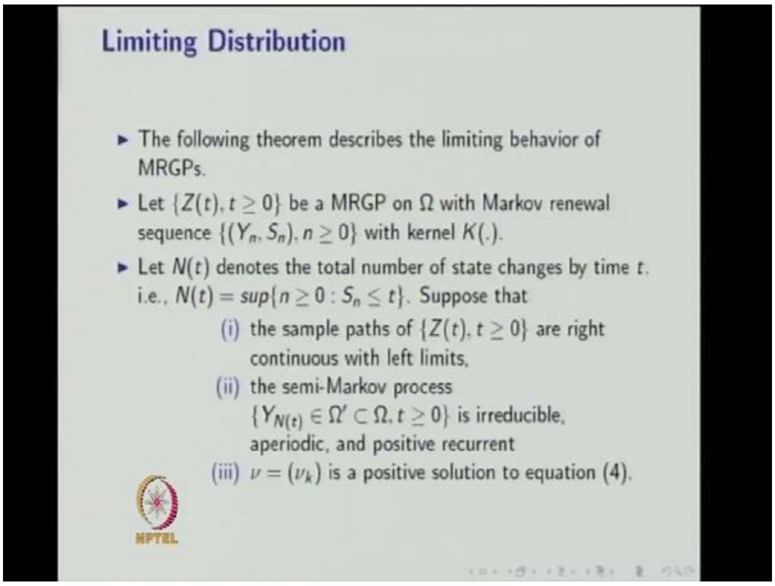

So, the limiting distribution is given in the following theorem. Let Z of t be the MRGP, with the Markov renewal sequence (Y n, S n). Let N t denotes the total number of states changes by time t, then the sample path of Z tare the right continuous with the left limits; and N of t is the semi Markov process, the Y of N of t is a semi Markov process, which is irreducible, aperiodic, and positive recurrent. And, v is a positive solution to the equation 4, that is this one, summation of v i is equal to 1 and v is equal to v P, if this properties are satisfied then the steady state probability vector pi whose elements are pi j's that is nothing but the limit the t tends to infinity probability of Z of t is equal to j, using this formula, where P K's are nothing but the summation of alpha K's.

So, as long as these three properties are satisfied; that means, the sample paths has to be right continuous, and the semi Markov process has to be irreducible, aperiodic and the positive recurrent, and you need a positive solution, this steady state probability vector, then you can get the steady state probability for the Markov regenerative process.