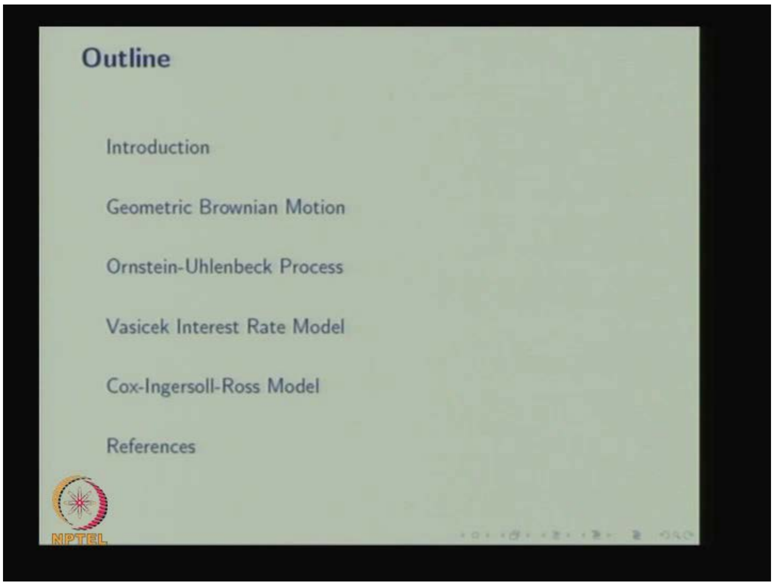
Stochastic Processes
Prof. Dr. S. Dharmaraja
Department of Mathematics
Indian Institute of Technology, Delhi

Module No - 7
Brownian Motion and its Applications
Lecture No - 6
Some Important SDE's and Their Solutions

This is stochastic processes, module 7, Brownian motion and its applications. This is lecture 6, some important stochastic differential equations and their solutions. The last 5 lectures we have discussed Brownian motion and it is properties. Then process derived from the Brownian motion, then we have discussed stochastic differential equation. That is related to Brownian motion, only then we have discussed the ito integral. That is the integral integration with respect to the Brownian motion.

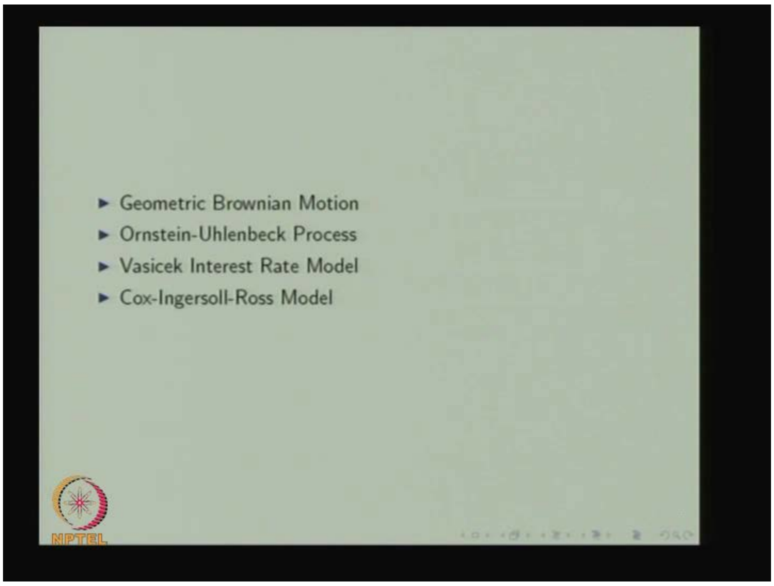
Then, we discussed ito formula that is the 1. We can use to solve the some special cases of stochastic differential equations at the end as the lecture 6. We are going to discuss some important stochastic differential equations with their solutions in this stochastic differential equations arises in the application of mathematical finance.

(Refer Slide Time: 01:38)



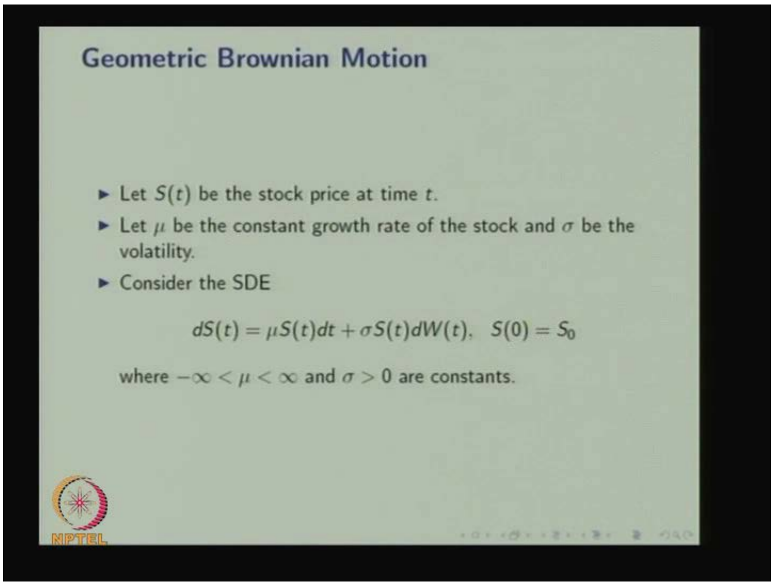
In this lecture, we are going to discuss four different stochastic differential equations. All are related to the applications of financial mathematics. The first one is geometric Brownian motion.

(Refer Slide Time: 01:58)



That is the stochastic process, this is this stochastic differential equation the corresponding stochastic differential equation is called Black-Scholes model. The second model is Ornstein-Uhlenbeck process. The third and fourth are the interest rate model. So, these four models we are going to discuss in this lecture in ditoil.

(Refer Slide Time: 02:34)

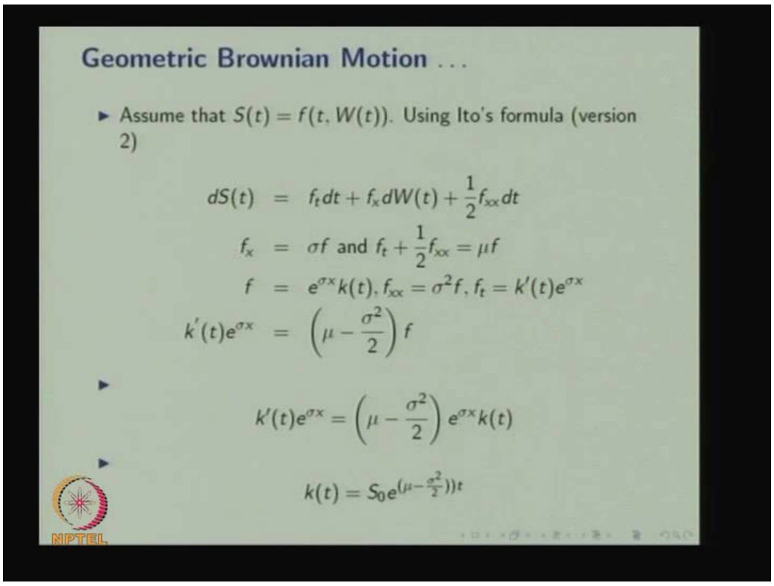


The first model this is nothing but Black-Scholes model. Let S (t) be the stock price at time t. Let mu be the constant growth rate of stock and sigma be the volatility. Consider the stochastic differential equation that is d S (t) is equal to mu times S (t) d t plus sigma times S (t) d W (t) at S and 0 you will know the value that is S naught. So, here we made the assumption mu sigma is a volatility, which is the constant but in the real world scenario, the volatile it is not a constant.

But for the this particular Black-Scholes model, we have chosen volatility is a constant in general volatility is a random variable. Therefore, over the time, that is a stochastic process. Also, it is a stochastic volatility the corresponding model is a stochastic volatility model. But, here the volatility we treated as a constant therefore, it is a sigma time S of t not a function of t and the growth rate that is mu. That is also a constant, so the mu can take the value from minus infinity to infinity. Whereas, sigma is volatility, so this is greater than 0, both are constant.

So, the corresponding stochastic differential equation, that is increment of S is mu times S of t d t plus sigma times S of t d W t because of this W t d W t, this equation is a stochastic differential equation. we are going to relate, we are going to find out the solution of this stochastic differential equation. The solution is a stochastic process, that is nothing but the geometric Brownian motion and corresponding this model is called Black-Scholes model.

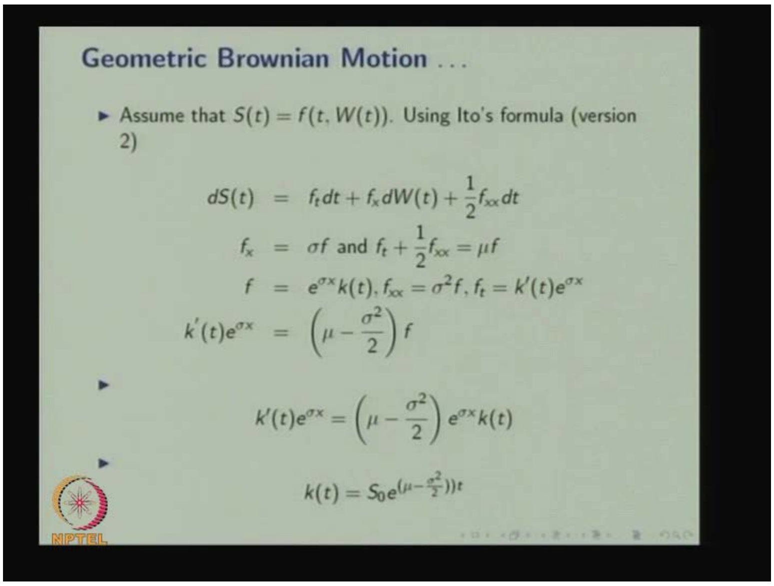
(Refer Slide Time: 04:58)



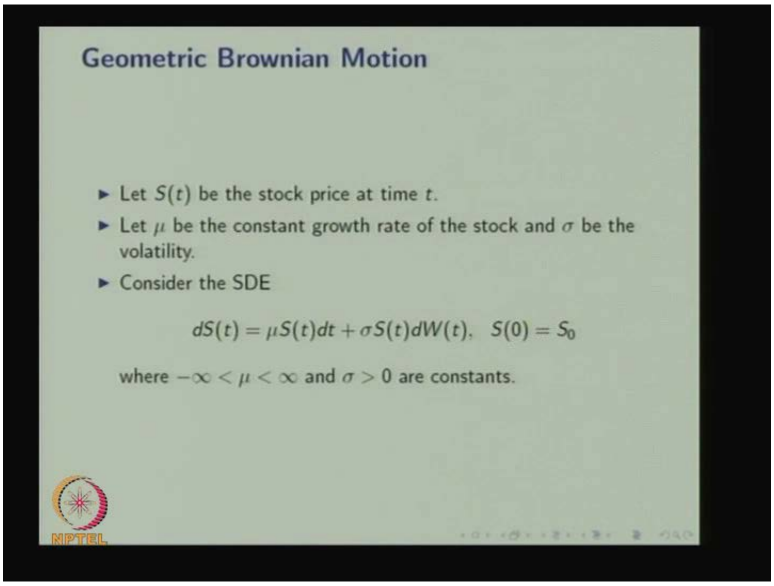
Assume that the solution S of t is a function of t comma W t, now you can use the ito formula version 2. That is nothing but d S t is equal to first derivative with respect to t d t, first derivative with respect to x d W t and half times first derivative with respect to second derivative with respect to x d t. Therefore, we will have a two terms with respect equation d t and one term with the factor d W t. Now, you can compare this with the given stochastic differential equation.

So, this is a given stochastic differential equation d of S t is equal to this one.

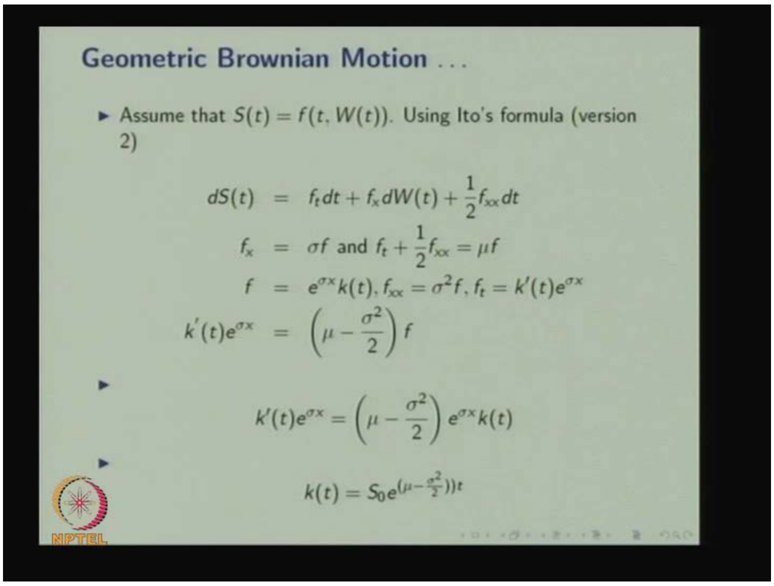
(Refer Slide Time: 04:58)



(Refer Slide Time: 05:45)



(Refer Slide Time: 05:53)

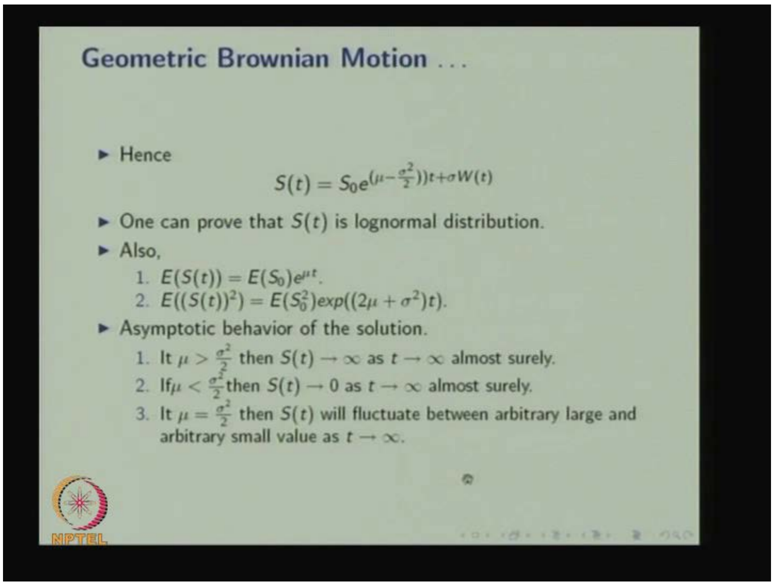


Whereas, by ito formula, the d of S t is return this one; so you can compare this equation with the given stochastic differential equation. So, you will get f the partial derivative with respect to x. That is nothing but the coefficient of d W t that is sigma times S of t. That is sigma times f. Here the coefficient of d t is f t plus half, so you can derivative with respect to x. So, therefore f t plus half times f double derivative with respect to x, that is mu times fu times S t that is u times f. So, you can solve this equation f x is equal to sigma times f by solving you will get f is equal to e power sigma times x some function of t because this is derivative with respect to x partial, partial derivative with respect to x. Therefore, the function is e power sigma times x some function of t.

You can take a second partial derivation with respect to x, therefore we will get f x. x will be sigma square f. Here you can take a partial derivative with respect to t. Therefore, we will get k dash of t e power sigma x. So, you can substituting f t here f x x here. Therefore, you will get k dash of t e power sigma x is same as mu times f minus half sigma square f.

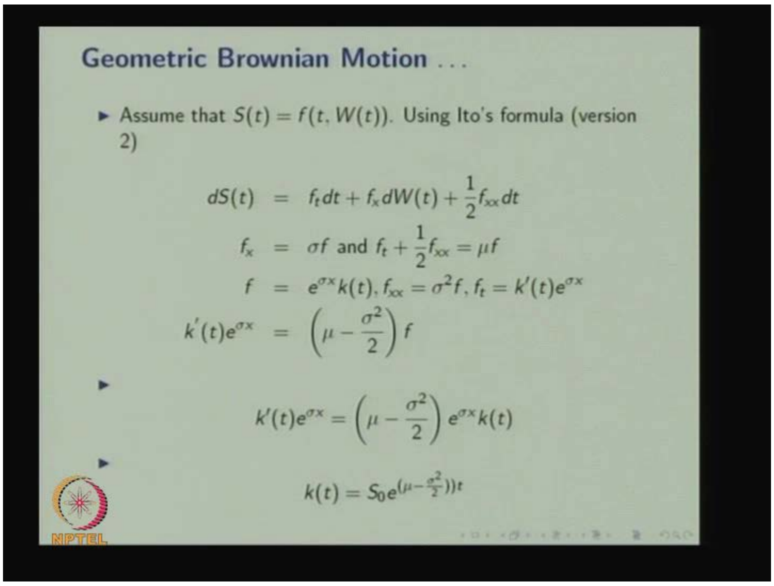
Therefore, substituting f t and f x x in this equation, you will get k dash of t e power sigma x is equal to mu minus sigma square by 2 f. Still k k of t is unknown, you have to find out the k of t. Then you can get the f. So, substitute f is equal to e power sigma x k of t. Therefore, you will have an equation, the only for the equation you get k of t. Therefore, we can solve that equation and get the k of t. So, k of t is nothing but s s naught, that is s at point. s initially at 0 e power mu minus sigma square t. So, you substitute this in the here, therefore you have f of t comma W t is e power sigma times W t, S naught e power mu minus sigma square t.

(Refer Slide Time: 08:45)



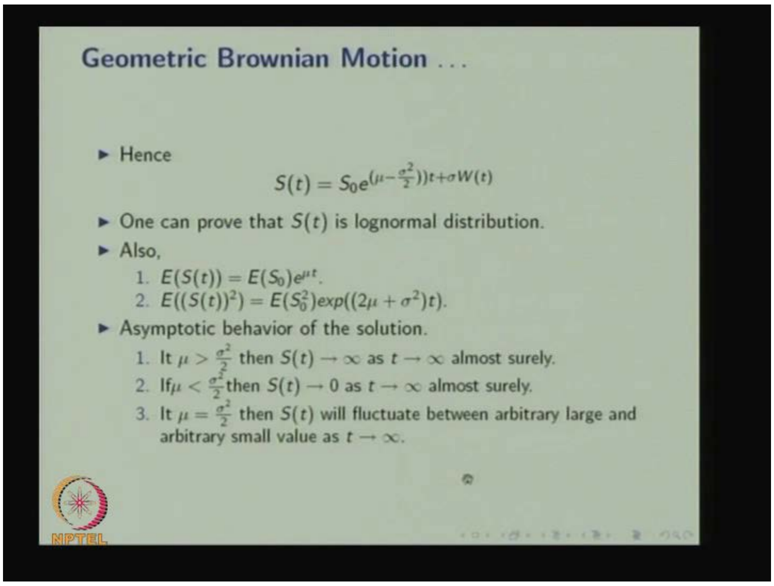
Therefore, S of t will be S naught e power mu minus sigma square t t plus sigma times W t, substituting k of t in the in this equation f is equal to e power sigma x k t, where x is nothing but W t.

(Refer Slide Time: 09:01)



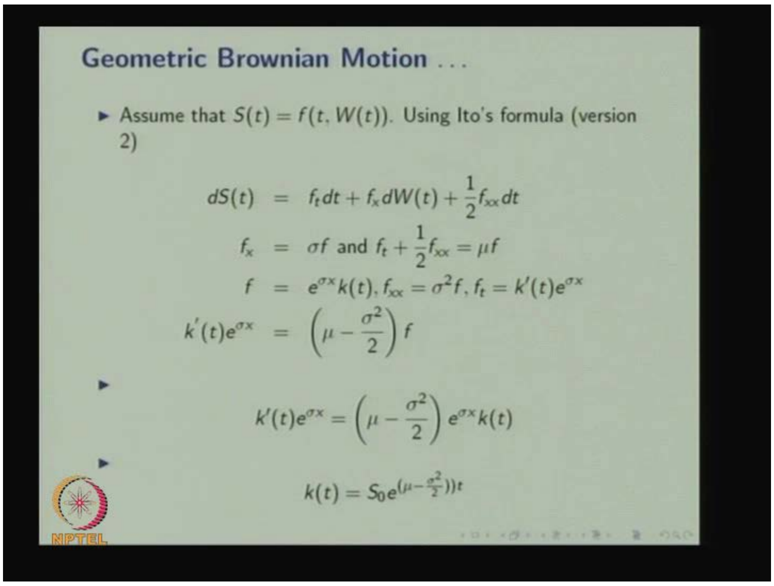
Therefore, your solution is S naught e power mu minus sigma square by 2 times t plus half W t.

(Refer Slide Time: 09:08)



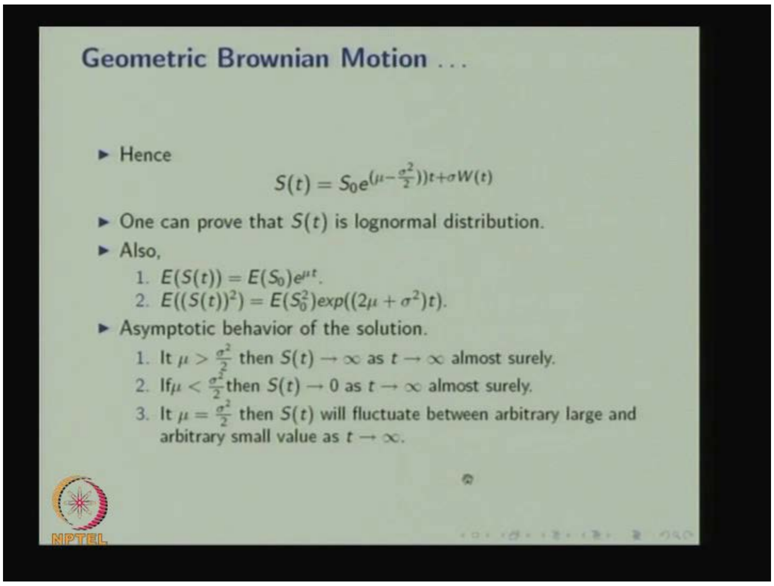
So, this is a solution of given stochastic differential equation. This will got by using the ito formula.

(Refer Slide Time: 09:27)



By using the ito formula version 2 without solving the stochastic differential equation by using the ito formula, we are finding the solution of given stochastic differential equation.

(Refer Slide Time: 09:39)

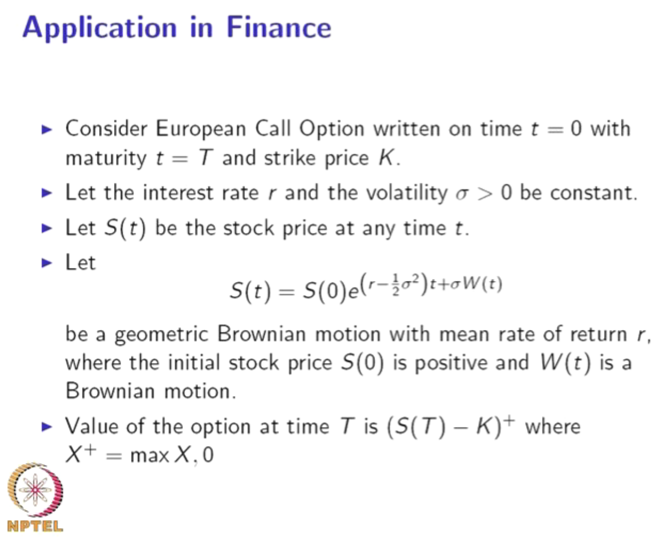


One can prove that S of t is log normal distribution because W t is normal distribution with mean 0. Variance t and this is of the form e power mu minus sigma square t plus sigma times W t. Therefore, the S of t will be lognormal distribution and also you can find out mean and variance. So, the mean will be expectation of S not e power mu t. Similarly, you can find out the expectation of S of t whole square using this two. We can find out the variance of S of t also. For this Black-Scholes model, we are finding the solution of S of t, also we are finding the distribution of S of t that is a log normal distribution. We know the mean and variance in terms of mu and sigma square.

We can also discuss, what is a asymptotic behavior of the solution because the solution is e power mu minus sigma square by 2 multiplied by t plus half time plus sigma times W t as a t tends to infinity, what happens? If mu is greater than sigma square by 2, if mu is greater than sigma square by 2 as t tends to infinity S of t will tends to infinity.

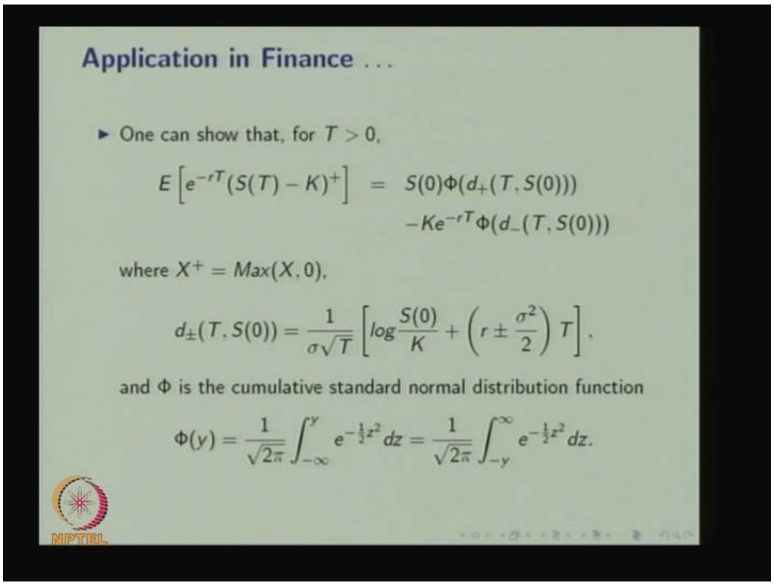
If mu is less than sigma square by 2, then as t tends to infinity S of t tends to 0 because the whole term is in the e power exponential of power. If mu is equal to sigma square by 2, as t tends to infinity S t will fluctuate between arbitrary large and arbitrary small value. Note that mu is, mu is a constant growth rate of the stock and sigma is a volatility. So, as t tends to infinity if based on these three conditions accordingly, S t tends to infinity or S t tends to 0 or S t will be fluctuating as t tends to infinity.

(Refer Slide Time: 12:24)



This is application in finance consider European call option written on t is equal to 0 with maturity t is equal to capital T and the strike price capital K. So, you can choose the mu is nothing but the interest rate with the notation r and the volatility is sigma that is the constant and the S t be the stock price. Therefore, S t the solution S of zero e power here replaced mu by r. Therefore, r is a interest rate here. So, r minus half sigma square t plus sigma times W t be the geometric Brownian motion with the mean rate of return r, where the initial stock price is S of 0 is positive and W t is a Brownian motion.

(Refer Slide Time: 13:59)

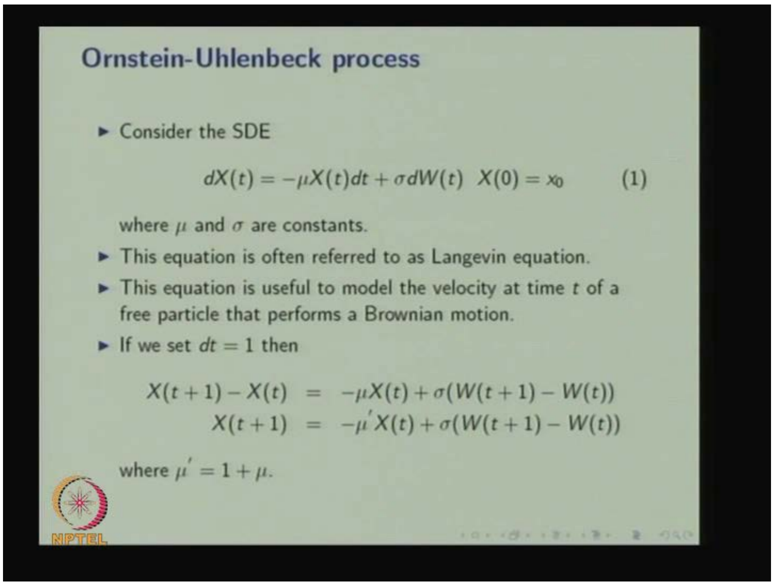


So, it has direct application in the finance. Suppose, we choose K is positive constant, which is strike price at the maturity date capital T, value of the option at time t at time capital T is S of capital T minus k superscript plus, where x superscript plus means maximum of x comma 0. Then, one can show that t is greater than 0.

The expectation of e power minus r times capital T, the random variable is S of capital T minus k power superscript plus that means is a maximum of x comma 0 for x superscript plus that means it will return the value 0 or whatever the x for the maximum. That is nothing but x of 0 psi of d suffix plus t comma S of 0 minus k times e power minus r of capital T psi of d minus T minus S of 0, where T plus and T minus are defined in this form and psi is the cumulative standard normal distribution function.

Psi of y is defined as a probability between minus infinity to till y in the standard normal distributed random variable. So, either minus infinity to y or minus y to infinity because of it is symmetric. So, it is expectation this is discounted 1 e power minus r T. So, the expected written will be calculated using the c d f of normal, standard normal distribution. You should know the strike price value capital K and maturity date capital T and also the interest rate r and the volatility sigma square also sigma also.

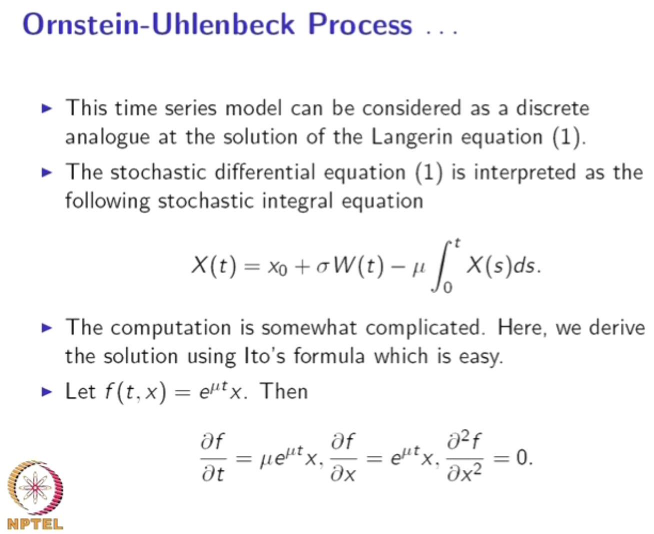
(Refer Slide Time: 15:50)



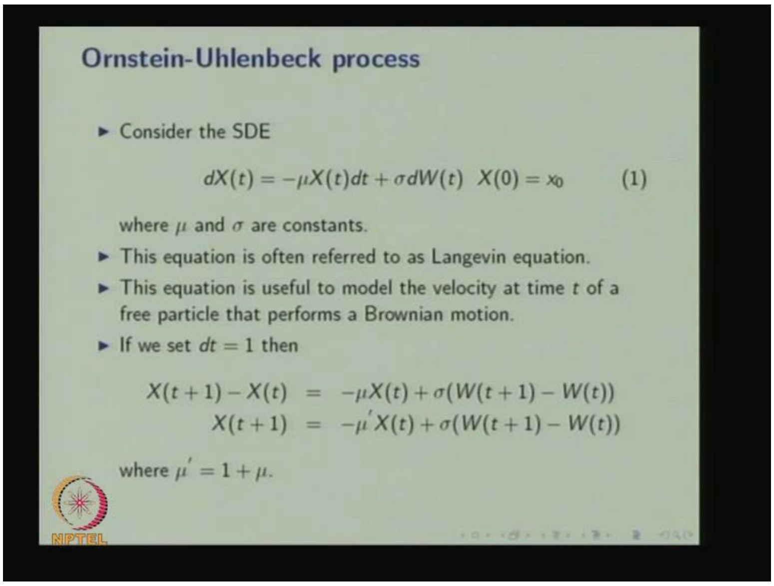
Now we are moving into the second model, that is Ornstein-Uhlenbeck process. Consider the stochastic differential equation d x t is equal to minus mu times x t d t plus sigma times d W t, where mu and sigma are constants. This equation is also referred as

Langevin equation. This equation is useful to model the velocity at time t of a free particle that performs a Brownian motion. If we described as the d t is equal to 1, then you will get the increment is nothing but x of t plus 1 minus x of t. That is d t is equal to 1. Therefore, this increment is W of t plus 1 minus W of t. Therefore, if you label mu dash t is equal to 1 plus mu, then you can transform the equation in the discretized version.

(Refer Slide Time: 17:01)

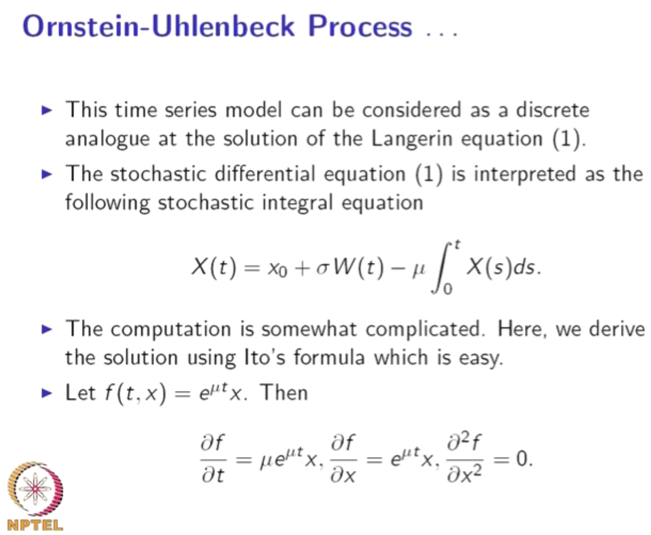


(Refer Slide Time: 17:32)



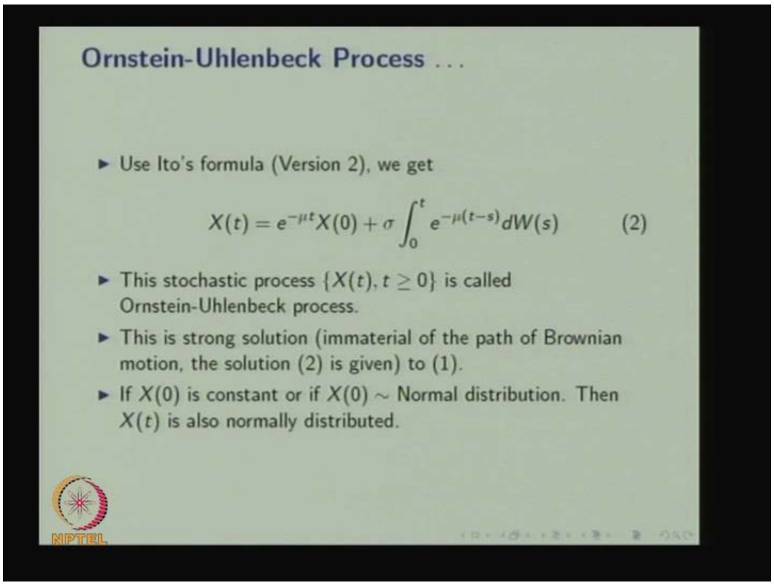
So, this time series model can be consider as a discrete analogue at the solution of the same equation 1. You can, you can in the you can develop the time series model. Then by discredizing the above stochastic differential equation and this stochastic differential equation can be interpret to the integral equation also. Because it is d x t is equal to minus mu times x t d t plus sigma times d W t.

(Refer Slide Time: 17:38)



Therefore you can write down this has the x t is equal to x at 0 plus sigma times W t minus mu times integration 0 to t x of S d S x of S x of t is unknown. Therefore, this is an integral equation. It is very difficult to get the solution, but now here we are using the ito formula, which is easy to get the solution. Assume that the solution is of the form e power mu t x. So, use the version 2 of ito formula; that means you have to find out first derivative with respect to t, first derivative with respect to x, second derivative with respect to x. Find out those things, then substitute, substitute in the equation.

(Refer Slide Time: 18:34)

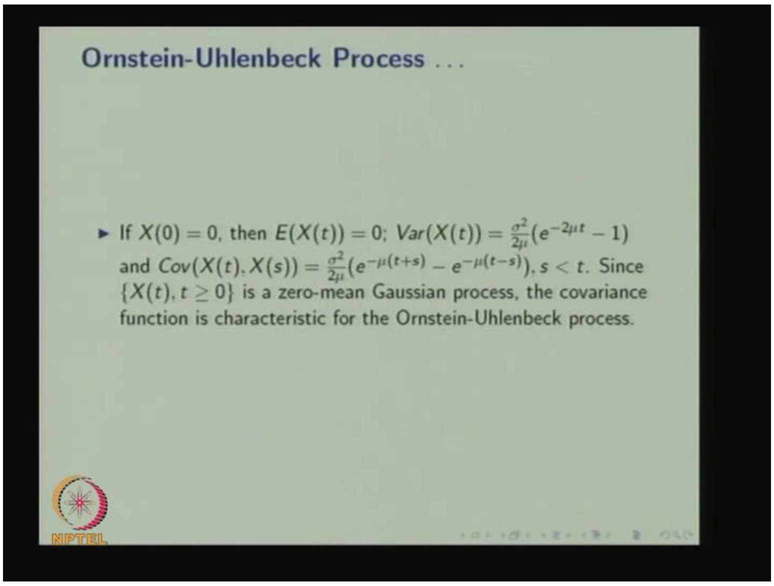


Then you can get the solution of x of t, that is x of t is equal to e power minus mu t x of 0 plus sigma times the integration 0 to t e power minus mu t minus S d W S. Note that this is a ito integral because this is a integration with respect to W S, but the integrant is non random function of non random function of time. Therefore, it is easy to get the solution, we have already discussed as a example how to solve the ito integral the integrant is a non random function?

So, you can refer that example and you can get the solution and this is strong solution immaterial of the path of Brownian motion, the solution for the equation one. We know that the, this is whenever the integrant is a non random function; the ito integral with the deterministic integrant is normally distributed with the mean 0 and the variance in in terms of integration.

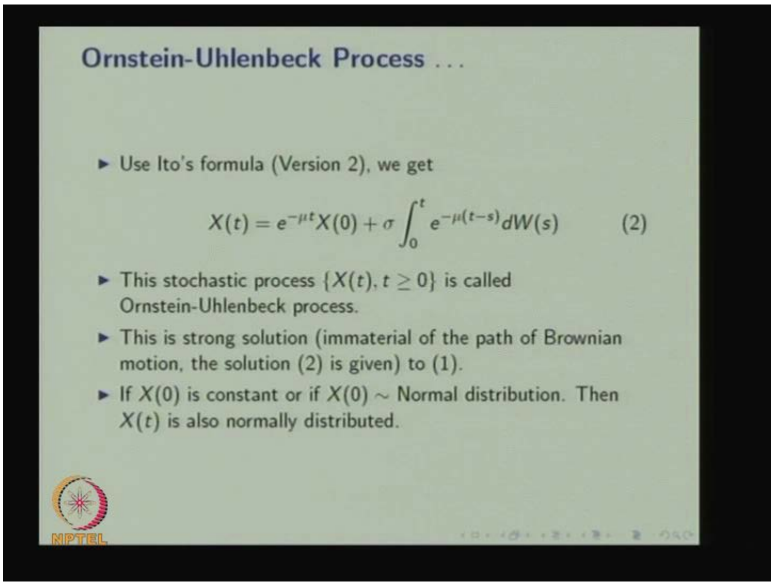
So, you can conclude when x of t is constant or x of 0 is normally distributed random variable, normally distributed random variable, you can conclude x of t is also normal distributed random variable. If x of 0 is constant or if x of 0 is normal distributed random variable, then x of t, one can prove that is also normally distributed random variable. You have to use the ito integral of non random function of time.

(Refer Slide Time: 20:43)



So, you can find if x of 0 is equal to 0, you can find the mean of x of t.

(Refer Slide Time: 20:55)



Mean of x of t, so this will be 0 and this is normally distributed, normally distributed random variable with mean 0.

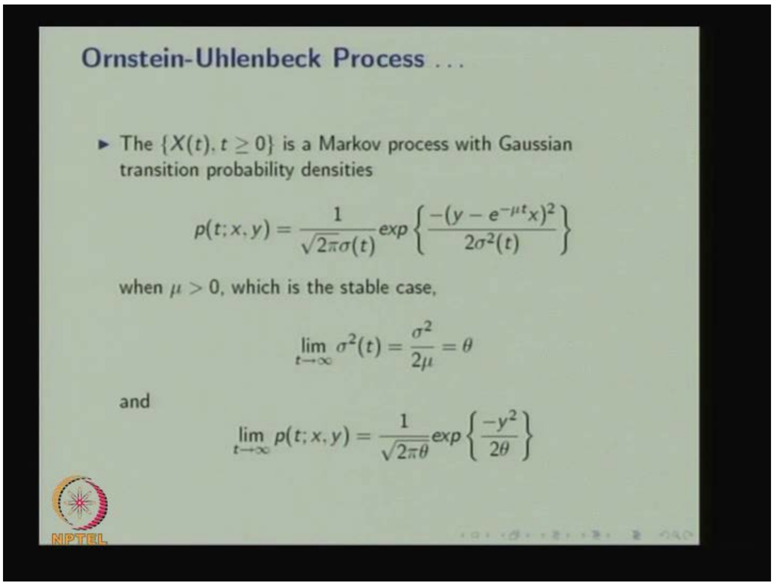
(Refer Slide Time: 21:04)

Ornstein-Uhlenbeck Process ...

- If $X(0) = 0$, then $E(X(t)) = 0$; $Var(X(t)) = \frac{\sigma^2}{2\mu}(e^{-2\mu t} - 1)$ and $Cov(X(t), X(s)) = \frac{\sigma^2}{2\mu}(e^{-\mu(t+s)} - e^{-\mu(t-s)}), s < t$.

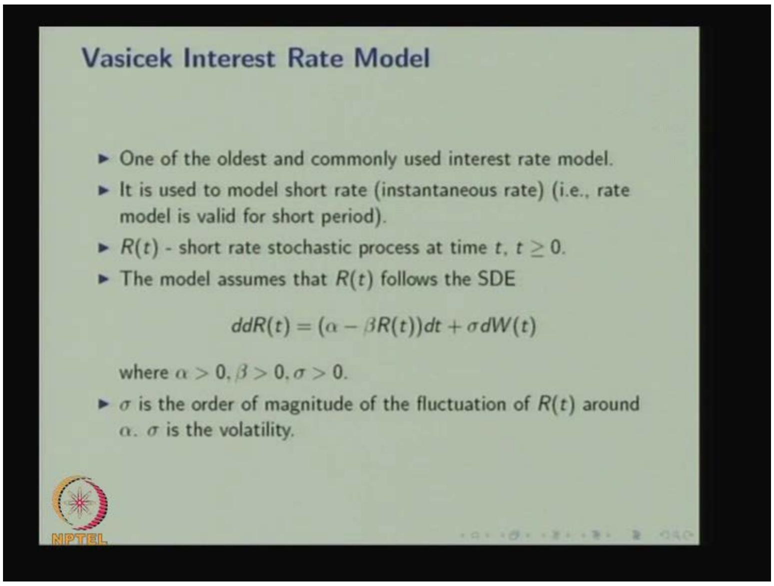
Therefore, the mean of x (t) will be 0 and the variance also one can find. Before that you should find out the expectation of x (t) whole square using the formula of variance of x t is equal to expectation of x (t) whole square minus expectation of t whole square. You can find out the variance of x (t) and also you can find out the covariance between two random variables x of t and x of S.

(Refer Slide Time: 21:37)



Also, we can prove this, this solution x t is a Markov process with a Gaussian transition probability density is in this form. The stable case hast tends to infinity you can make sigma square t is nothing but some theta, then you can find out the limiting distribution the transition probability density, that is standard normal distribution, sorry that is normal distribution with the mean 0 and the variance sigma the variance theta. So, you can you can prove the x t is a Markov process with Gaussian transition probability density, as well as t tends to infinity, you can prove the transition probability. So, the normal this is normal distributed random variable with the mean 0 and variance theta.

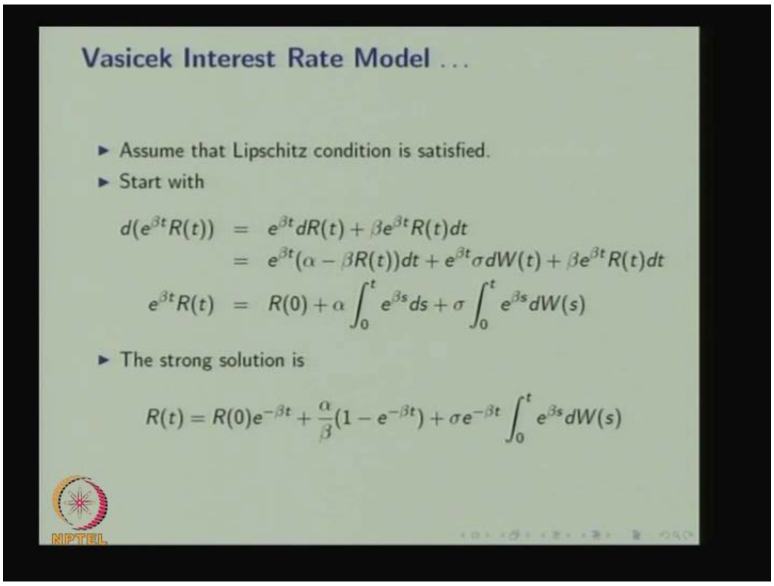
(Refer Slide Time: 22:52)



Now, we are moving into the third important stochastic differential equation. This is the one of the oldest and commonly used interest rate model Vasicek interest rate model. It is used to model short rate, that is the instantaneous rate; that is the rate model is valid for a short period. So, suppose capital R of t that is a short rate stochastic process at time t, then this model assumes the following stochastic differential equation.

That is d r t is equal to alpha minus beta times r of t multiplied by d t plus sigma d W t, where the alpha greater than 0, beta greater than 0, sigma greater than 0. Sigma is the order of magnitude of the fluctuation of r of t around alpha sigma is the volatility. So, we are going to find the solution of this stochastic differential equation.

(Refer Slide Time: 24:11)

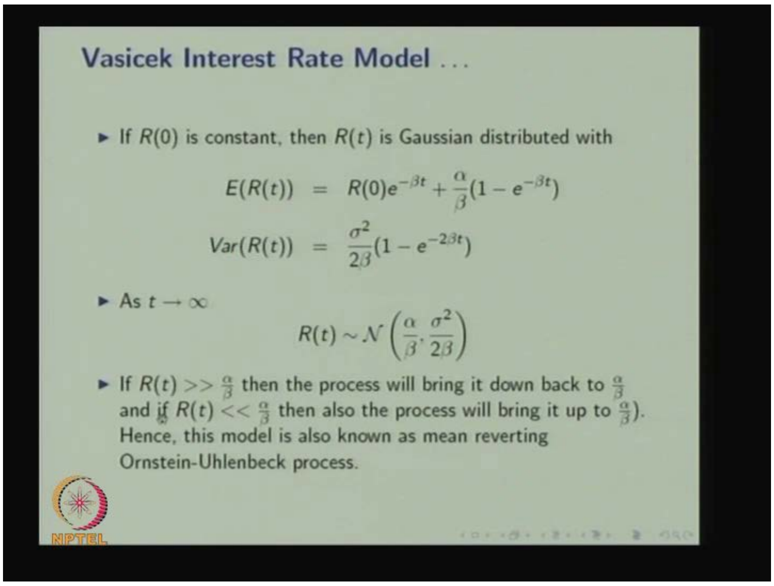


Assume that Lipschitz condition is satisfied, so here we are not going to use ito's formula, but we are going to use the increment concept. So, increment of e power b t R t will have 2 terms. Then substitute d of R t will form the stochastic differential equation. The previous is stochastic differential equation. We substitute d R t from here in this place. So, it is e power b t d R t is replaced by the given stochastic differential equation, so that is nothing but this one plus beta times e power beta t R t d t.

That is nothing but some terms will be cancelled out, so you will get e power beta t R t is nothing but because this is a differential form, so we are making a integral form. Therefore, e power beta t R t will be R of 0 alpha times integration 0 to t e power beta s d s. So, this is a Riemann integral, sigma times integration 0 to t e power b s d W s. This is a Riemann, this is a ito integral, but the integrant is a non random function of time. Therefore, this is a strong solution for a given stochastic differential equation.

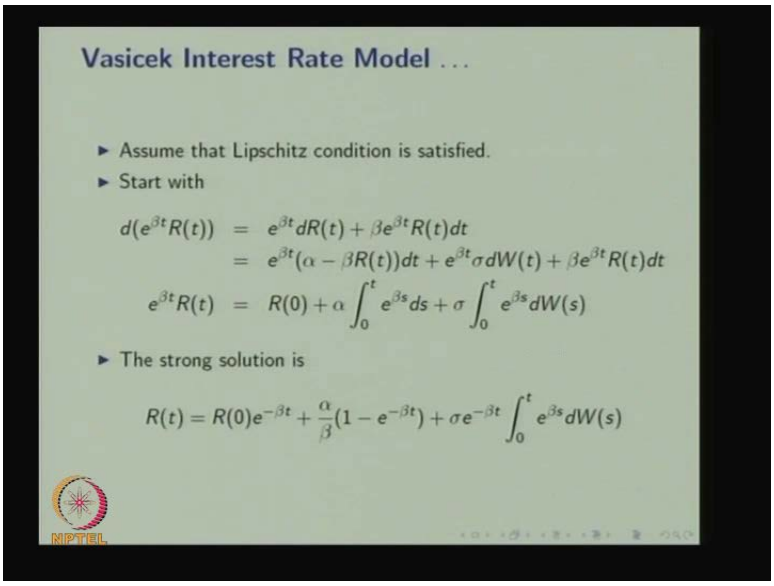
Therefore, the R of t will be R of 0 e power minus beta t plus this term and the last term, so this is a ito integral, but the integrant is the nonrandom function. You know that ito integral with non random function of time has a integrant, that will be a normal distributed random variable. Therefore, you can find out the distribution of R of t also.

(Refer Slide Time: 26:19)



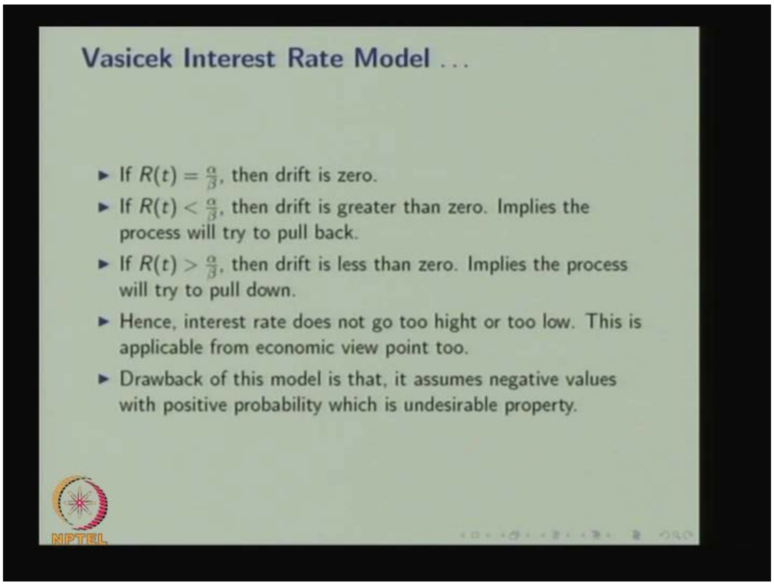
So, if R of 0 is constant, then you can find out the distribution of R of t which is Gaussian or normal distributed random variable with mean and variance is given in this form.

(Refer Slide Time: 26:37)



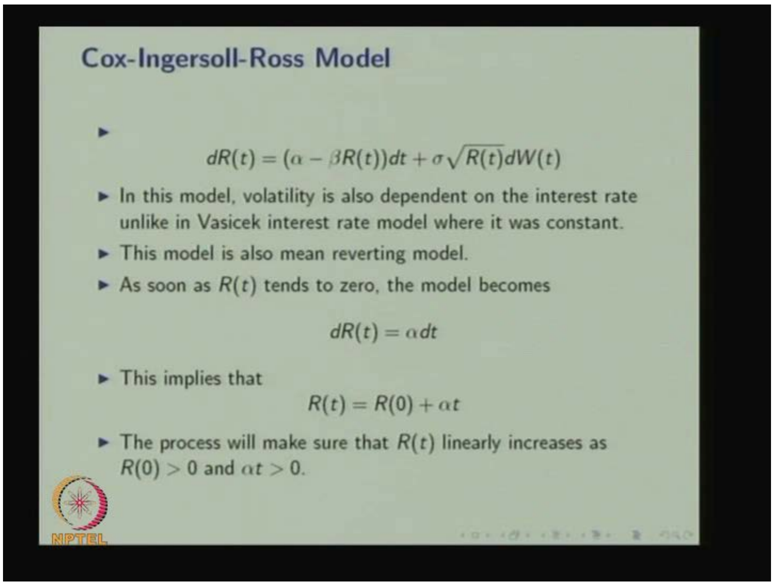
One can find out the mean and variance from. The, this solution term by term we can find out and we can find out the mean of R of t. So, here you can find the variance of R of t and you can discuss as t tends to infinity what is a distribution of R of t also. When R of t is much much greater than alpha by beta, then the process will bring down back to alpha by beta. If R of t is very very smaller comparing with alpha by beta, then also the process will bring it up to the alpha by beta. That means this model is known as mean reverting OU process because of this property. If R of t is much much higher than alpha by beta, the process will bring it down back to the alpha by beta. If R of t is much much lower than alpha by beta, then also the process will bring it up to the alpha by beta. Therefore, it is the it is called the mean reverting OU process.

(Refer Slide Time: 28:00)



If R of t is equal to alpha by beta, then the drift is 0. If R of t is less than alpha by beta then the drift is greater than 0, implies the process will try to pull back. If R of t is greater than alpha by beta, then the drift is less than 0 implies the process will try to pull down. So, these all are the behavior based on R of t is with the alpha by beta. Hence, interest rate does not go too high or too low. So, this is applicable from a economic view point also. The drawback of this model is that it assumes negative values with positive probability, which is undesirable property. The Vasicek interest rate model is a mean reverting OU process, but the drawback is it takes negative value with the positive probability which is undesirable property.

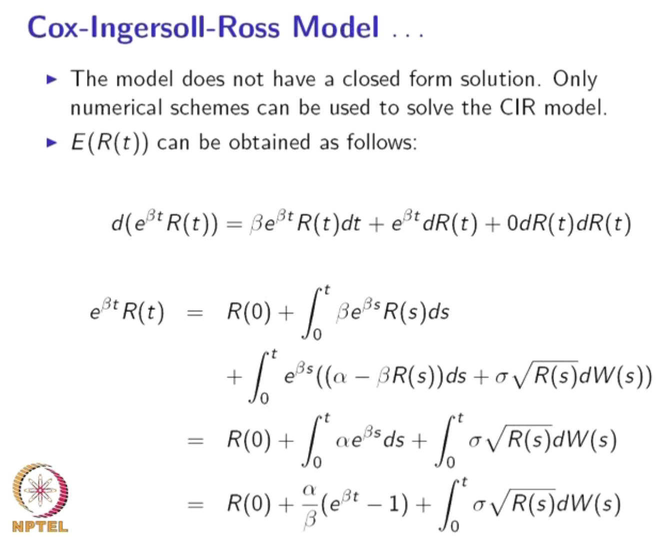
(Refer Slide Time: 29:15)



Now, we are moving into the CIR model, this is also the interest rate model and this also has application in financial mathematics. So, the underlying stochastic differential equation is d R t is equal to alpha minus beta times R t d t plus sigma square root of R t d W t. You cannot use the ito formula to solve the stochastic differential equation. This stochastic differential equation, but we are finding the solution to the increments. In this model the volatility is also depends on the interest rate. Earlier the volatility here it is sigma, but here it is sigma times square root of R t.

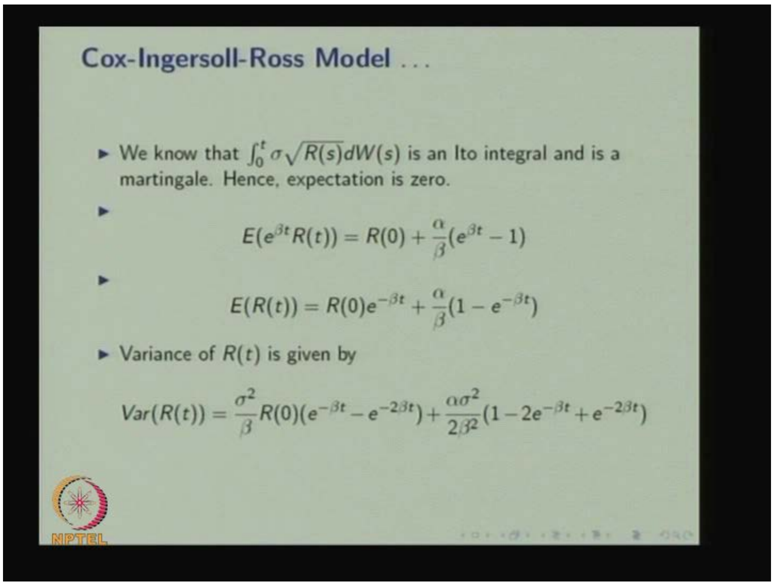
So, the volatility also depends on the interest rate unlike Vasicek interest model where it was constant. This model is also mean reverting model because of this alpha minus beta times R t d t. Therefore, this model is also called a mean reverting model. As soon as R t tends to 0, the model becomes d R t is equal to alpha times d t. This implies R of t, if you solve this equation you will get R of t is equal to R of 0 plus alpha t. That means this process will make sure that R of t linearly increases as R of 0 is greater than 0 and alpha times t is greater than 0. So, this is the observation as R t tends to 0.

(Refer Slide Time: 31:12)



Now, we are discussing the solution. This model does not have a closed form solution only the numerical schemes can be used to solve the CIR model. So, we are finding what is the expectation of R of t; so, we cannot use a ito formula to get the solution, only numerical schemes schemes can be used to solve the CIR model. Now, we are finding the expectation of R of t, for that we are taking the increment of e power beta t R t, get the terms. Then substitute R of d of R t from the previous equation.

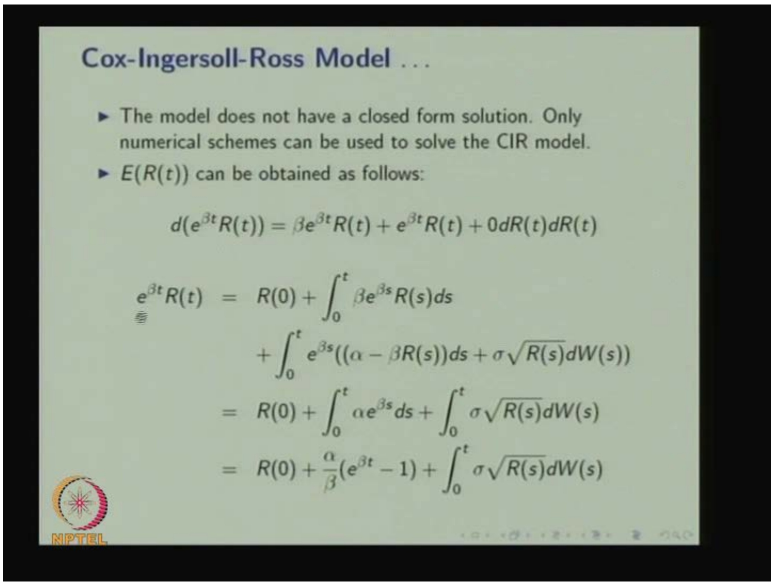
(Refer Slide Time: 32:38)



Therefore, you will get e power by converting into the ito integral, sorry by converting into integral equation you will get e power beta t b t R t is equal to R of 0 integration beta to times e power beta s R s d s.

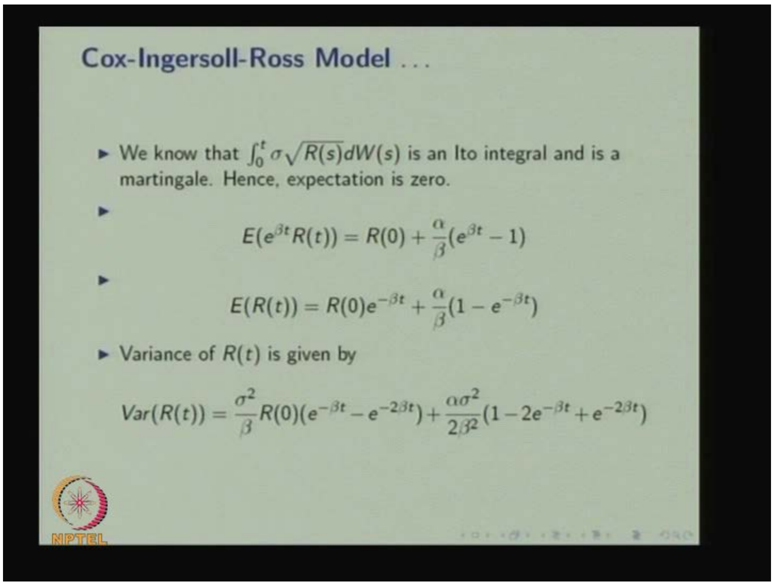
Substitute you can get the mean of, we know that… So, basically we should find a mean and variance of the ito integral 0 to t sigma times square root of R of s d w s. Since, this is ito integral that is also a martingale. Hence, expectation will be 0 because this integration, if you make it as i of t. Since, it is a martingale the expectation of i of t is same as expectation of i of 0, and the expectation of i of 0 is 0. Therefore, the expectation of i of t will be 0.

(Refer Slide Time: 33:25)



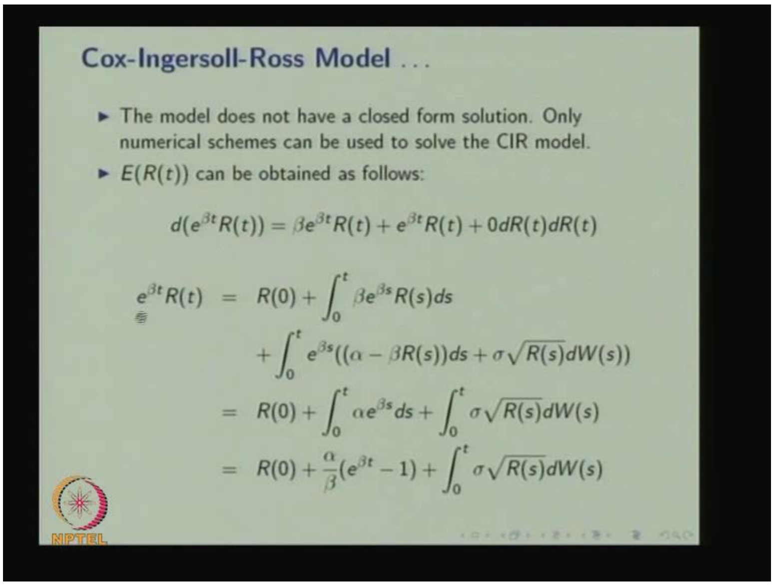
Therefore, from the previous equation expectation of R t will be R of 0 e power minus beta t. This term multiplied by e power minus beta t this whole ito integral multiplied by e power minus beta t. Since, this expectation is 0, therefore, expectation of R of t with the multiplication e power b t beta t will be R of 0 plus alpha by beta e power beta t minus 1.

(Refer Slide Time: 33:46)



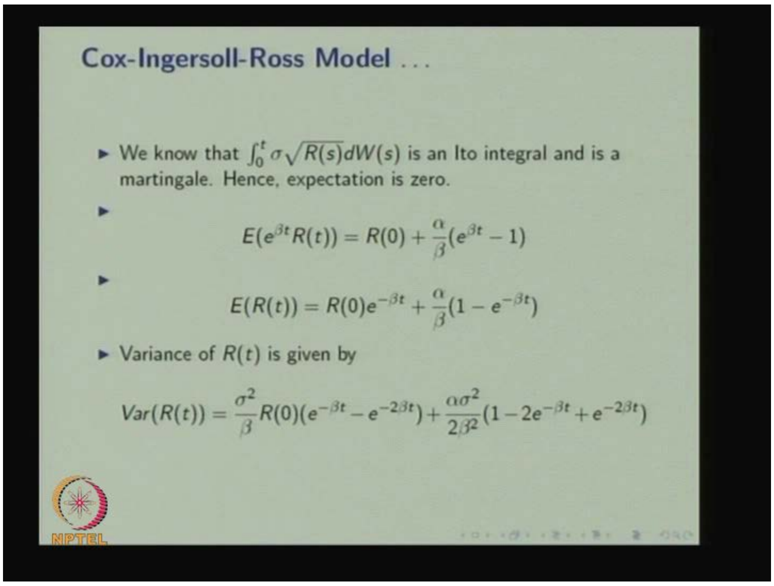
Therefore, expectation of R of t will be multiplied this side with the e power minus beta t, so nearly one can find out the variance of R of t.

(Refer Slide Time: 34:16)



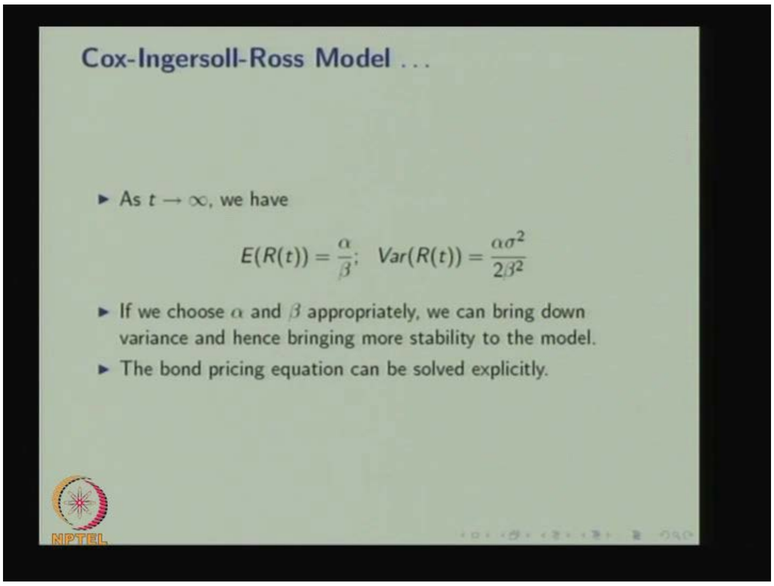
For that we have to find out expectation of R of t whole square. So, for that you have to find out the increment.

(Refer Slide Time: 34:21)

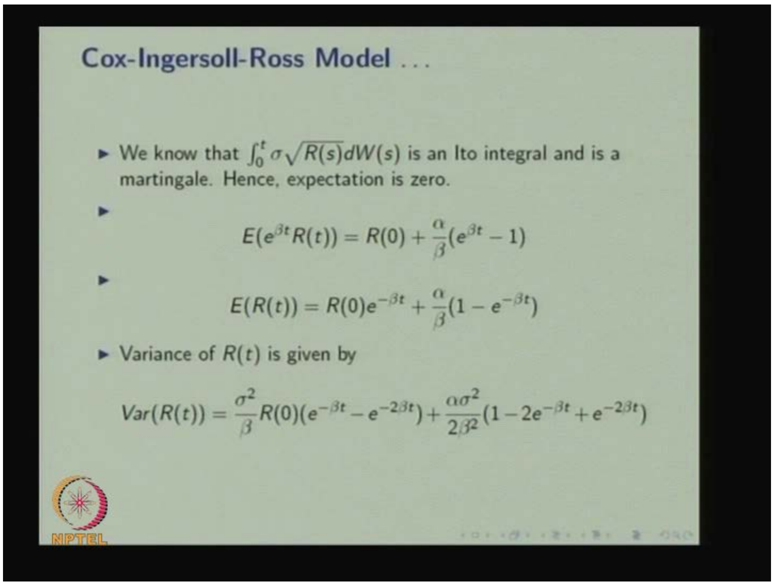


So, on secondary increment, then find out the expectation of R of t whole square, then you can find out the variance of R of t. So in this c, CIR model you have not giving the closed form solution because the ito formula does not satisfy, does not work, whereas, we are finding the mean and variance of R of t, using the using the mean and variance of ito integral. Now, we are discussing as t tends to infinity, what happens?

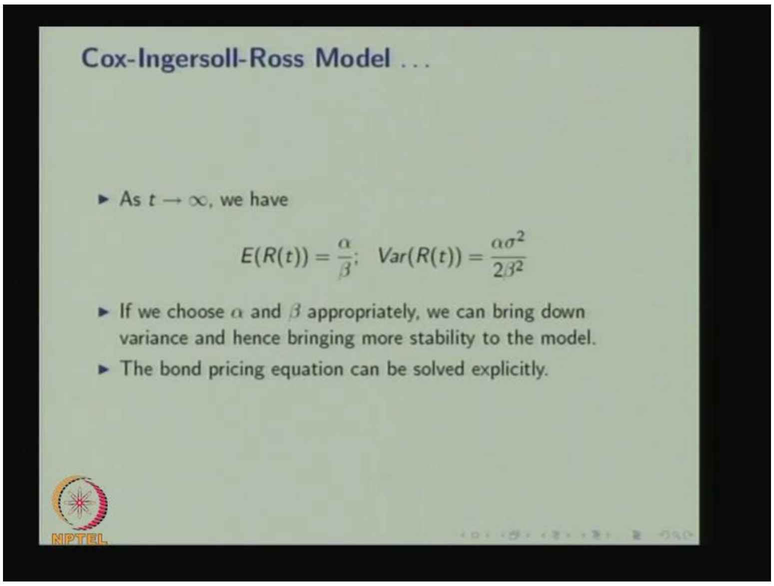
(Refer Slide Time: 35:00)



(Refer Slide Time: 35:10)

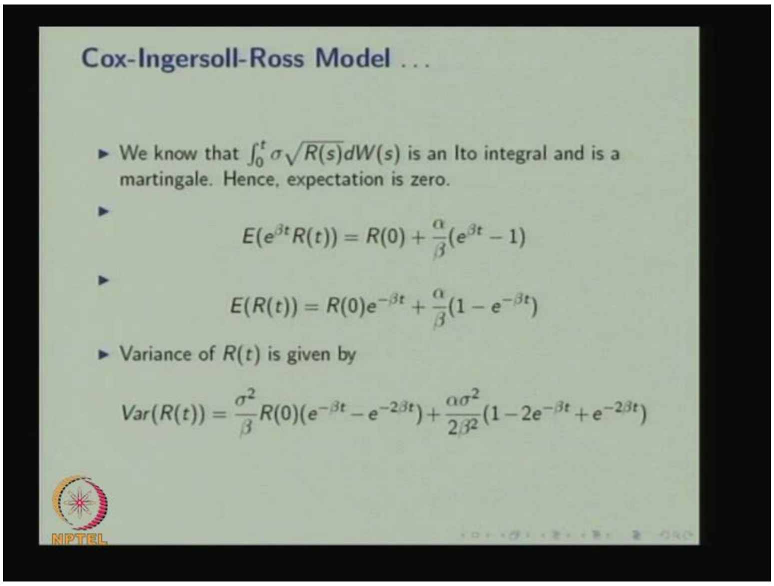


(Refer Slide Time: 35:20)



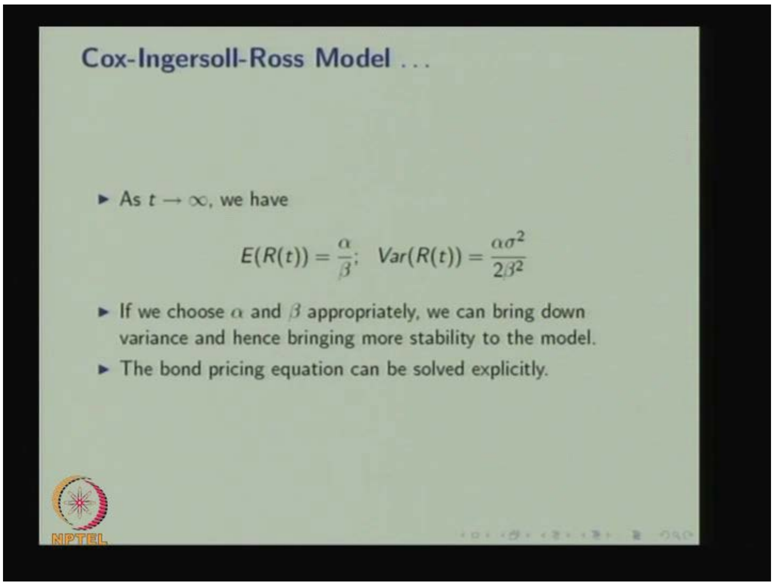
As a t tends to infinity from this equation you can make out this is expectation of R of t, as t tends to infinity, you will get with the assumption R of 0 is equal to 0. So, expectation of R of t is equal to alpha by beta. As t tends to infinity with R of 0 is equal to 0 expectation of t will be lambda by, sorry alpha by beta.

(Refer Slide Time: 35:43)



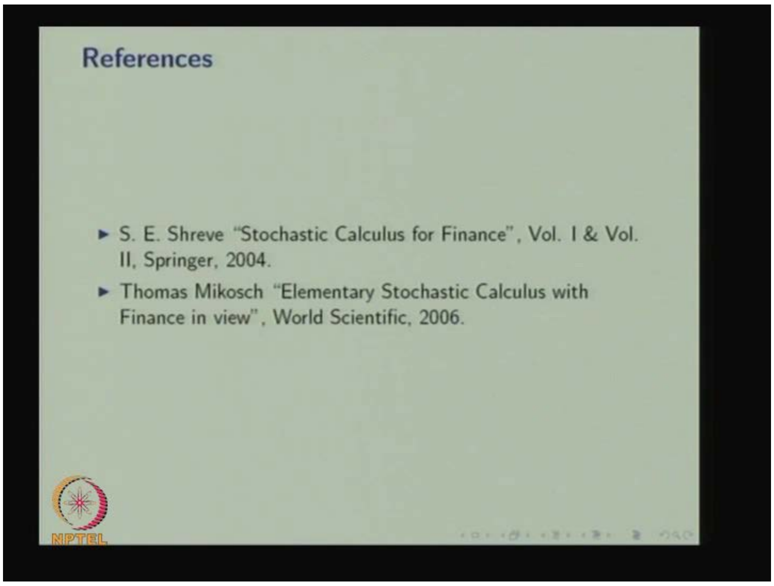
Similarly, you can find out the variance of R of t that is as t tends to infinity, you will get this term and R of 0 is equal to 0.

(Refer Slide Time: 35:52)



Therefore, you will get alpha times sigma square by 2 beta square. So, this is a symptotic behavior of R of t with the assumption R of 0 is equal to 0. If we choose alpha and beta appropriately, we can bring down variance because the variance is longer run, variance is in terms of alpha and beta. So, you can bring down the variance and hence bringing more stability to the model. You can make the model stable by properly choosing alpha and beta. This CIR model has lot of application in particular, the bond pricing model. So, using CIR model the bond pricing equation can be solved explicitly.

(Refer Slide Time: 37:01)



With this, we have completed 6 lectures in Brownian motion and its application, starting with the Brownian motion, definition and the properties. Then process derived from the Brownian motion. In particular we have discussed geometric Brownian motion, Levy processes and in the third lecture, we have discussed stochastic differential equation. The fourth lecture we have discussed ito integral andfew properties of ito integral. We have discussed few examples also in the ito integrals.

The fourth lecture we have discussed, in the the fourth lecture we have discusses ito integral and the fifth lecture we have discussed the ito formula. That is useful to solve some special cases of stochastic differential equation. In the last lecture, we have discussed some important stochastic differential equation and their solutions. Here the important means this stochastic differential equations are having applications in mathematical finance.

The first model is a Black-Scholes model we have discussed, the second model is OU process and the third and fourth one are the models for the interest rate. Some standard differential equations, we solved using this ito formula and in one model. We have not given the closed form solution, which does not exist in that case. We have discussed the mean and variance and a asymptotic behavior, with this the model 7 Brownian motion and its application is completed.