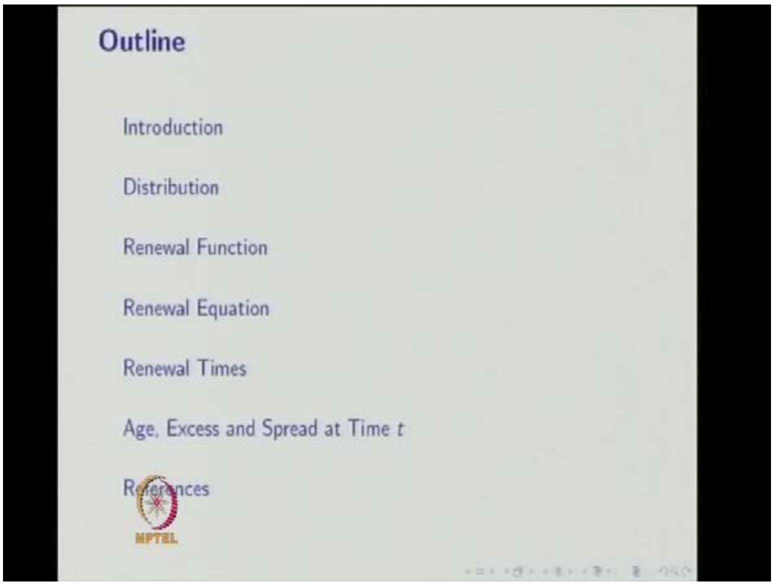
Stochastic Processes
Prof. Dr. S. Dharmaraja
Department of Mathematics
Indian Institute of Technology, Delhi

Module - 8
Renewal Processes
Lecture - 1
Renewal Function and Renewal Equation

This is stochastic processes module 8 renewal processes. In the module 1 we have presented the probability review and introduction to stochastic processes. In the module 2, we have presented the definition of stochastic process and its properties. Stationery processes we discussed in module 3, discrete time Markov chain is discussed in module 4, continuous time Markov chain is discussed in module 5, martingales are presented in module 6, Brownian motion and its properties are discussed in module 7.

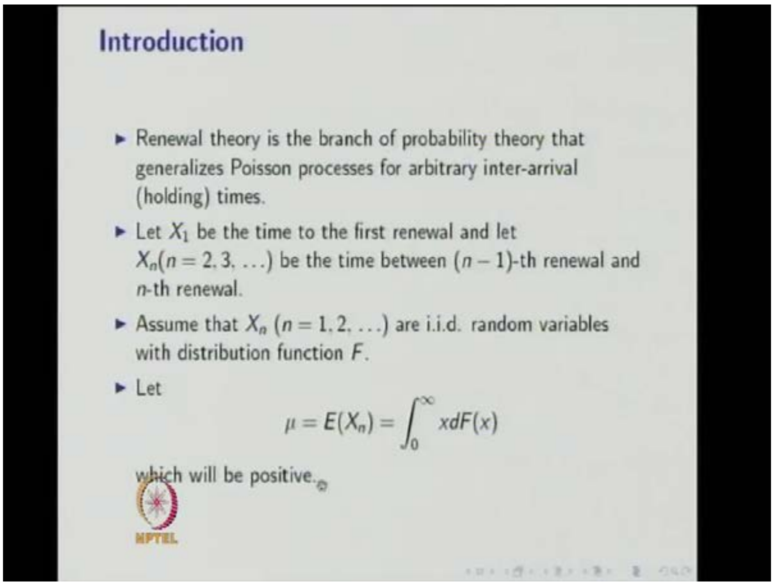
(Refer Slide Time: 02:19)



Now, we are giving the presentation on module 8 renewal processes. In this we are planning to give lecture for (( )) hours starting with the renewal function and renewal equation in the lecture 1. Generalized renewal processes and renewal limiting theorems in lecture 2 and in the lecture 3 we are planning to present as semi Markov process or Markov renewal process and Markov regenerative and in the lecture 4 we are planning to present non Markovian queues.

In the lecture 5 we are planning to present non Markovian queues with respect to the service distribution. In this lecture we are planning to give the definition of a renewal process and the related properties. Then we are going to present renewal equation and then we are going to present renewal times, then finally, we are going to explain, what is a meaning of age excess and spread at time t.

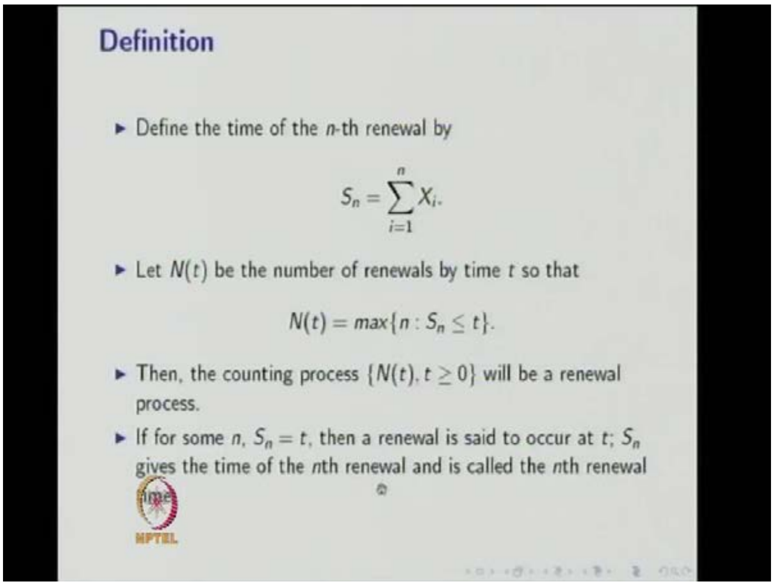
(Refer Slide Time: 02:51)



The renewal theory is the branch of probability theory that generalizes Poisson processes for arbitrary inter arrival or holding times. Renewal as the word suggests means that an event occurs again. In this when we say that X n X 1 is the time of first renewal then it refers to the first time an event occurs, where event may be defined as required by the user. Let X 1 be the time to the first renewal and let X n n running from 2, 3 and so on be the time between n minus 1 eth renewal and n eth renewal.

Assume that X i's are independent, identically distributed random variables with the distribution function capital F. The mean for the random variable X n that is denoted by the mu that is nothing but 0 to infinity x times integration with respect to d F of x which will be positive because X i's are nothing but the time between the n plus 1 eth arrival to the n th arrival, that is the random variable X n. Hence, the mean is will be a positive.

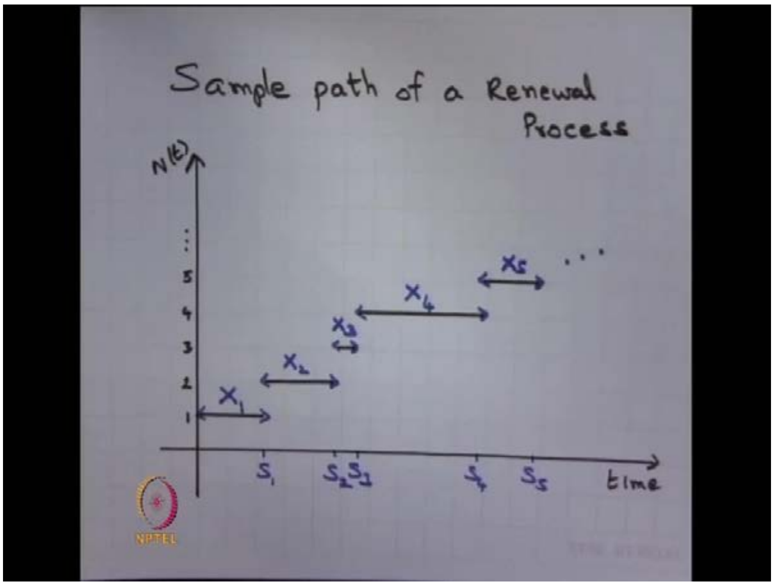
(Refer Slide Time: 04:33)



Define the time of the n eth renewal by S n is summation of a first n first X i random variables. The X i's are nothing but the time interval between the i minus 1 eth renewal to the i eth renewal and the S n is nothing but the time of n eth renewal that is denoted by summation i is equal to 1 to n X i. So, for every n you will get a one random variable S n, so, S 1, S 2, S 3 and so on. So, this is the time of n eth renewal that is a collection of random variables.

The another random variable let N t be the number of renewals by time t. So, that N of t is a maximum of n such that S n is less than or equal to t that is nothing but how many renewals takes place on or before time t that will be the random variable N of t. So, one can relate N t with S n with this formula maximum of n such that S n is less than or equal to t. So, the N of t is nothing but a counting how many arrivals, how many renewals takes place till time t. Therefore, this is the counting process and this counting process will be a renewal process. If for some n, S n is equal to t that means a renewal is said to occur a time t or S n is equal, S n gives the time of the n th renewal and is called the n th renewal time. S n is equal to t that is nothing but the n th renewal time.

(Refer Slide Time: 06:51)



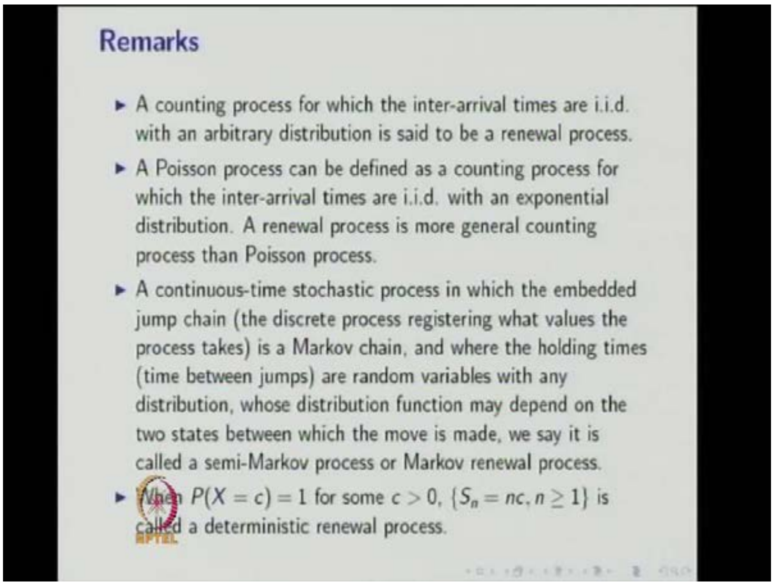
Let us see the sample path of renewal process. In this sample path of renewal process the x axis has the renewal time points, renewal time S 1 is the first renewal, the time in which the first renewal takes place. S 2 is the time in which the second renewal takes place, S 3 is the, S 3 is the time in which the second renewal takes place.

The X 1 is the time between the first renewal and the second renewal, the X 2 is the time between second renewal and first renewal and X i’s nothing but the iid random variables with the distribution function capital F of x. Since, it is in between the renewal time, between the renewal time therefore, the mean will be a positive. So, in this sample path we are giving the different values of X 1, X 2, X 3 and what is, what are all the different time points in which the renewal takes place that is S 1, S 2, S 3, S 4, S 5 and so on.

And in the y axis we are making a count N of t. So, till some time the renewal is only 1. After some times the renewal is 2. So, suppose you stopped it somewhere and asking how many arrivals takes place at this much time. Therefore, 2 arrival takes place this much time. After crossing suppose I stopped it here then there are 4 arrivals takes place in this much time. Therefore, the N of t is equal to maximum of N such that S n is less than or equal to n, S n is less than or equal to t makes the possible values of N of t.

So, N of t for possible values of t that will be a stochastic process, it is a counting process and this is a renewal process also and X i’s are the iid random variables and S n is the time of n th renewal. A common feature is this example thus is that rather than studying a random value of a sum of fixed number of random variables, one investigates the random number of terms required in order for the sum to attain a certain deterministic value. For non negative sum n's we are lead to the part of probability theory called a renewal theory and the summation, summation process S n is called the renewal process. So, this is the connection between the random walk and the renewal process.

(Refer Slide Time: 10:09)



Now, we are moving in to the remarks of renewal process. The first remark, a counting process for which the inter arrival times are iid random variables with arbitrary distribution is said to be a renewal process. Whenever you have a counting process in which inter arrival times are iid random variables with whatever be the distribution with arbitrary distribution is said to be a renewal process.

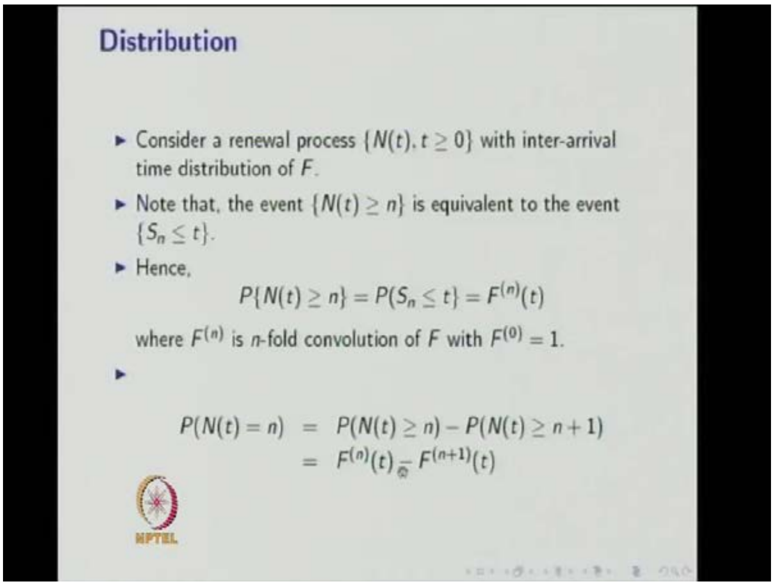
Second remark, a Poisson process can be defined as a counting process for which the inter arrival times are iid with the exponential distribution. So, the Poisson process is a special case of renewal process in which the inter arrival times are exponential distribution iid as well as each S exponential distribution, identical, independent and identically distributed randomly. A renewal process is more general counting process then Poisson process. A renewal process is more general counting process than Poisson process because in the Poisson process the inter arrival times are identical, independent and each one is exponential distribution whereas, in the renewal process it could be any arbitrary distribution, any counting process need not be particularly Poisson process.

A continuous time third remark, a continuous time stochastic process in which the embedded jump chain that is nothing but the discrete process registered what values the process takes, is a Markov chain and where the holding times that is nothing but the time between jumps or random variables are random variables with any distribution whose distribution function may depend on the two states between which the move is made, we say it is a semi Markov process or Markov renewal process.

If this condition is satisfied by a continuous time stochastic process, it has a embedded Markov chain and the holding times depends. It can be any distributions, but it depends on the two states between the, the system moves. Then the corresponding stochastic process is called a Markov renewal process or semi Markov process.

We are going to discuss Markov process or semi Markov process in lecture 3 in detail and the application of semi Markov processes in queuing models will be discussed in lecture 4 and 5. The remark number four, when the probability of X takes the value some constant c is equal to 1, for some constant C greater than 0 then the corresponding renewal process is called a deterministic renewal process. This X is nothing but the inter arrival time. For renewal process the inter arrival times are iid random variables.

(Refer Slide Time: 14:15)



Therefore, for one random variable we are defining it is a constant random variable. If this iid random variables are constant random variable for some constant c greater than 0 then the corresponding renewal process is called a deterministic renewal process. Now, we are going to discuss the distribution of renewal process. Consider a renewal process N t with the inter arrival time distribution F. We can relate the event N t greater than or equal to n with the S n less than or equal to t.

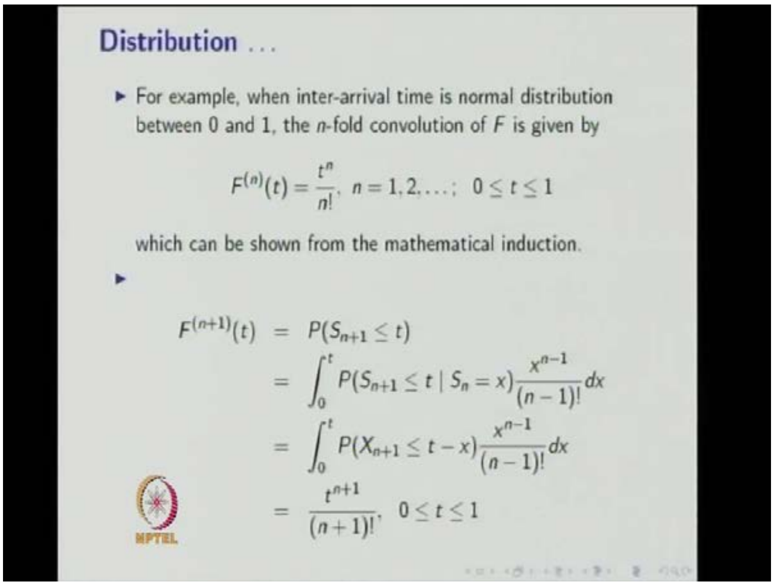
The event N t greater than or equal to n is nothing but the number of renewals takes place till time t is more than or equal to n. The event of S n less than or equal to t means the number of renewals, the n number of renewals takes place on or before time t. So, both the events are same. Therefore, the probability of N t greater than or equal to n is same as probability of S n is less than or equal to t. That we are going to denote as the F superscript n of t that means it is a n fold convolution of the distribution function F.

So, this is the distribution of the n eth renewal S n. The distribution of n eth renewal S n. So, using the the distribution function of n eth renewal of n eth renewal that is S n you can find out the distribution of N of t. Since, N of t is the counting process the possible values of N of t is 0, 1, 2 and so on. So, we can find the probability mass function for the random variable N of t for fixed t.

So, the probability mass function for the random variable N of t for fixed t is probability of N t is equal to n is nothing but what is the probability that more than or equal to N renewals takes place on or before time t minus what is the probability that N plus 1 renewals takes place more than or equal to N plus 1 renewals takes place on or before time t. That difference will be the probability of N renewals takes place on or before time t.

So, since we know that this relation N of t greater than or equal to n is same as the probability of a, probability of N of t is greater than or equal to n is same as probability of S n is less than or equal to t that is nothing but the n fold convolution of F. Therefore, the probability of N of t greater than or equal to n will be F superscript n fold convolution of F minus N plus 1 fold convolution of F at time t. So, once you, once you know the inter arrival time distribution F you can find out the n fold convolution.

(Refer Slide Time: 18:07)

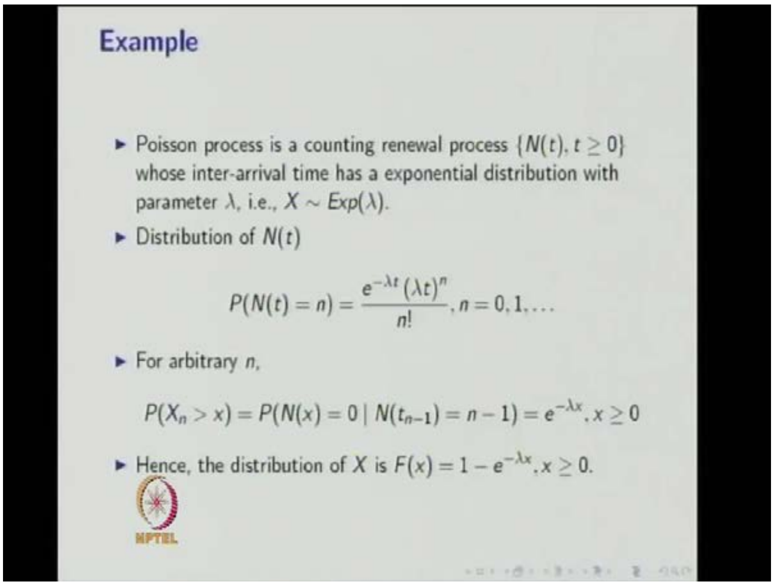


Once you know the n fold convolution of F using this formula you can find out the probability mass function of N of t for fixed t. For example, when inter arrival time is normal distribution. For example, when inter arrival time is uniform distribution between 0 and 1 the n fold convolution of F is given by, given by F superscript n of t is nothing but t power n divided by n factorial where t is lies between the interval 0 to 1. So, this can be proved by the mathematical induction.

So, we can start with n plus 1 fold convolution of F that is nothing but the distribution function of the random variable S n plus 1 that can be written in the conditional distribution. The probability of S n plus 1 is less than or equal to t given S n is equal to X that is same as what is the probability that the inter arrival time of the N plus 1 th renewal is takes less than or equal to t minus x.

Because the given S n is equal to x and what is the probability of S n plus 1 is less than or equal to t that is same as what is the probability of X n plus 1 is less than or equal to t minus x. You can substitute and after simplification we can get the n plus 1 eth convolution of F that will be t power n plus 1 divided by n plus 1 factorial. Hence, proved the n fold convolution of F will be t power n by n factorial for n is equal to 1, 2 and so on and t is lies between 0 to 1.

(Refer Slide Time: 20:25)



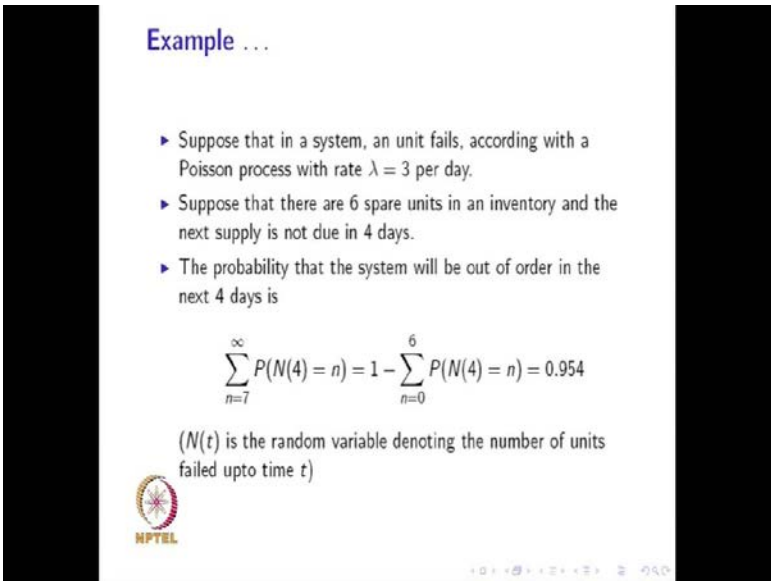
Now, we will discuss the other example and find the distribution of N of t. The Poisson process is the renewal process or counting process whose inter arrival time has the exponential distribution with the parameter lambda. Since, the inter arrival times are iid random variables each has the distribution exponential with the parameter lambda. Using the earlier, earlier form of writing distribution of N t in terms of n fold convolution you can find the distribution of N of t. So, here for arbitrary n; first, we know the distribution of N of t because it is a Poisson process. Therefore, it is e power minus lambda t lambda t power n by n factorial.

Since, you know the probability of N t is equal to n you can find out the inter arrival time distribution. The inter arrival time distribution for that you can go for finding the compliment CDF of the random variable X n. So, the probability of X n greater than x for arbitrary n that is nothing but since the inter arrival times are independent you can find out the probability of X n is greater than x is same as the probability of n x equal to 0 given N of t minus 1 is equal to n minus 1.

That means what is the probability that n minus 1 renewal takes place till the time point t n minus 1 and given this situation what is the probability that no arrival takes place in the interval t n minus 1 to t n minus 1 plus x or since it is inter arrival times are i.i.d and stationery also the N of x will be 0, finding the probability of N of x equal to 0 given that N of t minus 1 is equal to n minus 1. Since, you know the distribution of N of t you can substitute and you can get it is e power lambda x for x greater than or equal to 0 and this is valid for arbitrary n.

So, since the compliment CDF is e power minus lambda x the distribution of x will be exponential distribution with the parameter lambda. So, to conclude inter arrival time is exponential distribution, we are using the property of Poisson process and the definition of the probability mass function of Poisson process also.

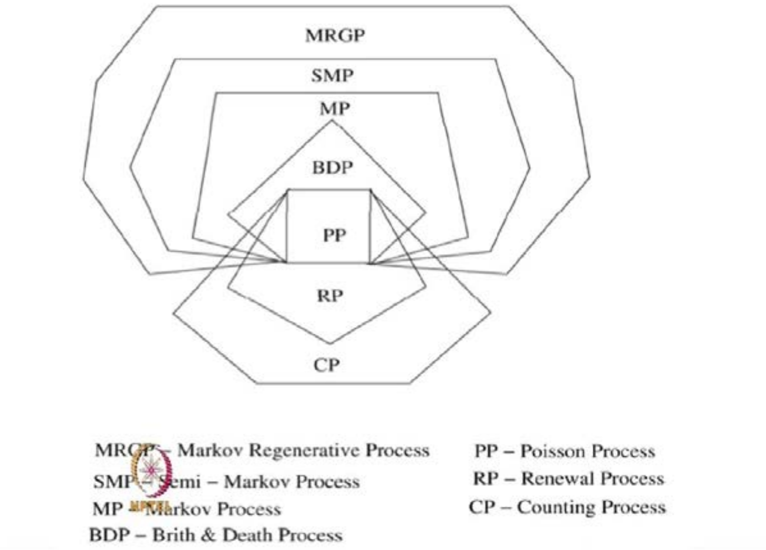
(Refer Slide Time: 23:43)



Next, we are going to consider one more example for the renewal process. Suppose, that in a system a unit fails according with the Poisson process with the rate lambda is equal to 3 per day. So, this is the example for Poisson process, a special case of renewal process. Suppose that there are 6 spare units in an inventory and the next supply is not due in 4 days. Our interest is to find out what is the probability that the system will be out of order in the next 4 days.

That is nothing but, what is the probability that at time point 4 the number of fail units in the system is greater than or equal to 7, that is the probability of system will be out of order in the next 4 days. So, this probability is same as 1 minus the summation n is equal to zero to 6 the probability that N of 4 equal to n. You substitute the probability mass function the lambda value and the t equal to 4 will get the required probability that will be 0.954. So, this is the example of Poisson process.

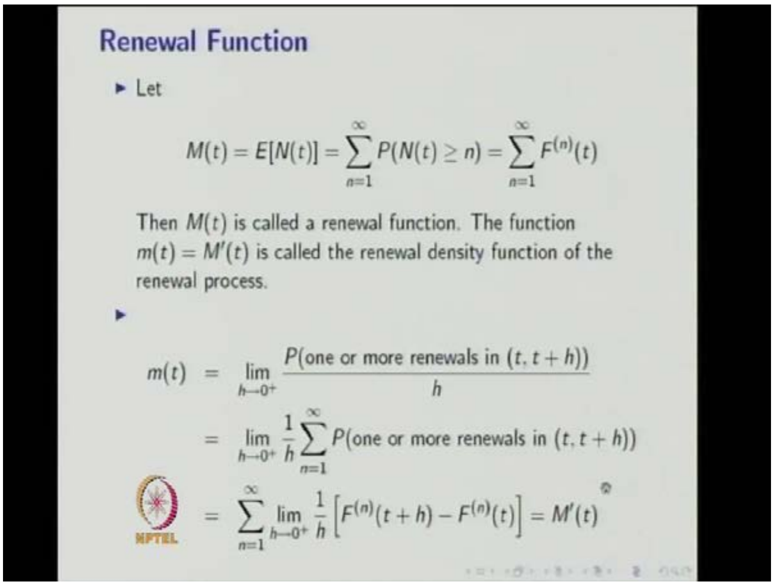
(Refer Slide Time: 25:29)



This is the pictorial representation of Poisson process and all other process in one graph. The Poisson process is in the PP. Poisson process is the special case of pure birth process and pure birth process is a special case of birth death process.

And birth death process is the special case of continuous time Markov chain and continuous time Markov chain is the special case of Markov process and Markov process is a special case of semi Markov process and the semi Markov process is the special case of Markov regenerating process. The other side, the other side Poisson process is the special case of renewal process. The renewal process is the special case of semi renewal process and the semi renewal process there is a special case of counting process. So, this is the way one can relate the Poisson process with all other stochastic processes.

(Refer Slide Time: 26:59)



Now, we are moving into the concept called renewal function. Let M of t is equal to expectation of N of t for fixed t, that is called the renewal function. Since, N of t is the counting process and possible values are 0, 1, 2 and so on. You can find out the expectation of N of t with the summation n is equal to 1 to infinity of probability N of t greater than or equal to n. Since, it is a discrete random variable with the possible values are 0, 1, 2 and so on.

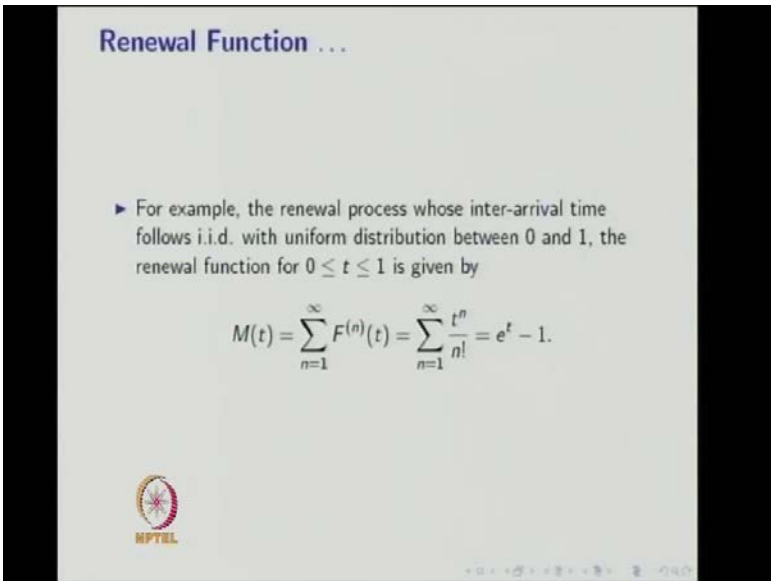
Since, for fixed t, N of t is a discrete random variable with the non negative integer values 0, 1, 2 and so on. The expectation is the sum of its complementary cumulative distribution function. Alternatively, the expectation can be computed as the sum of n times so P n of t that is same as sum of n times capital F n of t minus capital F n plus 1 of t which gives sum of F n of t.

The expected number of renewals up to time t first movement is referred to as a renewal function that is capital M of t that is equal to expectation of n of t. We know that the probability n of t greater than or equal to n that is same as the probability of S n is less than or equal to t and that is nothing but the CDF of S n that is nothing but the n fold convolution of F. Therefore, summation of probability F n of t greater than or equal to n is same as summation of n fold convolution of F were n is running from 1 to infinity.

So, using this one can find the renewal function and the derivative will be the renewal density function of the renewal process. So, the capital M of t is the renewal function and the small m of t that is nothing but the derivative of renewal function is called a renewal density function. A function small m of t specifies the main number of renewals to be expected in a narrow interval near t. One can find the renewal density function using limit h tends to 0 with the probability of one or more renewals occurs in the interval t to t plus h divided by h.

That is nothing but limit h tends to 0, 1 by h the probability of 0 or more renewals that is nothing but the summation of n is equal to 1 to infinity of probability of n renewals in the interval t to t plus h. The limit and the summation can be interchanged and probability of n renewals in the interval t to t plus h, where n is running from 1 to infinity that can be simplified in the form of n fold convolution of f between the interval t to t plus h.

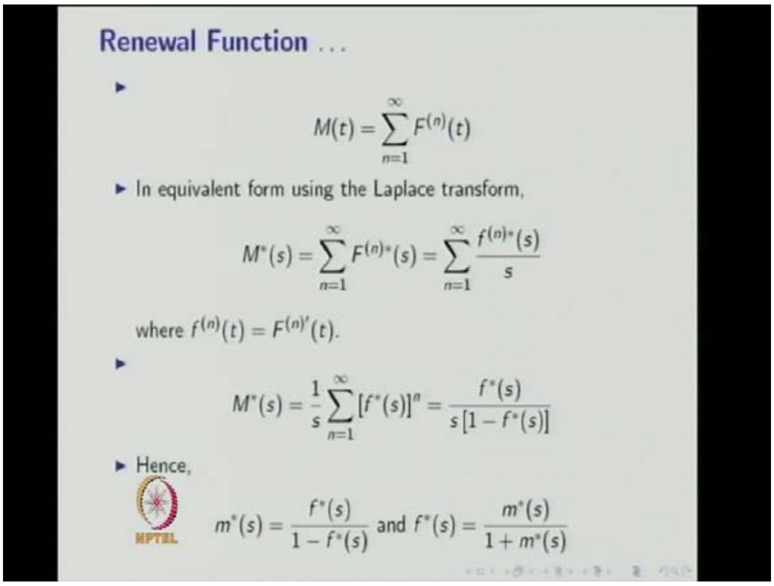
(Refer Slide Time: 31:13)



By using the M of t is nothing but the summation of F n of t. Therefore, the limit h tends to 0 1 by h of F n of t plus h minus f n of t is nothing but the derivative therefore, it is a capital M dash t. So, the derivative of renewal function is nothing but renewal density function. For example, the renewal process whose, whose inter arrival time follows iid with the uniform distribution between the interval 0 to 1, the renewal function is e power t minus 1 because inter arrival time is uniformly distributed between the interval 0 to 1.

Therefore, you know what is the F n of t n fold convolution of F; substitute that, that is t power n divided by n factorial. After simplification you will get e power t minus 1 that is the renewal function for a renewal process whose inter arrival times are i.i.d random variable with uniform distribution between the interval 0 and 1.

(Refer Slide Time: 32:08)

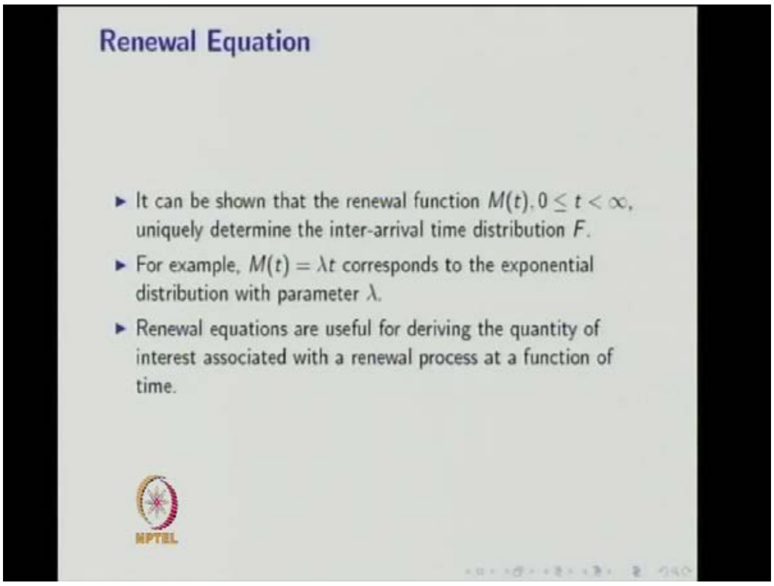


Now, we are going to relate the probability density function of inter arrival time, inter arrival time with the renewal density function and renewal function in Laplace form. We know that the renewal function is nothing but the summation n is equal to 1 to infinity F n of t n fold convolution of F, where F is the CDF of inter arrival time.

In equivalent form using the Laplace transform, the M star of s that is nothing but the Laplace transform of M of t renewal function, that is nothing but summation n is equal to 1 to infinity the n fold convolution of F in Laplace, in Laplace transform as a function of s, that is same as the small f of n fold convolution in Laplace form divided by s. We know that the derivative of n fold convolution of CDF is nothing but the n fold convolution of the probability density function.

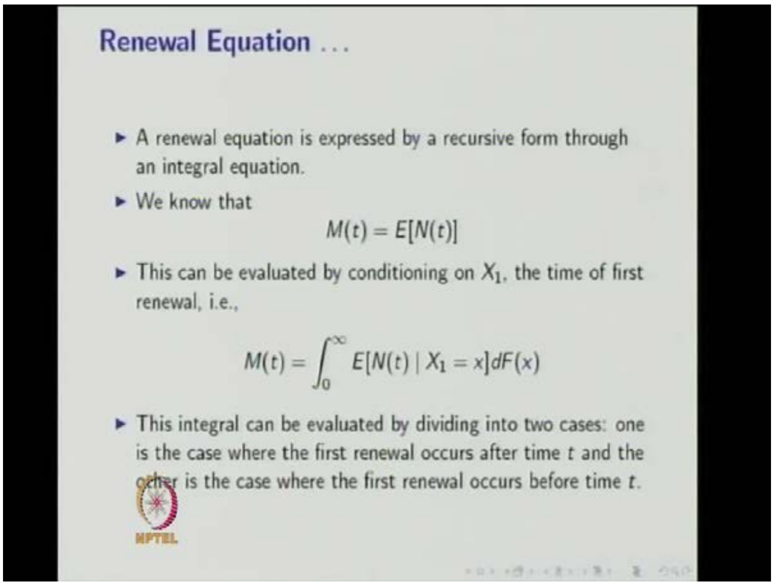
So, this can be written as 1 divided by s summation n is equal to 1 to infinity the Laplace transform of probability density function power n. You can simplify. After that you can find the renewal density function in Laplace, that is nothing but f of s divided by 1 minus f of s where f of s is nothing but the probability density function in Laplace and so on. From the same equation you can get the f of s in terms of m of s also where m of s is the renewal density function in Laplace.

(Refer Slide Time: 34:43)



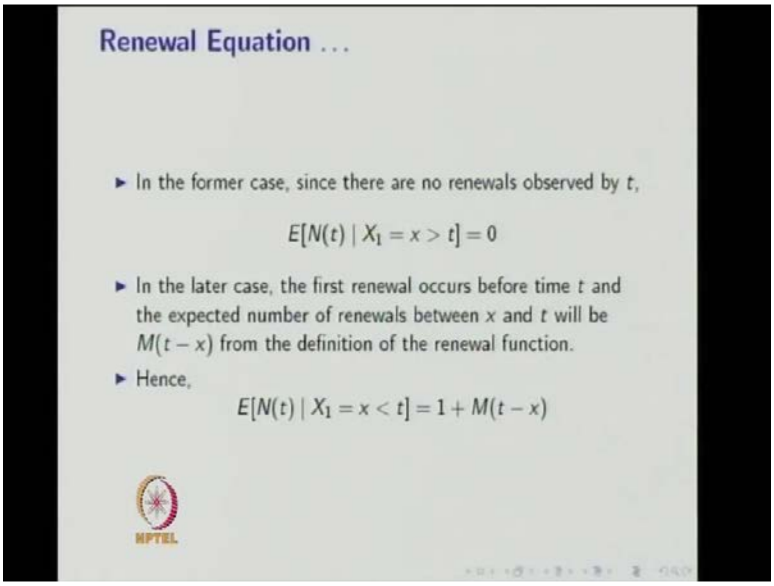
Now, we are moving into the next concept that is called the renewal equation. It can be shown that the renewal function M of t, uniquely determine the inter arrival time distribution F. For example, M of t is equal to lambda t corresponds to the exponential distribution.

(Refer Slide Time: 35:38)



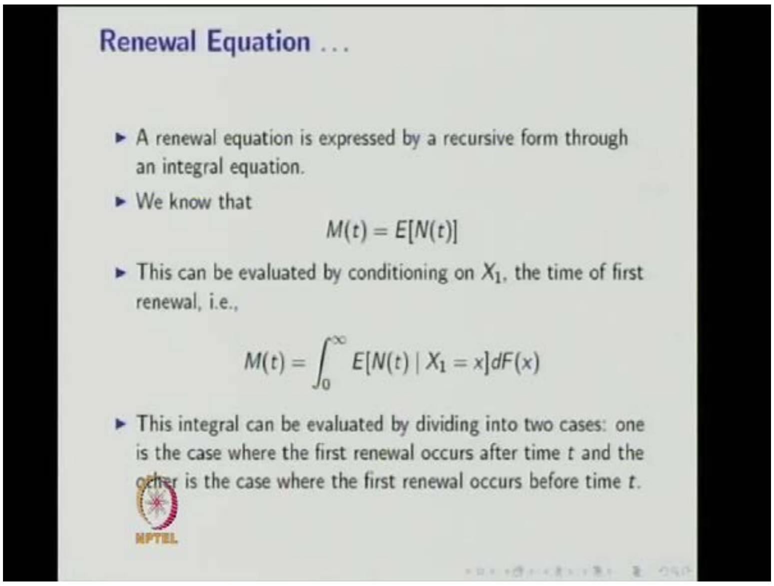
Break, break, now we are moving in to the next concept called renewal equation. A renewal equation is expressed by a recursive form through a integral equation. We know that the renewal function M of t is expectation of N of t. This can be evaluated by conditioning on X 1. That X 1 is nothing but the time of first renewal. Hence, the renewal function M of t is equal to integration 0 to infinity, expectation of N of t given X 1 is equal to x, integration with respect to the CDF of inter arrival time distribution. This integral can be evaluated by dividing into two cases. One is the case where the first renewal occurs after time t and the other is the case, where the first renewal occurs before time t.

(Refer Slide Time: 36:52)



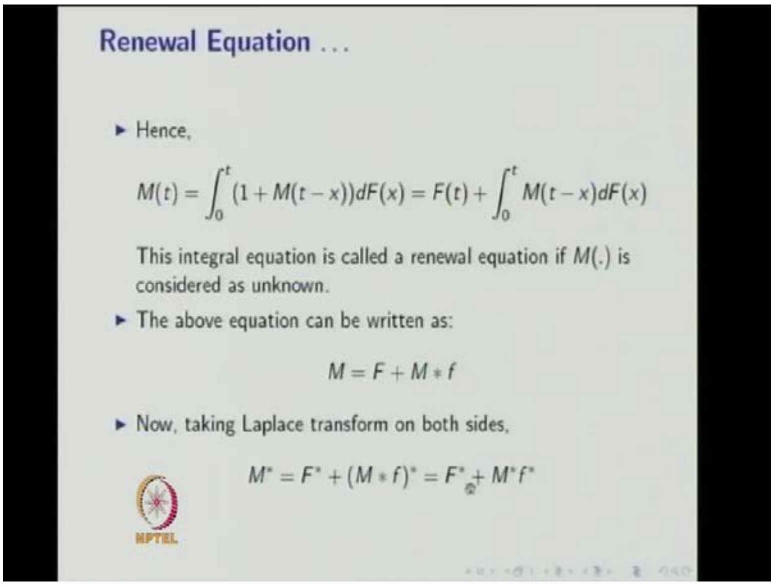
In the former case, since there are no renewals observed by time t the condition expectation will be 0. In the later case the first renewals occurs before time t and the expected number of renewals between X and t will be the renewal function M of t minus x from the definition of the renewal function. Hence, the conditional expectation is one renewal already takes place before time t therefore, 1 plus the expected number of renewals between the interval x to t will be M of t minus x. Therefore, the conditional expectation is 1 plus M of t minus x.

(Refer Slide Time: 37:51)



Therefore, the M of t that conditional expectation integration will be splitted into two.

(Refer Slide Time: 37:58)

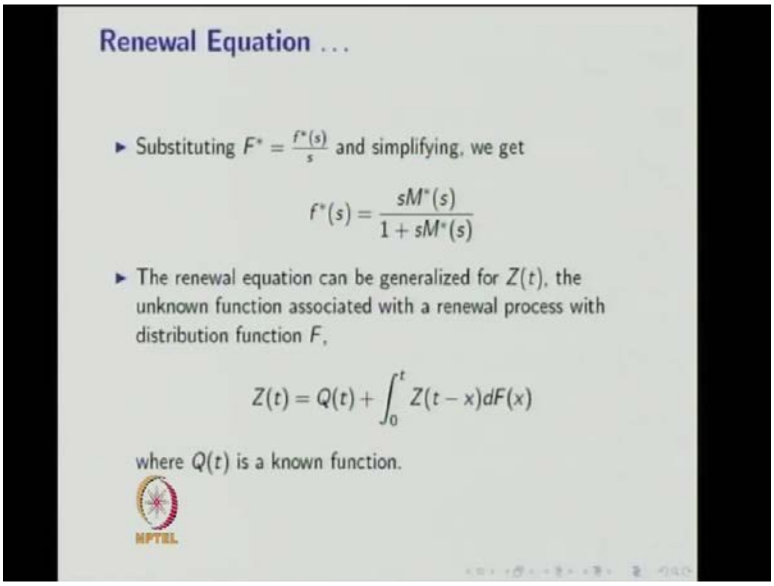


Hence, the one integration will be 0 because the conditional expectation is 0, the other integration will be 0 to t 1 plus M of t minus x d F of x that is same as capital F of t plus 0 to t M of t minus x d F of x. So, this integral equation is called the renewal equation if M is the, M is consider as a unknown, because the left hand side is the M of t that is a renewal function.

If M of t is the unknown, unknown then M of t is written in the form of M of t is equal to F of t plus integration 0 to t, the integrant is M of t minus x. Therefore, this is called the integral equation and this integral equation is called the renewal equation. Later, we are going to give the renewal limiting theorem as t tends to infinity for the integral also. M of t minus x d F of x can be evaluated as t tends to infinity if certain conditions are satisfied.

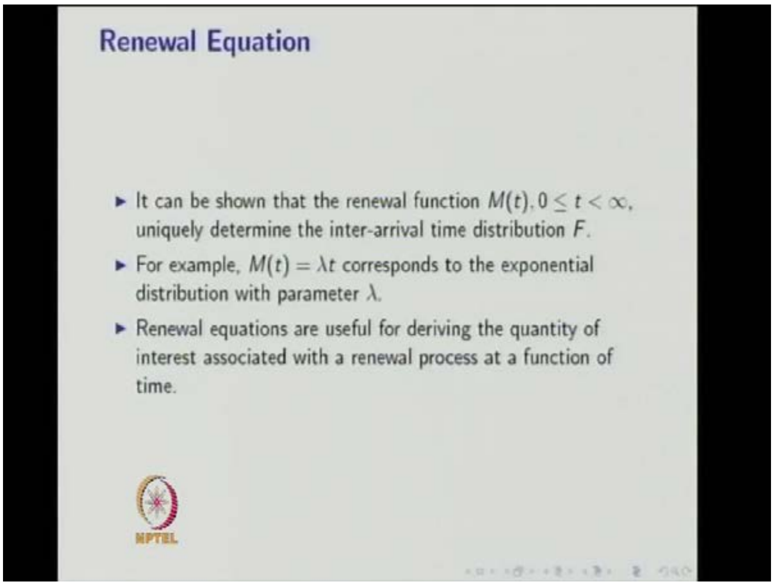
But here we are presenting the renewal equation where the renewal function is the considered as unknown. The above equation can be written as M is equal to F plus M star small f where small f is nothing but the probability density function of inter arrival time. Now, taking a Laplace transform on both sides you will get M star is equal to F star plus M star small f star.

(Refer Slide Time: 39:58)



You can relate F star is equal to f star by s where star is nothing denoted for Laplace form and simplifying we get f of s is equal to s times M star divided by 1 plus s times M star where capital M star is nothing but the renewal function in Laplace. The renewal equation can be generalized for some unknown Z of t, unknown function associated with a renewal process with the distribution function capital F as Z of t is equal to Q of t plus integration 0 to t Z of t minus x d F of x. So, the renewal function can be extended, can be generalized for any unknown function associated with the renewal process that is Z of t as Q of t plus integral 0 to t Z of t minus x d F of x, where Q of t is a known function.

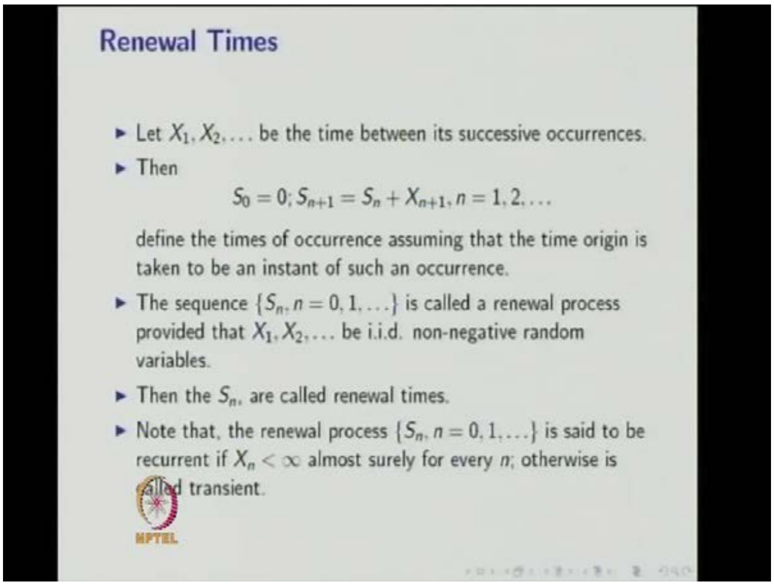
(Refer Slide Time: 41:24)



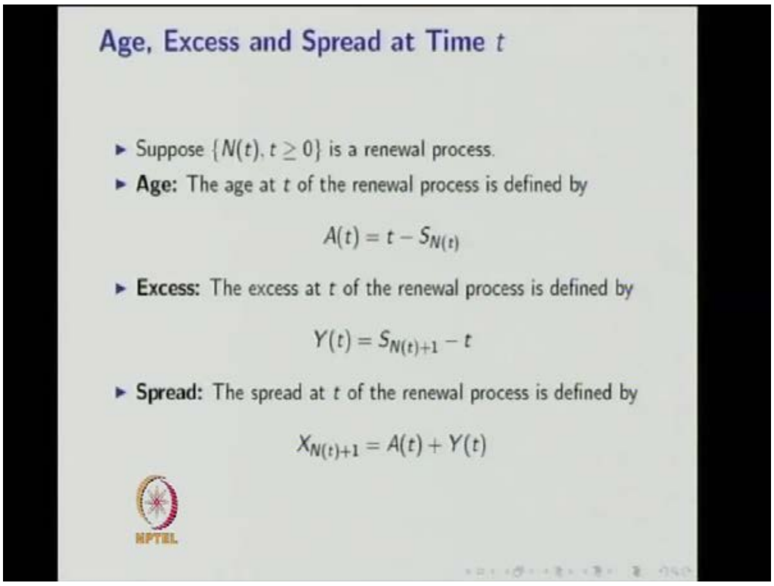
It can show that the renewal function uniquely determines the inter arrival time distribution. Once, you know the renewal function you can uniquely determine the distribution function. For example, if I know the renewal function is lambda t then you can say that the inter arrival time distribution is exponential with the parameter lambda. Hence, the renewal equations are useful for deriving the quantity of interest associated with a renewal process at a function of time. So, first we discuss the renewal equation in terms of renewal function and this can be generalized to any unknown function of time which is associated with the renewal process also.

Now, we are moving into the next concept renewal times. Let X 1, X 2 be the time between its successive occurrences. Then S naught is equal 0 and S n plus 1 is equal to S n plus X n plus 1 defines the times of occurrence assuming that the time origin is taken to be a instant of such a occurrence. Then the sequence S n is called the renewal process provided the inter occurrence times are i.i.d non negative random variables. Then each S n is called the renewal times. Note, that the renewal process is said to be recurrent if X n is a finite almost surely for every n, otherwise the renewal process is called transient.

(Refer Slide Time: 42:25)

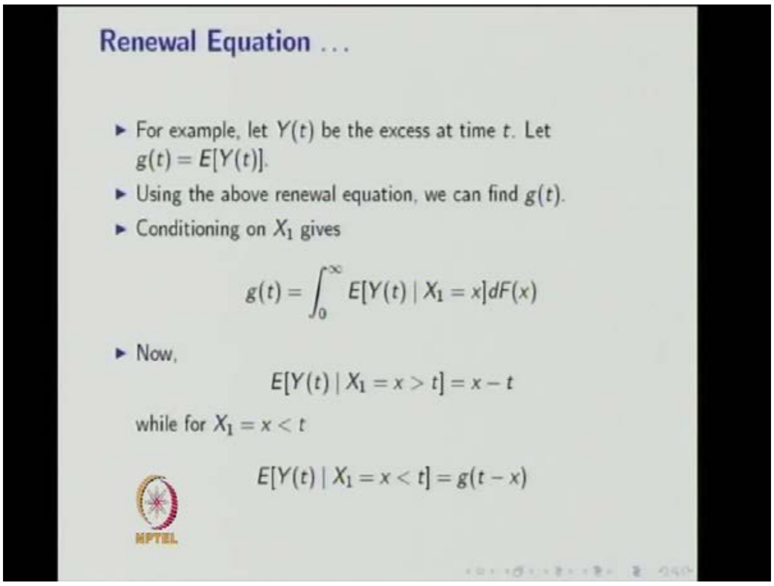


(Refer Slide Time: 43:32)



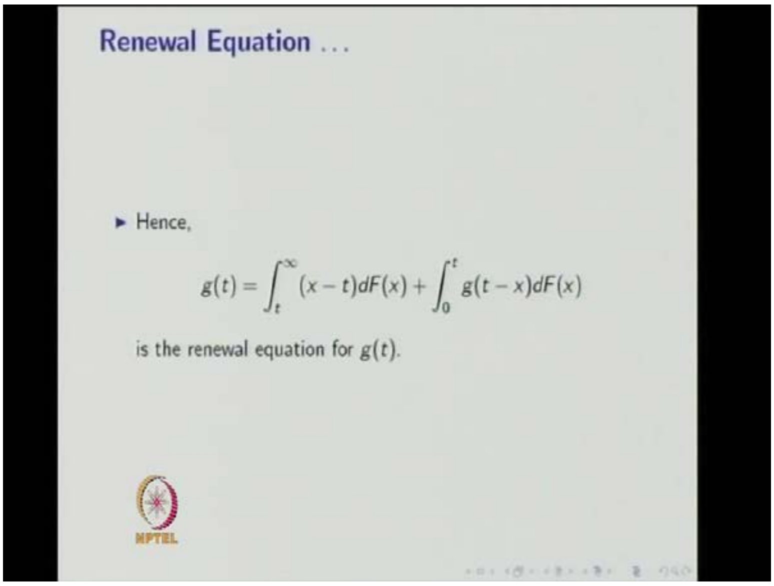
Now, we are moving into the last concept in the, in this lecture that is age, excess and spread. Suppose, N t is a renewal process and S n forms the renewal time you can define age at time t of the renewal process as A of t is equal to t minus S suffix N t. The excess a time t of the renewal process is defined as Y of t is equal to S of N plus S of N plus t plus 1 minus t, that is a excess at time t. Whereas, the spread is the addition of age and excess that is a spread of, spread a time t of the renewal process.

(Refer Slide Time: 44:28)



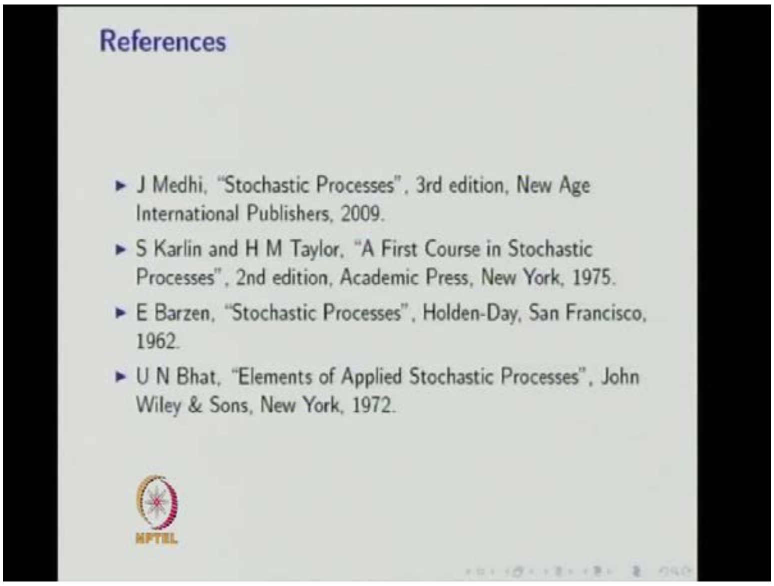
Here, we are going to calculate, what is the average excess for expectation of expectation of excess? Let y t be the excess a time t and let g t be the expectation of excess a time t.

(Refer Slide Time: 45:32)



Using the renewal equation which we have to discuss now, we are going to find the expectation of excess at time t. First we are making condition on x 1 and conditional expectation will be expectation of Y of t given x 1 is equal x which is greater than t will be x minus t. While x 1 is equal to x, which is less than t. Then the conditional expectation will be a unknown still g of t minus x. Therefore, the g of t will be 2 integration; 1 integration is from t to infinity and integrant x minus t and the second integration 0 to t and the unknown is g of t minus x.

(Refer Slide Time: 46:11)



Since, G of t is unknown function of time t, this is the integral equation. One can solve the integral equation; this is nothing but the renewal equation. One can solve the renewal equation and that time g of t. Here is the list of reference for lecture 1.