Stochastic Processes
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Module - 8
Renewal Processes
Lecture - 2
Generalized Renewal Processes and Renewal Limit Theorems

This is a stochastic processes, module eight renewal processes, lecture two generalized renewal processes and renewal limit theorems. In the lecture one, we have discussed the, a renewal processes definition and its properties, followed by renewal process definition we have discussed the renewal function, and then we have discussed the renewal equation, and also we have seen a few examples in the lecture one.

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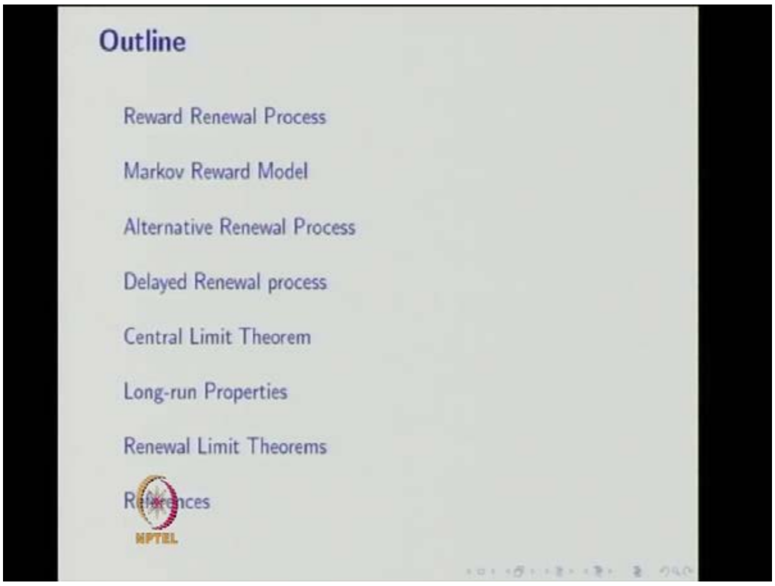

In the lecture two, we are planning to discuss the reward renewal process. As a special case we are going to discuss the mark reward models, then we are going to discuss two different renewal processes that is alternative renewal process and delayed renewal process, with this we are completing the generalized renewal processes. The second half we are going to discuss the central limit theorem, long run properties, and three important renewal limit theorems.

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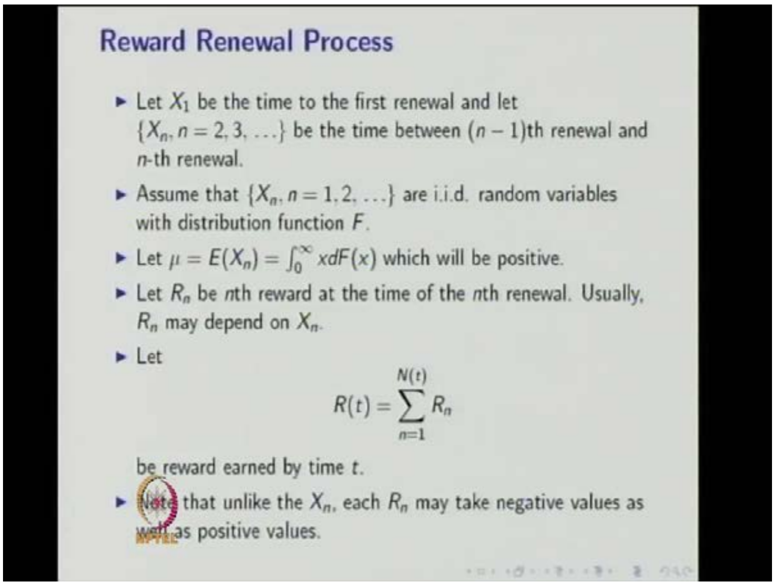

What is reward renewal process? Let x 1 be the time to the first renewal, and let x n be the time between n minus 1 th renewal and nth renewal, the same definition which we have used for the renewal process. And also you assume that x i's are i i d random variables with the c d f capital F of x. Since x i's are the inter arrival times, the mean will be a positive, the mean exist and it will be positive. Let R n be the nth reward at time t of the nth renewal. For each renewal you are attaching the reward at the time of the renewal. Usually R n may depend on X n.

Now, we are defining a new random variable R of t, function of t that is nothing but the summation of R n's, where n is running from 1 to n of t, where n of t is a renewal process. The collection of R of t for t greater than or equal to 0, will be called it as a reward renewal process, that R of t is nothing but reward earned by time t. So, from the renewal process we are attaching the reward, or each renewal, and by defining R of t is equal to summation of n is equal to 1 to n of t R n that will be called it as a reward renewal process. Note that, unlike the x n each R n may take negative values as well as the positive values. The x i's are nothing but the inter arrival time of the renewals; whereas, the rewards may take a negative values as well as positive values.

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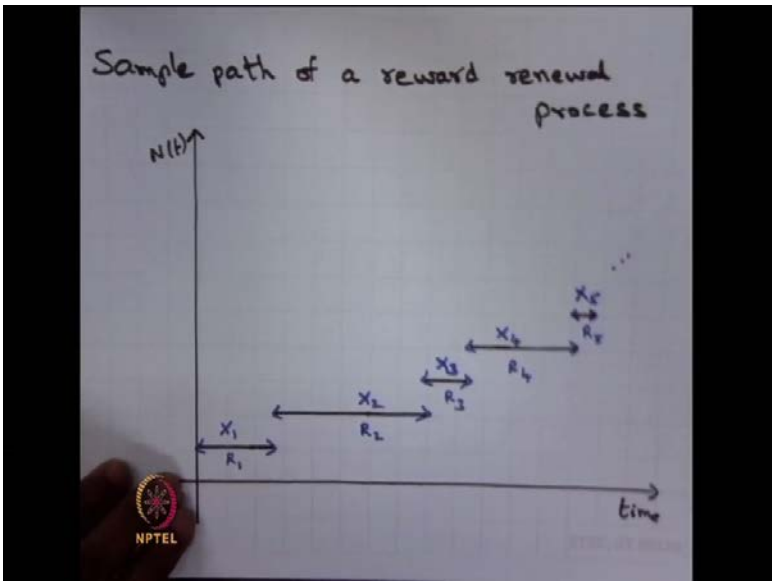

See the sample path of reward renewal process. So, the x axis is the time, the y axis is n of t, and these are all the time points in which the renewals takes place, the first renewal takes place, second renewal, and so on. So, this is the inter arrival time, and attaching the reward R i to the each x i. And this R i's may be negative or positive, then the collection of R i's with this form R t is equal to summation n is equal to 1 to N of t R n will be the reward renewal process. So, it is very difficult to show the sample path of R of t. So, here we are showing the sample path of renewal process with the rewards attached with the each x i's.

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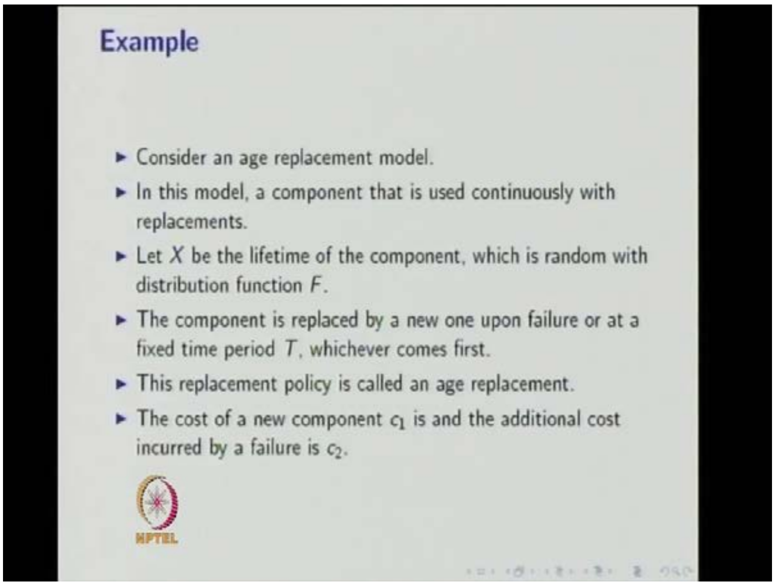

Now, we are moving into the simple example of reward renewal process. Consider an age replacement model. In this model a component that is used continuously with replacements, replacements. Let X be the life time of the component, which is random with distribution function F. The component is replaced by a new one upon failure, or at a fixed time period capital T, whichever comes first. The replacement policy is called a age replacement, because the component is replaced by a new one upon failure or at a fixed time period t. The cost of a new component is c 1 and the additional cost incurred by a failure is c 2.

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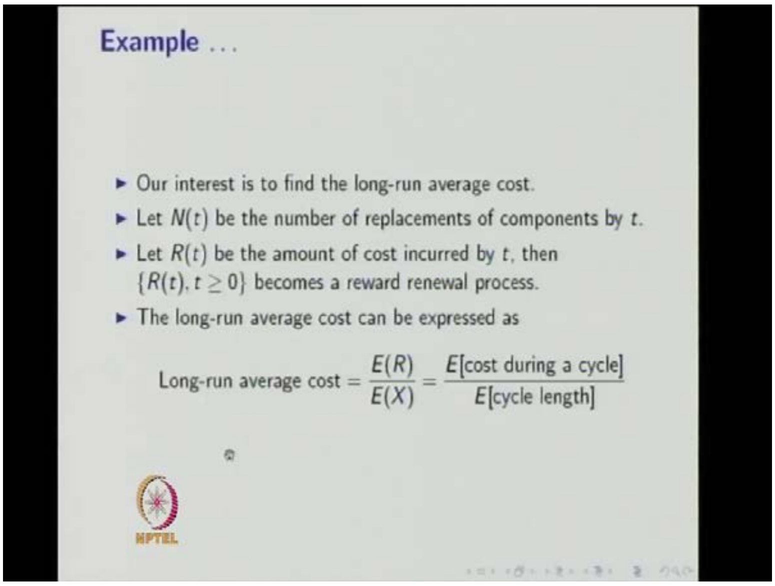

Our interest is to find the long- run average cost. Let N t be the number of replacements of components by time, N of t is a renewal process. Let R of t be the amount of cost incurred by time t. So, this is the reward renewal process. So, the long-run average cost can be expressed as the ratio of the expectation of R by expectation of X that we are going to conclude later. Now, we are using the long run average cost is expectation of R divided by expectation of X, expectation of R is nothing but expectation of cost during a cycle, and expectation of X is nothing but expectation of cycle length.

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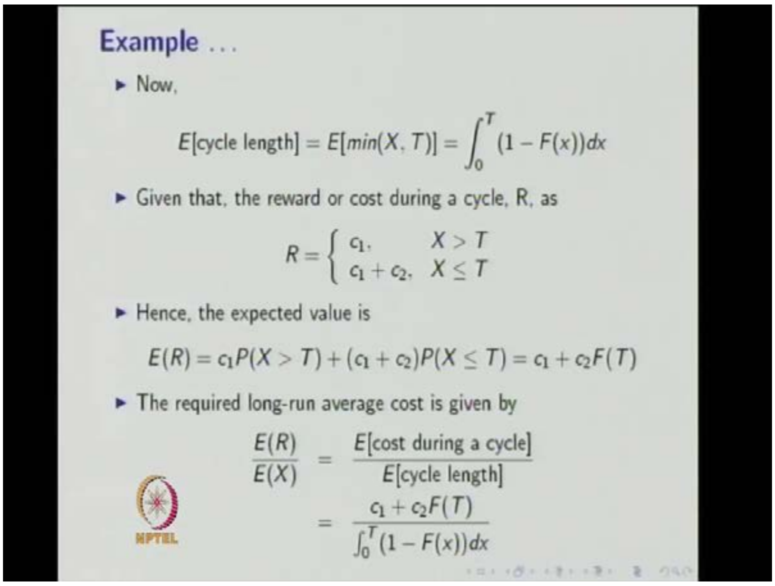

Now, one can find the expectation of cycle length that is nothing but the expectation of the cycle length is a random variable, which is nothing but the minimum of X or T. Here, the component is replaces, either upon a failure, or at age capital T. Time between replacement is called a cycle that is nothing but the integration 0 to T, 1 minus F of x d x. Given that, the reward or cost during the cycle R, therefore R will be c 1, if X is greater than T if it fails after T; if it fails before T then there is additional cost c 2, therefore, the cost will be R c 1 plus c 2, if failure is occurs before the fixed time T.

Hence, the expected value that is a expected cost is either c 1 with the probability X is greater than T, or c 2 plus, c 1 plus c 2 with the probability X is less than or equal to T; that is nothing but c 1 plus c 2 times F of T. Hence, the long run average cost is nothing but expectation of R divided by expectation of X, that is nothing but expectation of cost during a cycle divided by expectation cycle length, substitute the values we will get the long run average cost.

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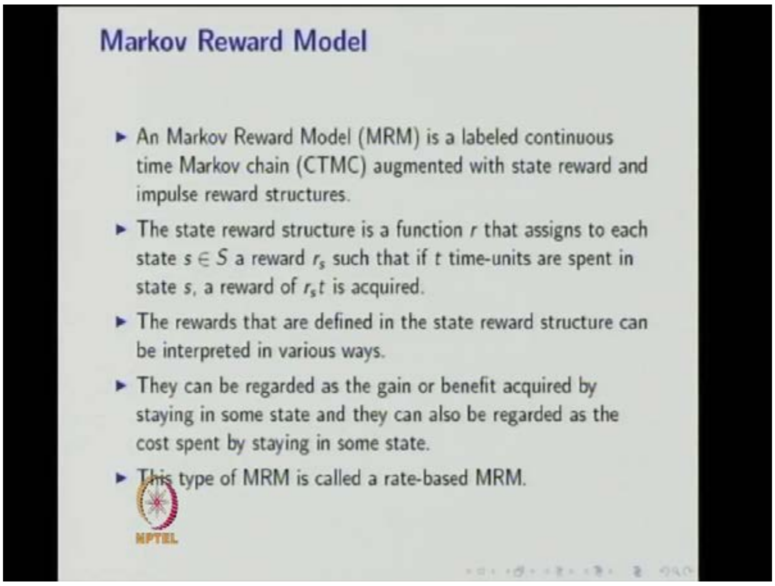

Now, we are moving into the special case of reward renewal process that is a Markov reward model. A Markov reward model is a labeled continuous time Markov chain augmented with state reward and impulse reward structures. The state reward structure is a function r that assigns to each state, a reward r suffix s such that if t time units are spent in state s, a reward of r s times t is acquired. The rewards that are defined in the state reward structure can be interpreted in various ways.

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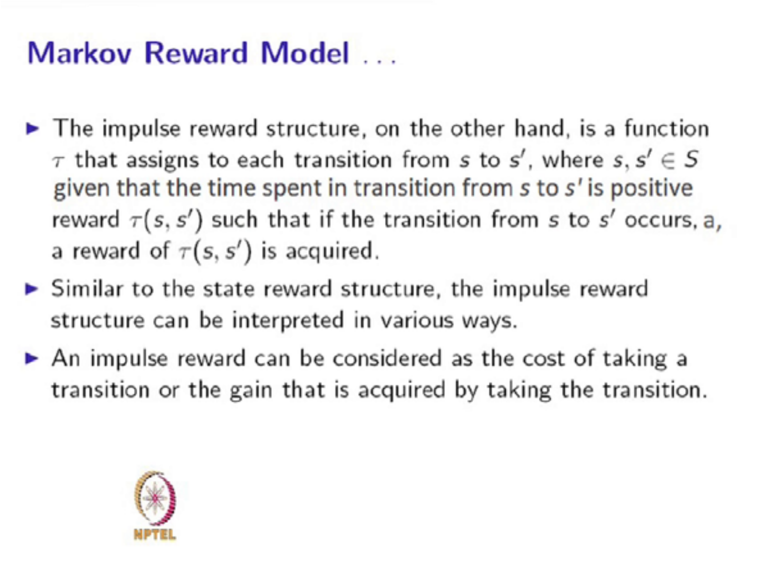

They can be regarded as the gain or benefit acquired by staying in some state and they can also be regarded as the cost spent by staying in some state. And, this type of Markov reward model is called rate based Markov reward model. The impulse reward structure, on the other hand, is a function of, function tau that assigns to each transition from s to s dash, where s and s dash are belonging to the state space s, and spending time in transition of s to s dash is positive, and the reward tau of s to s dash such that if the transition from s to s dash occurs, a reward of tau s of s comma s dash is acquired. Similar to the state reward structure, the impulse reward structure can be interpreted in various ways. An impulse reward can be considered as the cost of taking a transition or the gain that is acquired by taking the transition.

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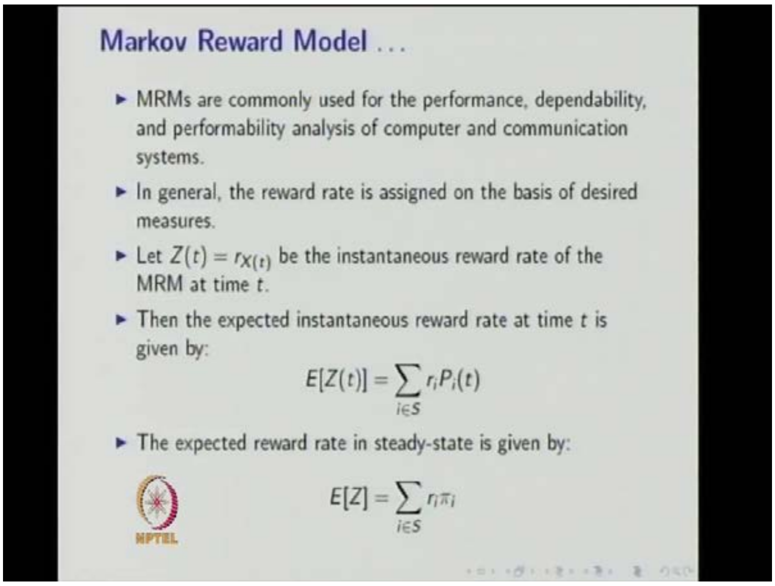

Markov reward models are commonly used for the performance, dependability, and perform ability analysis of computer and communication systems. In general, that reward rate is assigned on the basis of desired measures. Let Z of t is nothing but r suffix X t be the instantaneous reward rate of the Markov reward model at time t. Then the expected instantaneous reward rate at time t is given by expectation of Z of t that is nothing but summation of r i's P i of t. The expected reward rate in steady state is nothing but expectation of Z is summation of r i pi i. Suppose in the perform availability model our interest is to find out the availability of the system, then one can assign the rewards for the upstate's are 1 and the rewards for the downstate's are 0's then the expected reward rate will be the availability for the system by making a summation of r i’s and the probability of being in those states.

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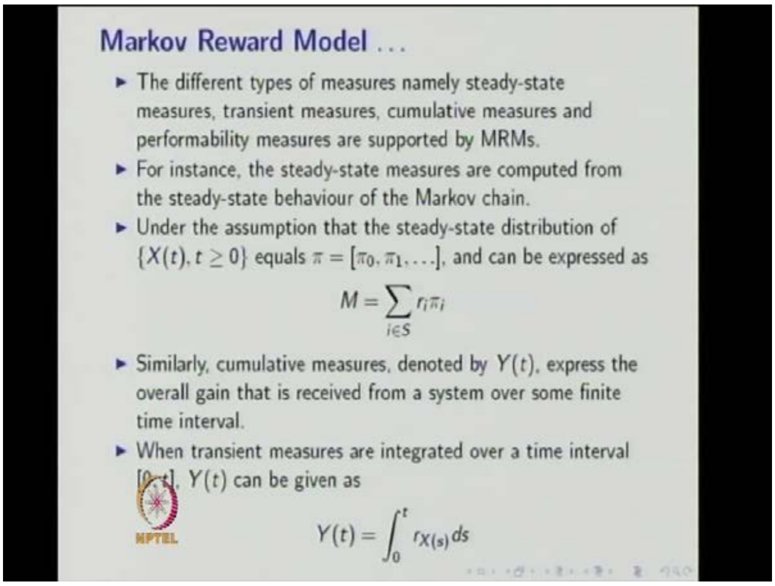

The different types of measures namely steady-state measures, transient measures, cumulative measures and a performability measures are supported by Markov reward models. For instance, the steady-state measures are computed from the steady-state behavior of the Markov chain. So, for the availability model if you know the steady-state behavior of the Markov chain such as a steady state probability of being in this system, then i assigning the rewards 1 to the upstate’s and 0 to the downstate’s, one can get the steady state availability of the system. Similarly reverting the rewards you can get the steady-state unavailability of the system also.

Under the assumption that the steady state distribution of X t one can find the steady-state measures by multiplying r i’s with the pi i’s, where i is, i belonging to the state space S. Similarly, the cumulative measures, denoted by Y of t, express the overall gain that is received from a system over some finite time interval. When transient measures are integrated over the time interval 0 to t, then Y of t can be given as integration 0 to t r suffix X s of d, d s.

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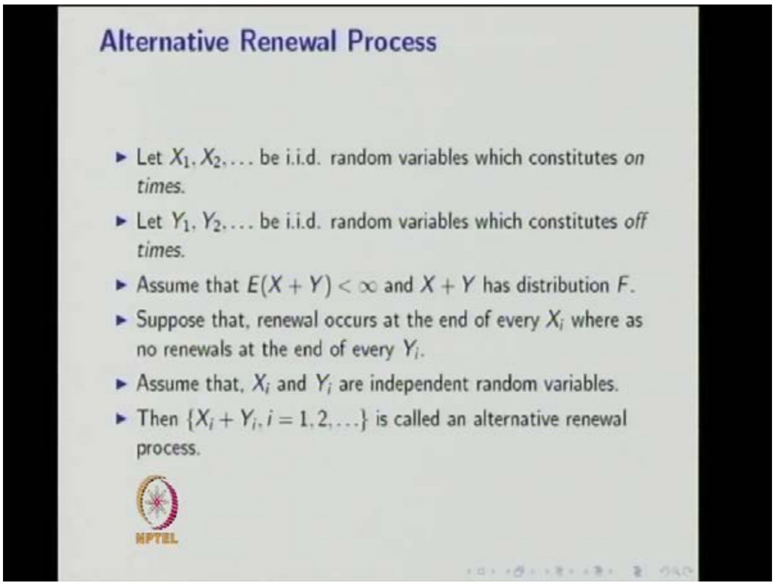

Now, we are moving into the one type of renewal process that is called alternative renewal process. Let x i's are i.i.d. random variable which constitutes on times, and Y i's are i.i.d. random variable which constitutes off times. Assume that mean is exist and it is finite, and X plus Y has the distribution capital F. Suppose that, the renewal occurs at the end of every X i's where no, whereas no renewals at the end of every Y i's. Assume that, X i's and Y i's are independent random variables also. Then the X i plus Y i are called the alternative renewal process.

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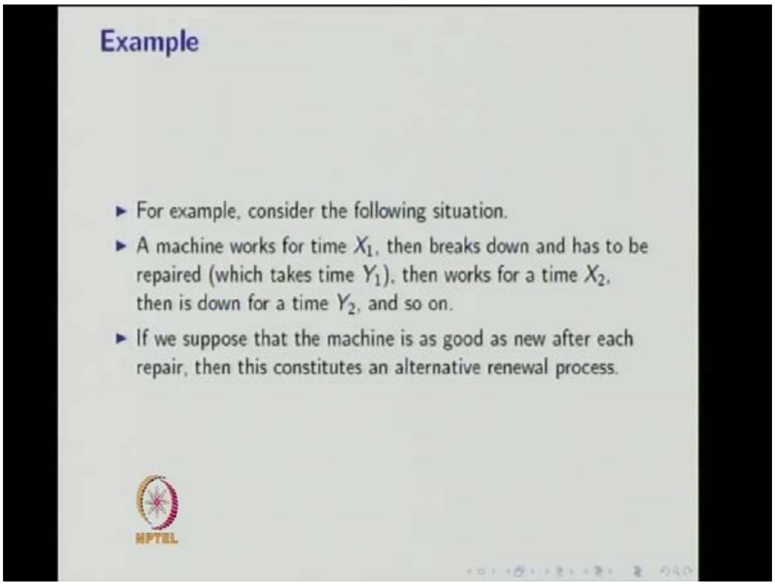

See the example for this situation. A machine works for the time X 1, and then breaks down and has to be repaired which takes the time Y 1, then works for the time X 2, then it is down for a time Y 2 that is a second repair time, and so on. So, X i's are nothing but the machine works, and Y i's are nothing but the repair time. If we suppose that the machine is good as new after each repair, then this constitute a alternative renewal process. So, this is the example of alternative renewal process with proper assumptions.

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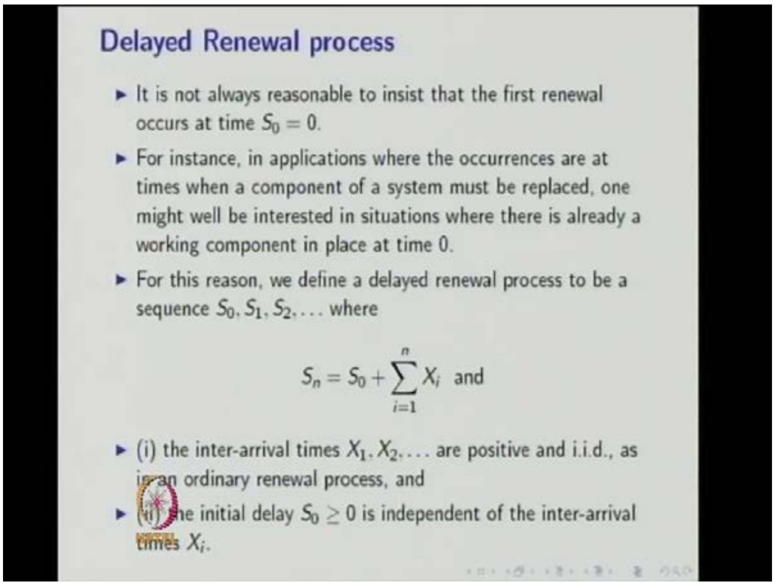

Now, we are moving into the next renewal processes, that is delayed renewal processes. It is not always reasonable to insist that the first renewal occurs at time S naught that is equal to 0, in the originate itself; it is a time 0 itself. For instant, in applications where the occurrences are not, are at times when a component of a system must be replaced, one might well be interested in situations where there is already a working component in place at time 0. For this reason, we define a delayed renewal process to be a sequence S naught, S 1, S 2 and so on, where S n is nothing but S naught plus summation of first n X i's, and the inter arrival times X i's are positive, and i.i.d. random variables as in the ordinary renewal process; and initial delay S naught which is greater than or equal to 0 that is independent of inter-arrival times X i's.

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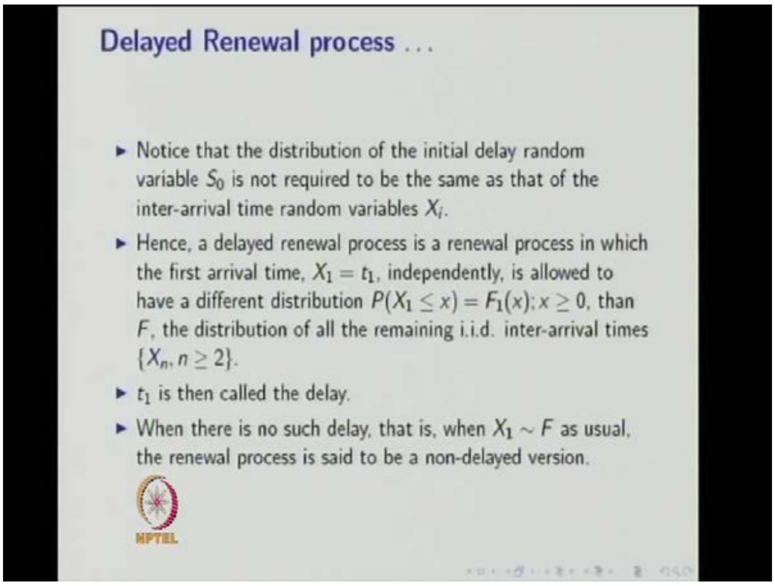

Notice that the distribution of initial delay random variables S naught is not required to be the same as that of the inter-arrival time random variables X i's. Hence, a delayed renewal process is a renewal process in which the first arrival time, X 1 is independently and is allowed to have a different distribution that is F 1, the distribution of all remaining i.i.d. random variables that distribution is capital F. So, F 1 is different form F. The X 1 is equal to t 1 that is nothing but the delay. When there is no such delay, then X 1 is also distributed in the same way, so the distribution of x 1 is capital F as usual, then the renewal process is said to be a non-delayed version.

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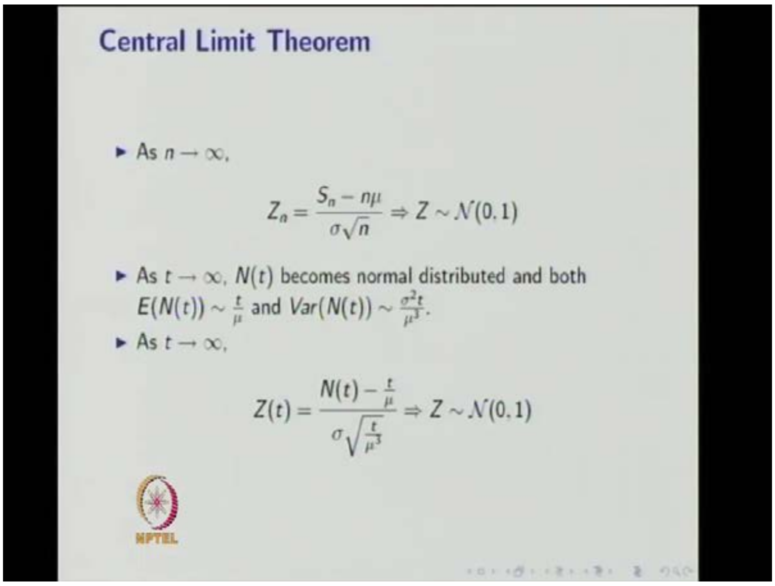

Now we are discussing the central limit theorem on the renewal process. As n tends to infinity, the random variable S suffix n that is nothing but the nth time renewal, minus n times mu, divided by sigma times square root of n, will be normal distribution with the mean 0 and variance 1. So, this convergence takes place in distribution. So, this is a weak distribution, weak convergence. So, as n tends to infinity, the random variable S n and the mean of S n is n times mu, and the variance of S n is sigma square n, and the random variable minus their mean divided by the standard deviation is normally, normal distributed with the mean 0 and variance 1, as n tends to infinity.

As n tends to infinity, the N of t that counting process, it is renewal process becomes a normal distribution with the mean t divided by mu and the variance sigma square t divided by mu cube, where mu is the mean of inter arrival time. As a t tends to infinity, the random variable N of t minus t by, t by mu divided by sigma times square root of t by mu cube will be normal distribution with the mean 0 and the variance 1. So, this is also the convergence in distribution.

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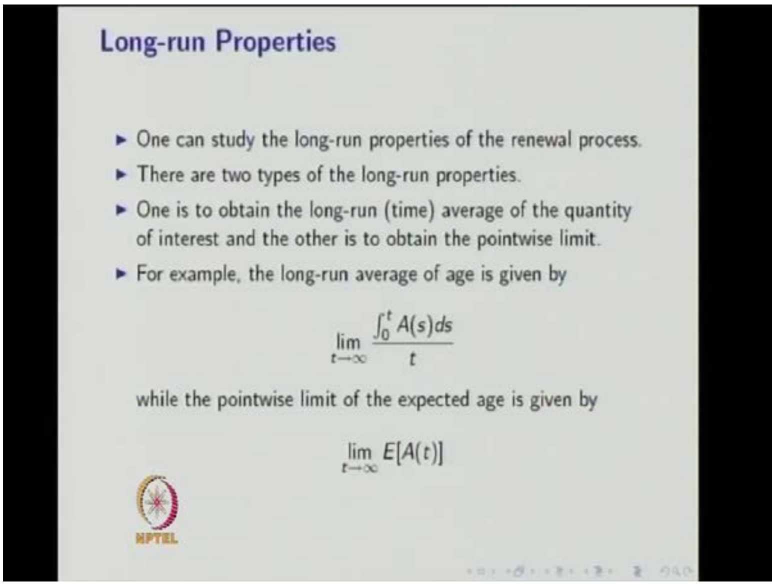

Now, we are going to discuss the long run properties of renewal process. There are two types of long run properties. One is to obtain the long-run average of the quantity of interest, the other one is to obtain the pointwise limit. For example, the long-run average of age that is limit t tends to infinity, the integration 0 to t of A of s, A of s is the age, divided by t, while the point wise limit of the expected age that is limit t tends to infinity expectation of A of t.

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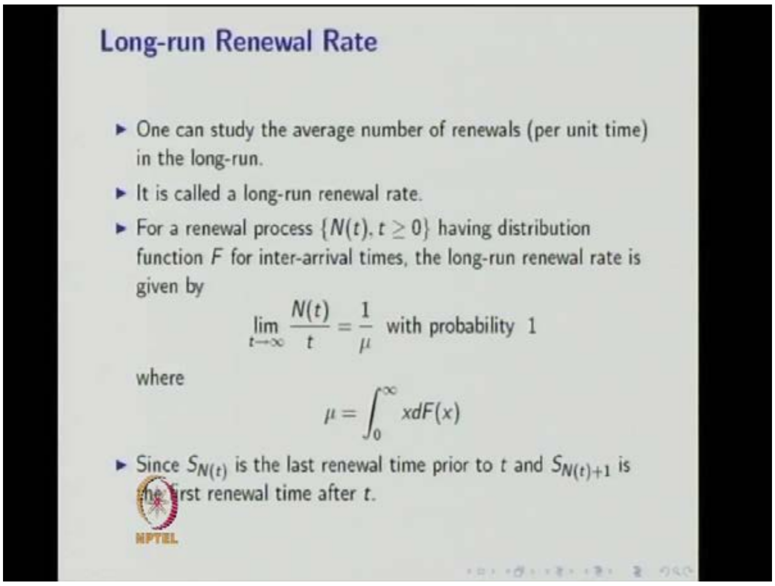

One can study the average number of renewals per unit time in the long-run. It is called the long-run renewal rate. For a renewal process having a distribution F for the inter-arrival times, the long run renewal rate is nothing but limit t tends to infinity N of t divided by t, that will be 1 divided by mu with the probability 1, where mu is nothing but the mean of inter arrival time.

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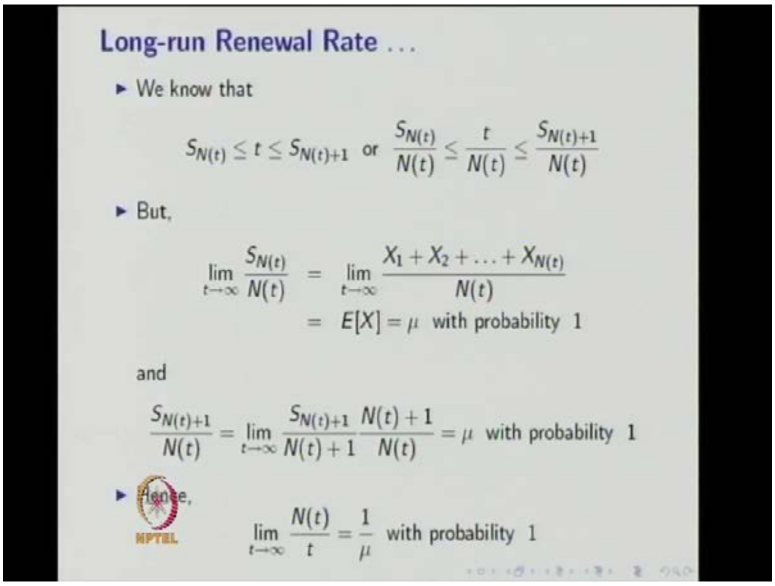

Since, S suffix N of t is a last renewal time prior to t and S suffix N of t plus 1 is the first renewal time after t. We know the relation of S of N of t with a S of N of t plus 1, and that lies, the t is lies between those two renewal times. You can divide by t, you can divide by N of t, therefore, S of N of t divided by N of t less than or equal to t divided by N of t, that is less than or equal to S of N plus t plus 1 divided by N of t. Now, we can evaluate the first one, S of N of t divided by N of t limit t tends to infinity, that is nothing but the numerator is nothing but the summation of X i's till N of t, denominator is N of t, that is nothing but expectation of X in a long-run, in the long run, the summation of X 1, X 2 till X of N of t divided by N of t that is nothing but expectation of X, that is same as the mu with probability 1.

Similarly, one can evaluate the last value that is S of N of t plus 1 divided by N of t, that is also will be mu with the probability 1. Since, t is lies between S of N of t and S of N of t plus 1 and the total divided by N of t in all 3, therefore as limit t tends to infinity N of t divided by t will be 1 divided by mu with probability 1.

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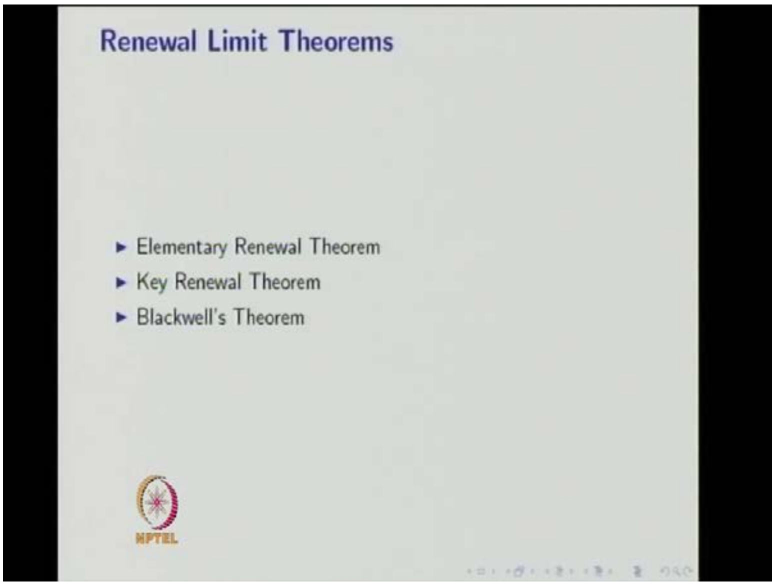

This is a last part of, in this lecture that is a renewal limit theorems. We are going to discuss only three important renewal theorems, without proof.

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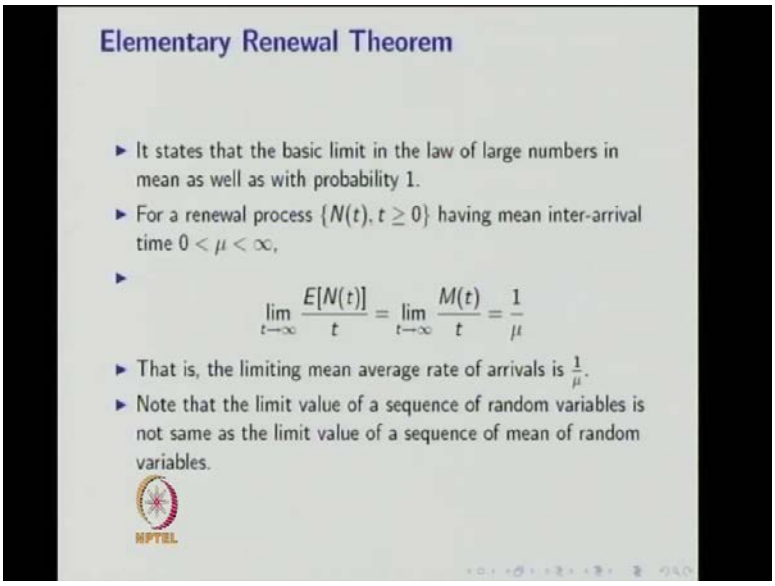

The first one is a elementary renewal theorem. It states that the basic limit in the law of large numbers in mean as well as with probability 1. For a renewal process having the mean inter-arrival time where mu is lies between 0 to infinity, the limit t tends to infinity the expectation of N of t divided by t that is nothing but the limit t tends to infinity of expectation of N of t is nothing but the renewal function capital M of t divided by t that is same as 1 divided by mu. Since, N of t goes to infinity almost surely and by the strong law of renewal process we get this result. This result shows that, for large t, the number of renewals per unit time converges to 1 divided by mu that is the limiting mean arrival rate of arrivals, mean average rate of arrivals is 1 by mu. That is, the limiting mean average rate of arrivals is 1 by mu.

Note that the limit value of a sequence of random variables is not same as the limit value of a sequence of mean of random variables. This is true in general also, here we discuss particularly for renewal process. The N of t is a random variable, expectation of N of t for fixed t is a constant. So, here, we are finding limit t tends to infinity, the expectation of N of t divided by t, that is 1 by mu.

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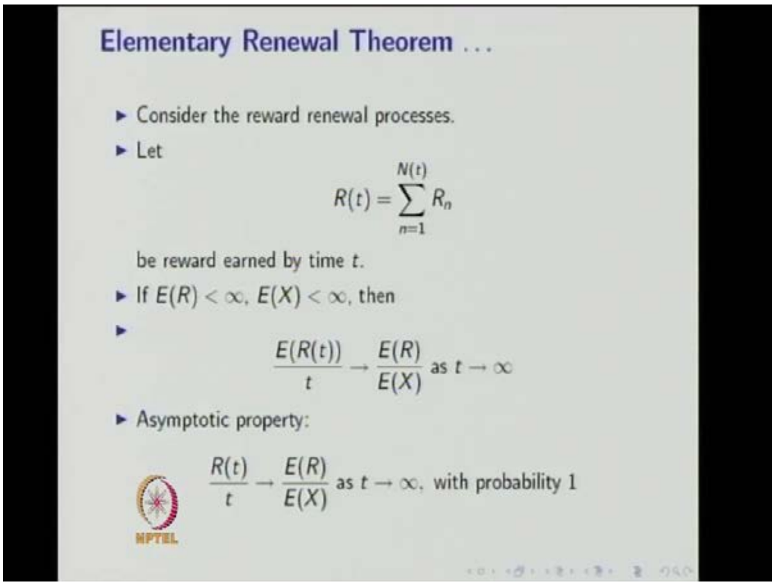

If you consider the reward renewal process R of t, when the expectation of R is finite as well as expectation of X is finite, limit t tends to infinity, the expectation of R of t divided by t that will tends to expectation of R divided by expectation of X, using the elementary renewal theorem. If you know the value of expectation of R, and if you know the value of expectation of X as t tends to infinity, expectation of R of t divided by t will be expectation of R divided by expectation of X.

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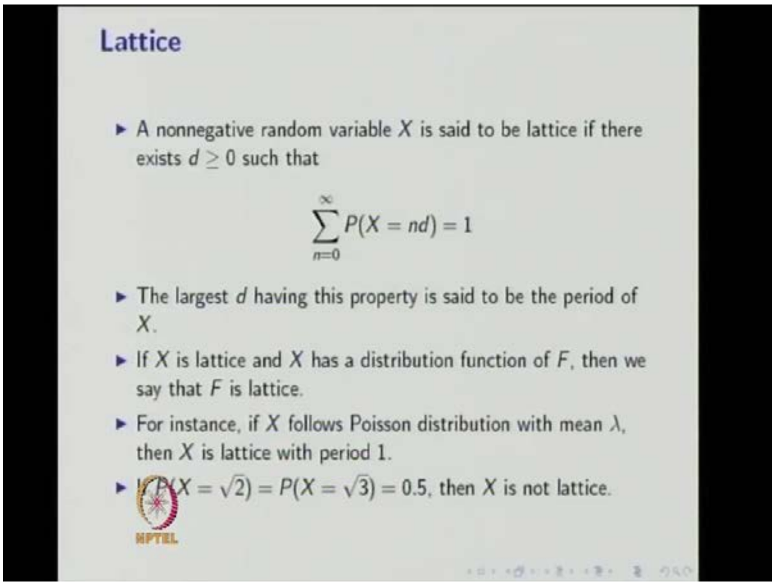

So, this result we have used it in one example. The asymptotic property of reward renewal process is as follows: as a t tends to infinity, the R of t divided by t will converge to expectation of R by expectation of X with a probability 1. This is called a asymptotic property of reward renewal process; both will be used in applications.

Before moving into the next renewal limit theorem, we need the definition of a lattice. A non negative random variable X is said to be a lattice, if there exists d greater than or equal to 0 such that the probability of X is equal to n times d for n is various from 0 to infinity, that will be 1. If this condition is satisfied then we say the d is called the period, and the corresponding random variable is called a lattice. The largest d having this property is said to be a period of X. If X is lattice and X has a distribution F, then we say the F is lattice.

For instance, X follows Poisson distribution with mean lambda, then X is lattice with period 1. Because the summation n is equal to 0 to infinity, the probability of X is equal to n, is equal to 1; that means, d will be 1. Whereas, in the second example, if P of, probability of X equal to square root of 2 is equal to 0.5, the probability of X equal to square root of 3 is 0.5, then you cannot find d such that summation of n is equal to 0 to infinity, the probability of X equal to n times d is equal to 1. This condition is not satisfied by the second example; therefore, x is not lattice.

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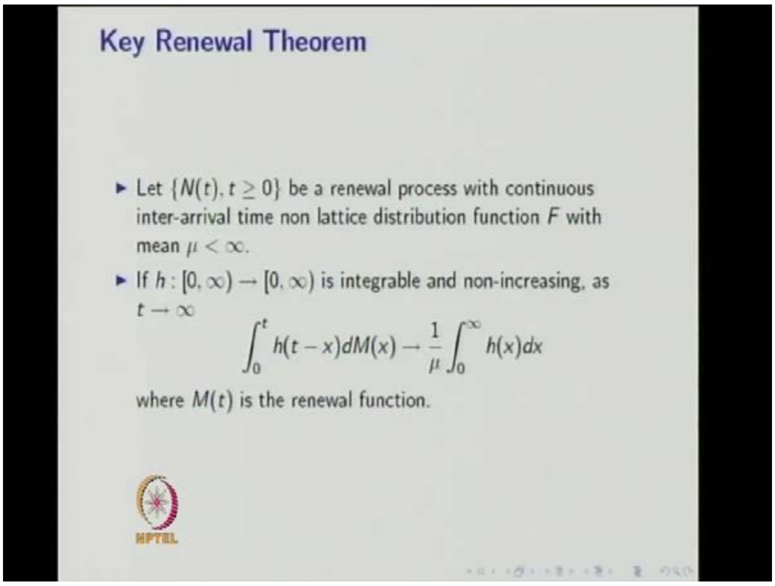

So, using lattice we are going to discuss the second renewal limit theorem that is key renewal theorem. Let x t be a renewal process with continuous inter-arrival time non lattice distribution function F with the mean mu. So, this is the renewal process, and has a inter-arrival time, which is, the distribution is non lattice and it has the finite mean. If h is a integrable and non increasing function, as t tends to infinity, the integration of 0 to t h of t minus x integration with respect to the renewal function capital M of t will tends to 1 divided by mu the integration 0 to infinity h of x d x.

So, this type of integral comes in to the, comes in the integral equation which we discuss in the renewal equation. So, as t tends to infinity, whenever the integrant is integrable and non-increasing function, and integration is with respect to the renewal function, and the corresponding renewal process has a non lattice distribution for the inter arrival time, then as a t tends to infinity this integral will be 1 divided by mu times integral 0 to infinity h of x d x, whenever h is a integrable and non decreasing function.

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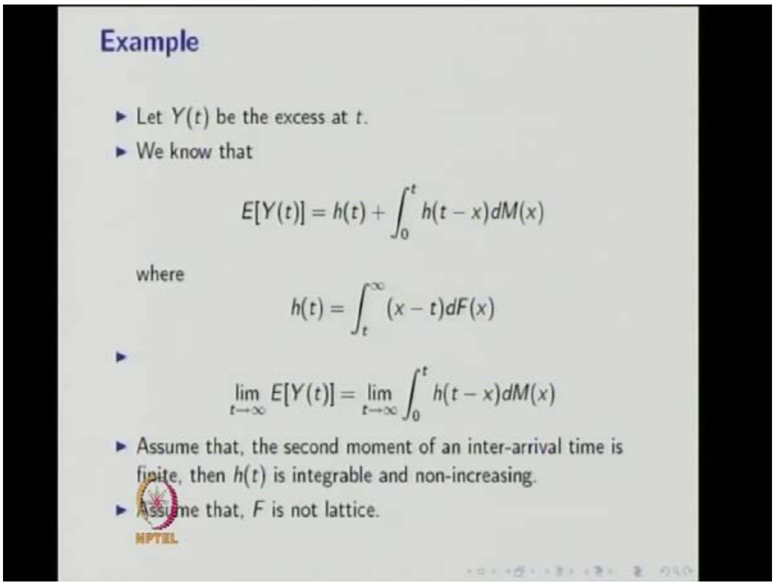

We are going to consider one simple example for that. Let Y t be the excess at time t. And, we know that the expectation of excess is nothing but h of t plus integration 0 to t h of t minus x integration with respect to the renewal function, where h of t is nothing but t to infinity x minus t integration with respect to the distribution function F. Therefore, limit t tends to infinity in the expectation of excess, will be limit t tends to infinity of the integration 0 to t h of t minus x d M x, since the first value will be 0 as t tends to infinity.

Assume that, the second moment of the inter-arrival time is finite, then h of t is a integrable and non-increasing function. Therefore, you can use a key renewal theorem by making additional assumption, F is non lattice distribution.

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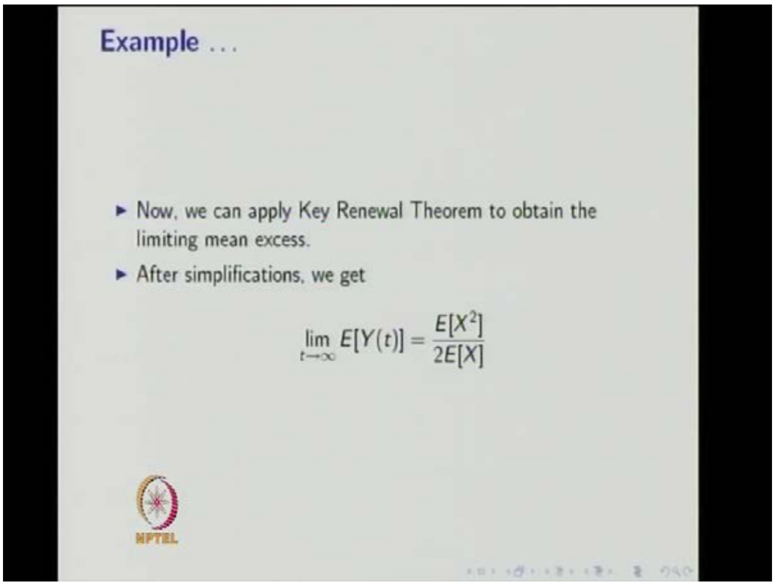

Now, we can apply key renewal theorem to obtain the limiting mean excess. After simplification, one can get a limit t tends to infinity expectation of Y of t is same as expectation of X square divided by 2 times expectation of X. So, to get this value we made a assumption, the second moment of inter-arrival time is finite, hence h of t is integrable and non increasing function. And also, we made a assumption F is a non lattice distribution. Hence, we are able to apply the key renewal theorem, and getting the limiting mean excess as a expectation of X square divided by 2 times expectation X.

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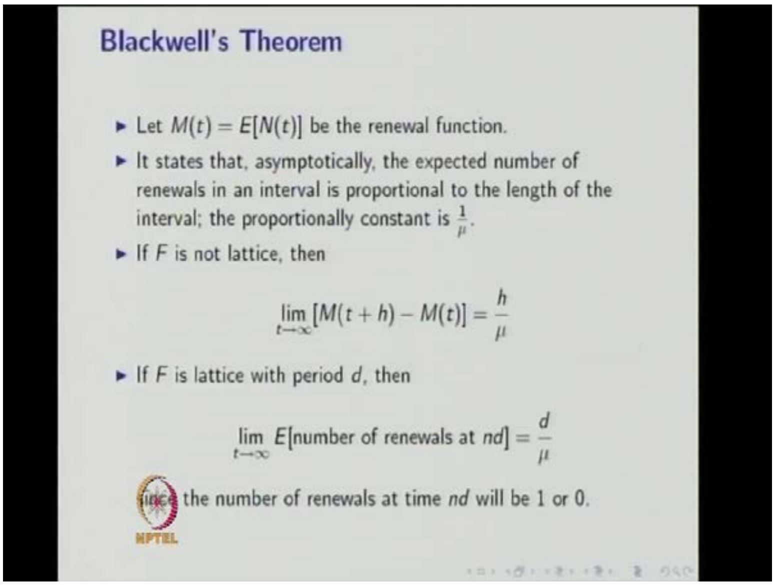

The last renewal limit theorem is a Blackwell's theorem. Let M t be the renewal function. And the Blackwell theorem says, asymptotically, the expected number of renewals in an interval is proportional to the length of the interval; that is the proportionality constant is 1 divided by mu. If F is not lattice, then limit t tends to infinity, the renewal function evaluated at t plus h minus renewal function at t, same as the interval length, multiplied by 1 by mu, 1 by mu is a proportionality constant, the length is h, because we are evaluating the renewal function between the interval M t to t plus h.

If F is a lattice distribution with a period d, then the limit t tends to infinity, the expected number of renewals at n d will be d divided by mu, because the probability of number of renewals at n d will be d by mu, and the number of renewals will be either 1 or 0; therefore, the limit t tends to infinity expected number of renewals at n d is d by mu. Here is the reference for lecture two in module 8.