Stochastic Processes
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Module - 7
Brownian Motion and its Applications
Lecture - 5
Ito Formula and its Variants

This is stochastic processes module 7 Brownian motion and its applications; this is lecture 5 Ito formulas and its variance. In the first lecture, we have discussed the Brownian motion definition and properties. In the second lecture, we have discussed the process derived from the Brownian motion. In the third lecture, we have discussed stochastic differential equation. In the fourth lecture, we have discussed the Ito integrals; and in all the four lectures, we have seen few examples also as the applications.

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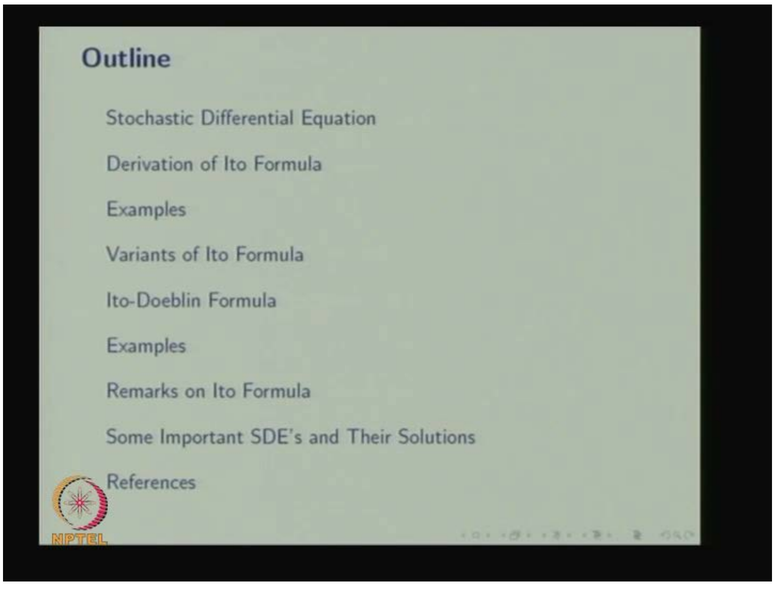

In the lecture 5, we are going to discuss the derivation of a Ito formula. Then, we are going to give few examples. Followed by the examples, we are going to discuss the variants of Ito formula in particular Ito Doeblin formula. Then, we are going to give a 2, 3 examples also and followed by examples we are going to discuss the remarks on Ito formula. In the next lecture we are going to discuss some important SDE's and their solutions.

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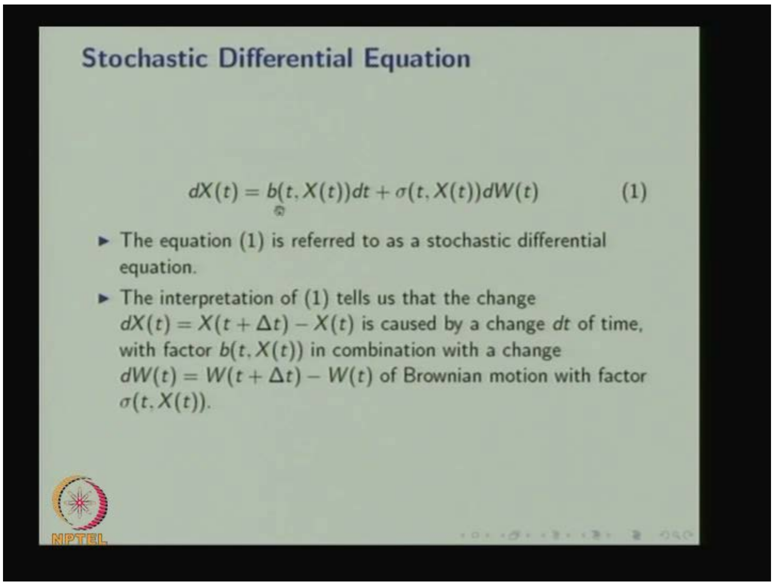

Recall, what is the meaning of stochastic differential equation? A differential equation is of the form d X(t) is equal to b function of t, X(t) multiplied by d t plus some other function that is sigma of t, X(t) multiplied by d of W(t). You know that W(t) that collection is stochastic process that is a standard Brownian motion, which is nowhere differentiable and unbounded variation.

Therefore, this differential equation is a different from of the ordinary differential equation. And, the differential equation with the term d W(t) this type of equation is called a stochastic differential equation. So, the interpretation of one tells us that the change d of X(t) that is nothing but in t to t plus delta t. The change is X plus X(t) plus delta t minus X(t) that is caused by the change dt of time with the factor b of t, Xt in combination with the a change dW(t) that is in the interval t to t plus delta t the change is W of t plus delta t minus W of t that is of Brownian motion with the factor sigma of t, X(t). So, in this lecture we are going to discuss; how one can solve the stochastic differential equation of some special cases using Ito formula.

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Informal Derivation of Ito Formula

- Assume that $X(t) = f(W(t))$.
- Assume that f is atleast thrice differentiable.
- Borrowing ideas from the classical Taylor series, we can informally write:

$$
\begin{aligned}
f(W(t+\Delta t)) &= f(W(t)) + (W(t+\Delta t) - W(t))f'(W(t)) \\
&\quad + \frac{1}{2!}(W(t+\Delta t) - W(t))^2 f''(W(t)) + \ldots
\end{aligned}
$$

- We know that

$$dW(t)dW(t) = dt; dW(t)dt = 0; dtdt = 0$$

The derivation of a Ito formula is as follows. You assume that the real valued function X(t) has function of t is a function of W(t); where W(t) is a standard Brownian motion. Assume that f is at least thrice differentiable function borrowing idea from classical Taylor series. We can informally write series expansion of the function f of dw f(Wt). So, that means f(W(t) plus delta t minus f(W(t)) is equal to you get the difference. Then, the first derivative of the function f(t) f dash plus 1 divided by 2 factorial the difference whole square.

The second derivative of the function f plus 1 by third factorial difference whole cube f 3 factorial and f power 3 third derivative of the function and so on. So, this is the Taylor series expansion of the function f with assumption that f is twice differential.

Our interest is to solve this stochastic differential equation. So, for that we are making we need solution X(t) it satisfies this equation. So, for that we are making the assumption X(t) is going to be a function of W(t) and also we are making the assumption f is twice differential. Now, we are using the Taylor's theorem. So, that we are able to get this Taylor's series expansion. Since, W(t) is Brownian motion. So, we know that dw(t) is equal to dt that is nothing but the quadratic variation of W(t) is dt. W the quadratic variation of W(t) is t in the interval 0 to small t. Whereas, the cross variation of W(t) with t that will be 0. Similarly, the quadratic variation of t will be 0. So, we are using this in the above Taylor series expansion. Therefore, we would not have the third and fourth terms and so on.

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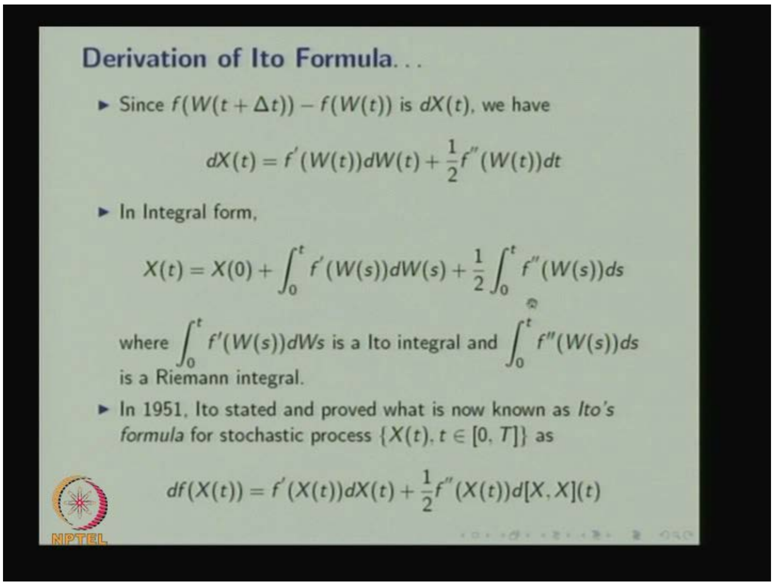

So, that derivative those terms will be vanishing. So, we will have the dx(t) that is nothing but the f(W(t)) plus delta t minus f(W(t)) that is dX(t) that is nothing, but f dash with dW(t) that is nothing but if f dash W(t) the increment that is nothing, but the d of W(t). Therefore, this difference will come this side that would be dx(t). So, the dX(t) is equal to f dash dW(t) 1 divided by 2 this is dW(t) multiplied by t W(t) f double dash. We know that dw(t) is equal to dt.

Therefore, the second term will be dt this is because of dW(t) is equal to dt whereas, the further terms will vanish. So, this can be written in the integral form that is X(t) is equal to X(0) plus 0 to t f dash W(s) dW(s) plus 1 by 2 0 integration 0 to t; f double dash W(s)ds. In given that these 2 integrals; the first integral is a Ito integral because f is a twice differentiable. So, f dash of W(s) integrant function of W(s) the integrant is f dash of W(s) and integration with respect to W(s). You can see the conditions of Ito integral that is satisfied. Therefore, this is going to be a Ito integral whereas, the second integral integration with respect to s therefore this is a Riemann integral.

Now, I am going for the general definition of Ito 1 that is in 1951 Ito stated and proved. What is known as a Ito formula? For the stochastic process X(t) that is nothing but d(f(X(t))) is equal to f dash of X(t) dx(t) half f double dash of X(t) the quadratic variation of X of interval 0 to t for the stochastic process X(t). You can make a Ito formula that is nothing but the increment of a function is same as f dash dx(t) plus 1 by 2 f double dash the quadratic variation of the interval 0 to t of the function X.

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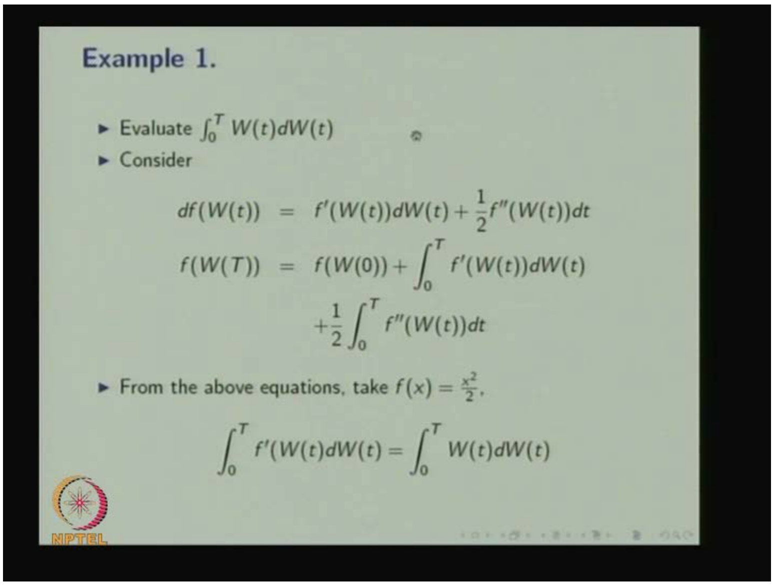

Let see the simple example. Remember that the Ito formula is valid for only special cases of stochastic differential equation not for all the not for every stochastic differential equation can be solved using the Ito formula. Let us evaluate the integration 0 to T; W(t) dW(t) integrant is W(t) integration with respect to W(t). Therefore, this is a Ito integral; you can verify the conditions of Ito integral then W(t) is adapted process with respect to the filtration f(t). And, also this is mean square integral therefore this is a Ito integral. So, you can evaluate this Ito integral.

So, instead of evaluating this Ito integral by the definition that we have discussing the lecture 4 that is nothing, but m into n tends to infinity the summation i is equal to 0 n minus one W of si multiplied by W of si plus 1 minus w of si. That is the way you can evaluate the Ito integral by definition.

But here we are going to evaluate the same integral using the Ito formula. How? Let us see the Ito formula the increment of f is f dash dW(t) plus half time of f double dash W(t) dt. This can be written in the integral form f(W(t)) minus f(W(0)) that is equal plus integration 0 to T; f dash dw(t) plus 1 by 2 integration 0 to T f double dash.

We here we are evaluating 0 to T. Therefore, this is function evaluated at the point T is equal to function evaluated at the point 0 plus integration 0 to T plus half times integration 0 to T. So, if you want to solve this equation you can compare that equation with the integral form. Then, you can find out what should be the function? Such that f dash of W(t) will becomes W(t). If you want to solve the integral equation Ito integrals or the corresponding stochastic differential equation. We have to use Ito formula such a way that the integral the Ito integrals are going to be matched.

So, you have to find down the function f such that this integration is same as this integration. That means, you have to find out the function f such that f dash of W(t) is equal to W(t). That means, f is going to be X square by 2 then only the f dash of W(t) will becomes W(t). So, take f(X) is equal to X square now substitute in the above Ito integral form the Ito formula in integral form. Therefore, you can evaluate this integration without actually evaluating the Ito integral. So, by choosing the function f(X) is equal to X square by 2 this Ito integral is matched with this Ito integral.

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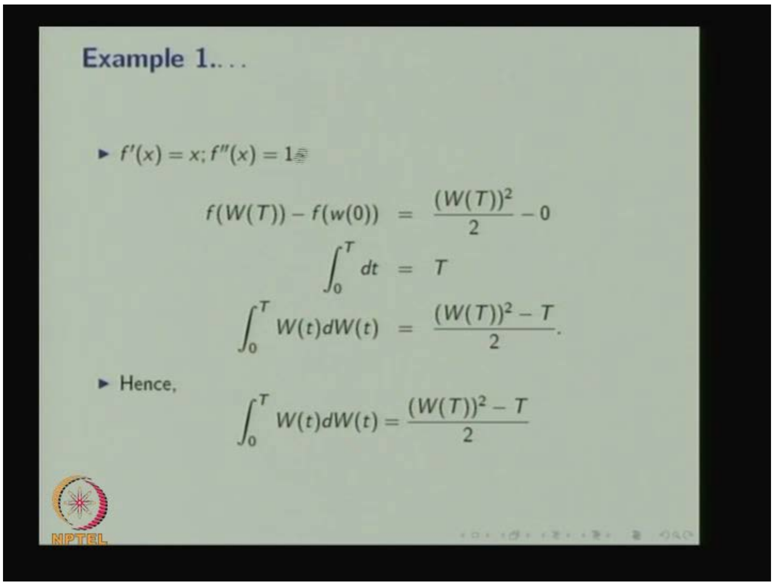

Therefore, f dash(X) equal to X; f double dash of x equal to 0 because f(X) is equal to X square by 2. Therefore, if you see the integral form of Ito formula f of W(T) minus f of W(0) is equal to integration 0 to T; that is the required integration W(t) dW(t) plus half times this is a Riemann integral. That means, the Ito formula which connects the increments and Riemann integral includes the Ito integral.

Therefore, Ito integral can be evaluated using the Ito formula. So, for that you have to evaluate the Riemann integral and the value evaluated at the both the limit; upper limits and the lower limit. Hence, f of W(t) f(X) is X square. Therefore, it is W(t) whole square by 2 minus W(0) whole square by 2 is equal to this integration plus half the Riemann integral. So, the Riemann integral is nothing, but 0 to t; f double dash is equal to 1 therefore a Riemann integral is 0 to T 1 dt.

So, that is nothing, but therefore, the Ito integral is nothing but W(T) whole square minus T by 2. So, in this example we are not actually evaluating the Ito integral using the definition, but we are using the Ito formula to evaluate the Ito integral or the corresponding stochastic differential equation.

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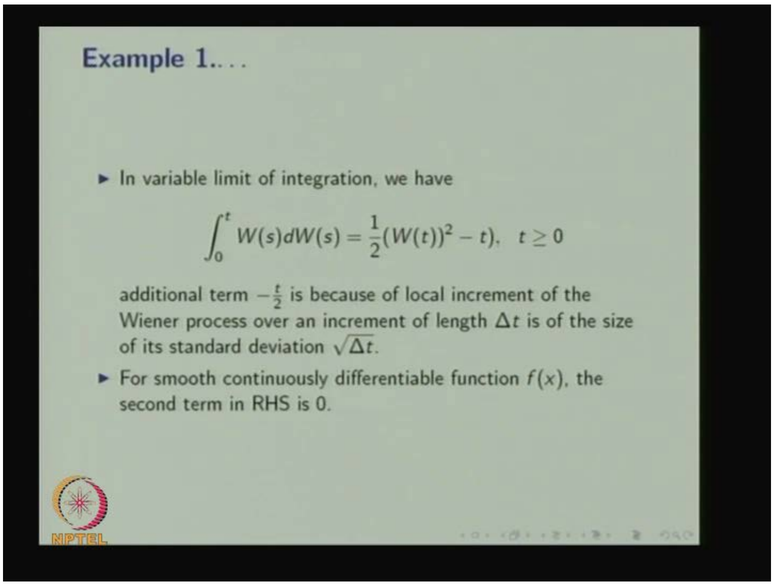

Instead of integration from 0 to T one can go for the variable limit t also in that case it is one divided by 2 W(t) whole square minus t for t greater than or equal to

0. The additional term minus t by 2 is because of the local increment of the wiener process over the increment of the length delta t is of the size of it is standard deviation square root of delta t. Because this is a Ito integral the extra terms becomes because of the vernier process quadratic variation is t. If it is a Riemann integral then it is simply 1 by 2 W(t) whole square. So, for smooth continuously differentiable function f(X) the second term in the right hand side is 0. So, this term will be 0. This term will be 0 if the integration with respect to the function which is smooth continuously differential.

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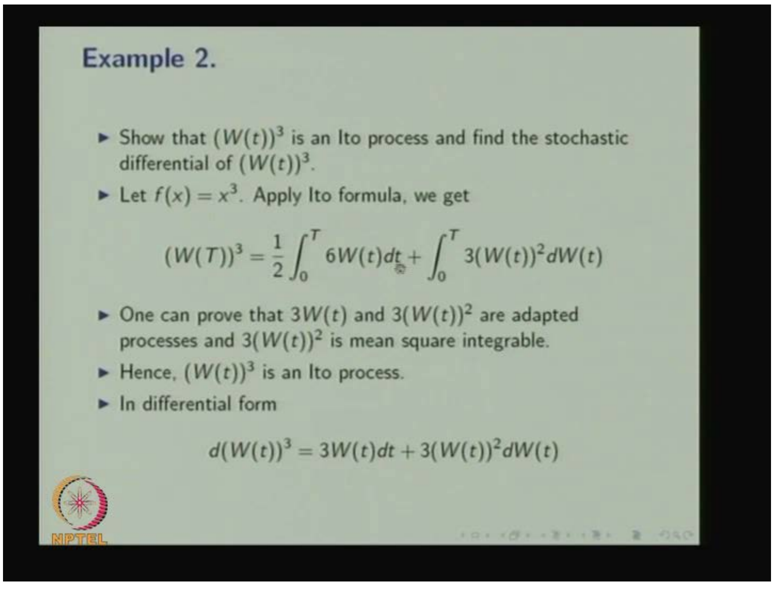

Now, we are moving into the second example. In this example we are going to show that W(t) whole cube is a Ito process and also it is asking find the stochastic differential of W(t) whole cube. So, you can choose the function f(S); x is equal to X cube we can apply the Ito formula.

In that W(t) whole cube is equal to 1 by 2 integration 0 to T 6 times W(t) dt plus integration 0 to T 3 times W(t) the whole square dW(t). So, this is the Ito integral and this is a Riemann integral. Therefore, the differential form will be d(W(t)) whole cube that is nothing, but 3 times W(t) dt plus 3 times W(t) whole square dW(t) that is a differential form; the integral form is this one.

You know that W(t) is a adapted process. Therefore, 3 times W(t) whole square that is also adapted process. And, also 3 times Wt whole square is mean square integral because the W(t) is a standard Brownian motion. The increments are independent stationary and increments are normal distributed random variable with the mean 0 and variance t.

Therefore, the 3 times W(t) whole square that is also mean square integral. Therefore, you can conclude this is a Ito process. The W(t) whole cube that is a Ito process. So, instead of the fixed limit T if you go for the variable limit W(t) whole cube then the integration is 1 by 2 0 to t 6 W(s) ds plus 0 to t 3 W(s) whole squared W(s) that is going to be the Ito process.

In the corresponding differential stochastic differential equation is d(W(t)) whole cube is equal to 3 times W(t) dt plus 3 times w(t) whole square dW(t). So, in the lecture 4 we have discussed when we can say the given stochastic process is a Ito process. So, you have to verify those conditions are satisfied then you can concluded W(t) whole cube is a Ito process.

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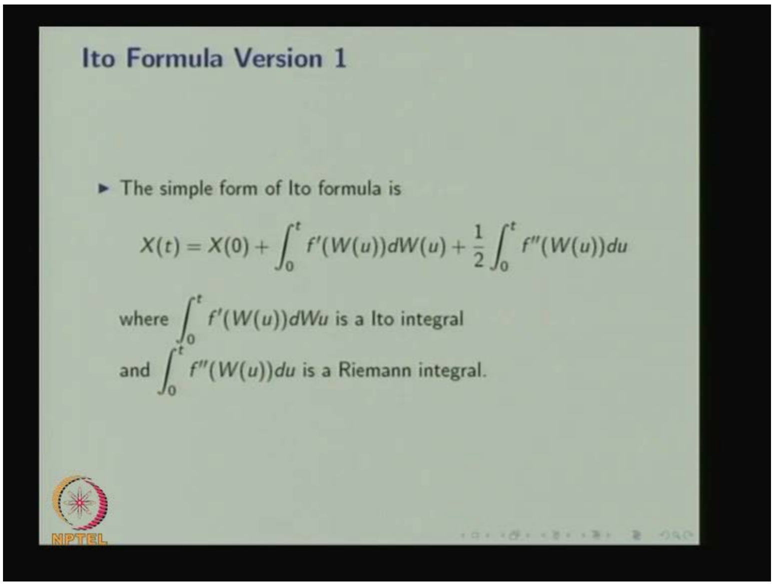

Now, we are going to discuss variance of Ito formulas. I said Ito formula is valid, Ito formula will be useful to solve some special cases of stochastic differential equation or the Ito integrals. Not all the not every stochastic differential equations. So, the simplest form of Ito formula is X(t) is equal to X(0) plus integration 0 to t f dash w(u) dw(u) plus 1 by 2 integration 0 to t; f double dash W(u) du where this integration is Ito integral and this is a Riemann integral.

Since, it relates the differences in terms of Riemann integral and the Ito integral. One can solve the Ito integral as long as the function is twice differentiable then you can evaluate the Ito integral using the Ito formula. And, here we have used the quadratic variation of W(t) is equal to t. Therefore, you will end up; you will have the integration with respect to u that is a Riemann integral.

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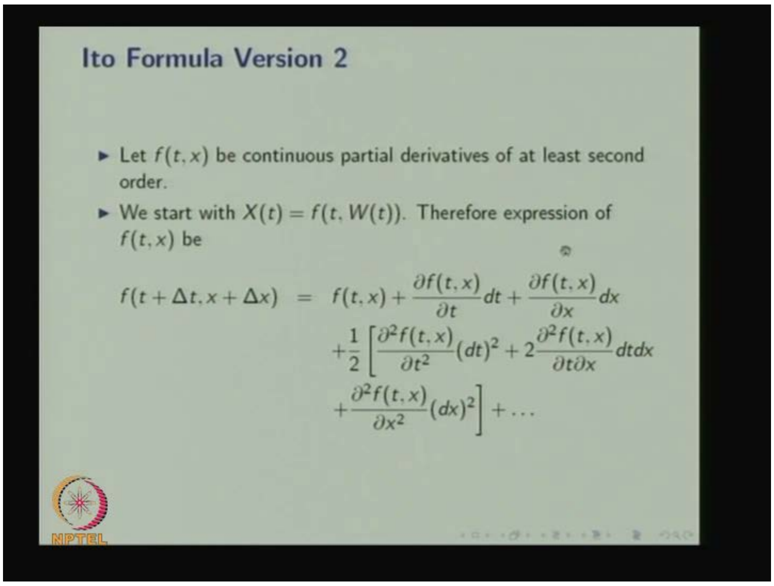

Now, we are moving into second conversion or version 2 of Ito formula. In this we have assumed the function with the 2 variables t, x; where x is a function of t. So, here we start x(t) is equal to function of t, W(t) there we have assumed x(t) is equal to function of W(t) alone, but now in the version 2 we are making x of t is equal to function of t with W(t).

Therefore, the expression of t(f(t)) comma x be the increment in the t increment in x is same as f(t) comma x plus the partial derivative with respect to t dt plus partial derivative with respect to x dx. So, these 2 terms are related to the partial derivative plus 1 by 2 factorial that is 1 by 2; you have to include all the second derivative. So, second derivative with respect to t second derivative with respect to t as well as x then second derivative with respect to x. There are 3 terms plus 1 by 3 factorial like that you should go for 3 partial derivatives then 4 partial derivative and so on.

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Ito Formula Version 2 ...

- $$f(t+\Delta t, W(t+\Delta t)) = f(t, W(t)) + \frac{\partial f(t, W(t))}{\partial x} dW(t) + \frac{\partial f(t, W(t))}{\partial t} dt + \frac{1}{2}\frac{\partial^2 f(t, W(t))}{\partial x^2} dW(t)dW(t) + \ldots$$
- Interpret $dW(t)dW(t)$ as dt, $dW(t)dt = 0 = (dt)^2$, we get

$$f(t+\Delta t, W(t+\Delta t)) = f(t, W(t)) + \frac{\partial f(t, W(t))}{\partial x} dW(t) + \left[\frac{\partial f(t, W(t))}{\partial t} + \frac{1}{2}\frac{\partial^2 f(t, W(t))}{\partial x^2}\right] dt$$

$$df(t, W(t)) = \frac{\partial f(t, W(t))}{\partial x} dW(t) + \left[\frac{\partial f(t, W(t))}{\partial t} + \frac{1}{2}\frac{\partial^2 f(t, W(t))}{\partial x^2}\right] dt$$

Again you can use the quadratic variation of W(t) is equal to t and cross variation is 0 and dt, dt is equal to 0 and so on. Therefore, in that so f(t) plus delta t comma y of y of comma W(t) plus delta t is same as f(t) of W(t) plus partial derivative of f with respect to x multiplied by dW(t) plus partial derivative of f with respect to t multiplied by dt plus 1 by 2; second order partial derivative of f with respect to x multiplied by dw(t) and so on.

Interpret dW(t) as dt dW(t) dt is equal to 0 also dt is equal to0. We get f of t plus delta t comma W(t) plus delta t that is same as f(t) comma W(t) plus partial derivative of with respect to x multiplied by dW(t) plus partial derivative of f with respect to dt plus half times; second order partial derivative of f with respect to x both the terms multiplied by dt.

Therefore, the left hand side becomes the derivative of f.. Whereas, the right hand side is you can simplify as partial derivative of f with respect to x multiplied by d Wt and there are 2 terms for the dt. The first term is partial derivative of f with respect to t plus half time second order partial derivative of f with respect to x multiplied by dt.

So, you will get only 1 term which has dW(t) whole square that is dW(t) term that will be the dt. Therefore, we will have dt with 2 terms one is this one and other one is the second derivative one term that is dou dou square f by dou W(t) whole square. So, these 2 terms has the term dt and other term is dW(t).

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Ito-Doeblin Formula

In general, Ito process $\{X(t), t \geq 0\}$ where $X(t)$ is given by

$$X(t) = X(0) + \int_0^t A_{(1)}(s)ds + \int_0^t A_{(2)}(s)dW(s)$$

where $A_{(1)}(t)$ and $A_{(2)}(t)$ are stochastic processes which are adapted to natural filtration $\{\mathcal{F}(t), t \geq 0\}$ and $A_{(2)}(t)$ is mean square integrable.

$$\begin{aligned} df(t, X(t)) &= \frac{\partial f(t, X(t))}{\partial x} dX(t) + \frac{\partial f(t, X(t))}{\partial t} dt \\ &+ \frac{1}{2} \frac{\partial^2 f(t, X(t))}{\partial x^2} dX(t) dX(t) \end{aligned}$$

Therefore, the increment d of ft is nothing but partial derivative with respect to X dX(t) plus partial derivative with respect to t dt; and partial derivative with respect to dX(t) dX(t) 1 x(t) is W(t) then it is dt.

So, in general for Ito process X(t) it is going to be d of X(t) into d of X(t). And, Ito process can be 2 time in the general form X(t) is equal to X(0) integration 0 to t a function of S A1. And, another function of S that is A2 dW(s) where both the stochastic process are adapted to natural filtration f(t). As well as this integrant A2 of s is a mean square integral.

Then, the stochastic process is Ito process. For any Ito process you can have a version 2 Ito formula that is dW df; increment of f is equal to partial derivative terms of first order and second order terms.

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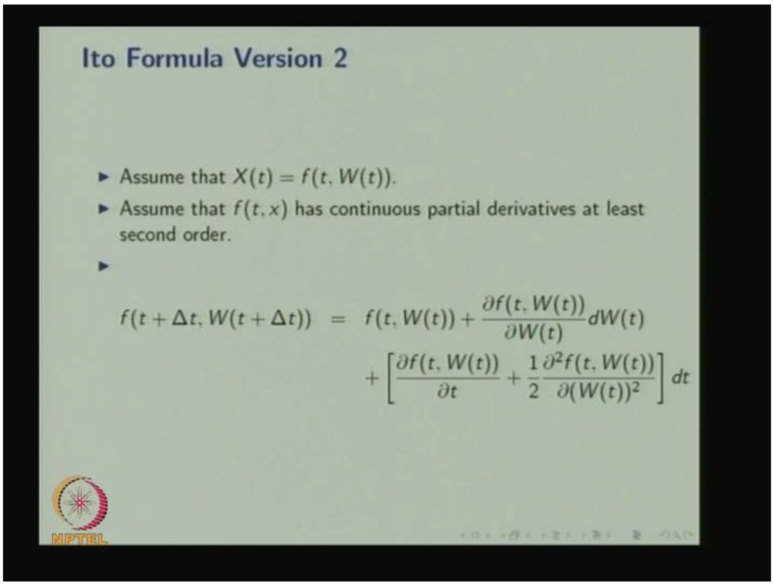

Here, the assumption is the function has continuous partial derivatives at least a second order. Then, only you can write down with the first order partial derivative and second order partial derivative. So, the increments the difference is equal to first order partial derivative multiplied by dW(t) and the first order partial derivative with respect to t plus 1 by 2 second order partial derivative with respect to W(t) into dt. In the assumption the X(t) is function of t with del W(t) and it has continuous partial derivative at least second order.

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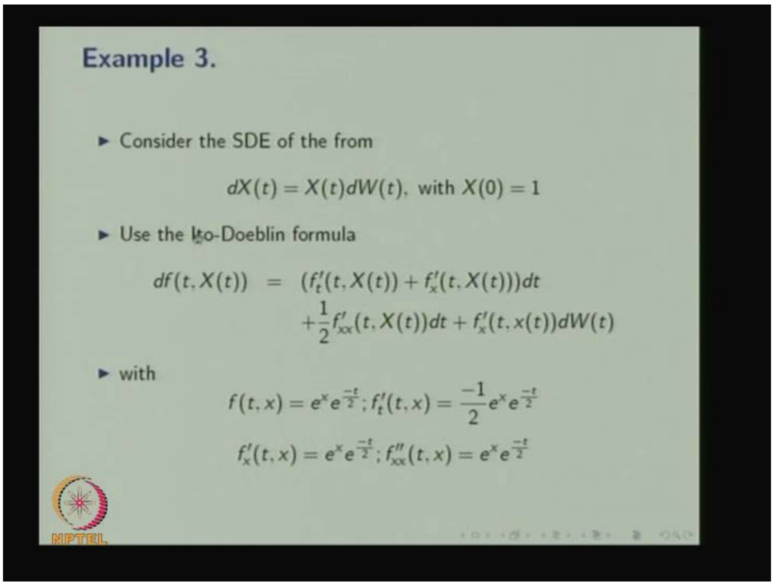

Let see the simple example. Consider the stochastic differential equation is of the form dX(t) is equal to X(t) dW(t). So, this is a stochastic differential equation because so increment with respect to vernier process with the initial condition x(0) is equal to 1. So, you can use the Ito Doeblin formula that is the increment of the function f is a derivative with respect to t, derivative with respect to X dt plus half times f double dash plus f dash dW(t).

Now, you can substitute f(x) f(t)comma x as e power x e power minus t by 2. Therefore, the f dash(t) that is nothing but minus 1 by 2 times f, f dash with respect to x that will be the f itself f double dash that is also again f.

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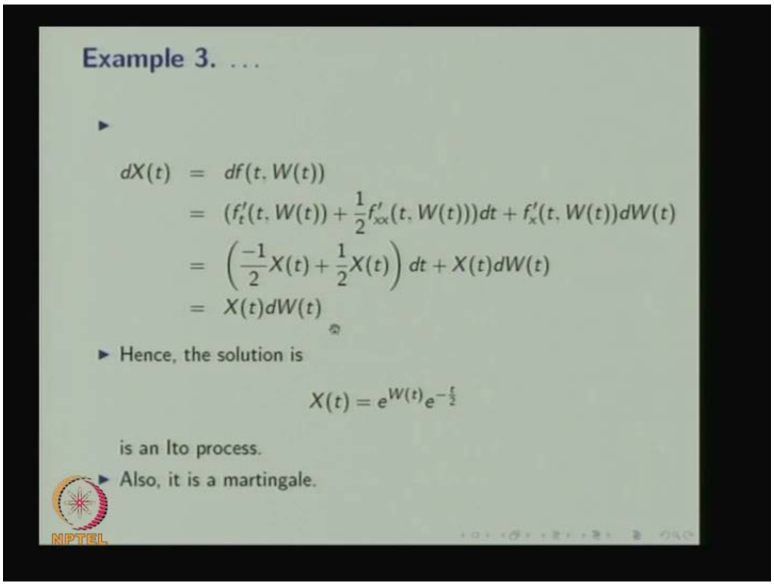

Therefore you can the df is equal to the df is equal to f dash(t) plus half times f double dash(dt) plus f dash(x) dW(t). So, you substitute all the values which you got it in the previous one that is f dash(t), f double dash(x), f dash(x). You substitute all the values; simplify you will get the equation that is x(t) dW(t).

So, since it is satisfying the differential stochastic differential equation d(X(t)) is equal to X(t) dw(t). Therefore, the solution f of that is e power x times e power minus t by 2 that is a solution of given stochastic differential equation. Hence, the solution is e power W(t) e power minus t by 2. you can prove that also you can prove that this is also a Ito process by writing this as the Ito integral; you can prove that this is a Ito process.

Already, we made one proper observation. Ito process is a martingale means Ito process satisfies the martingale property. Therefore, this is also a martingale; e power W(t) time's e power minus t by 2 that is a martingale also.

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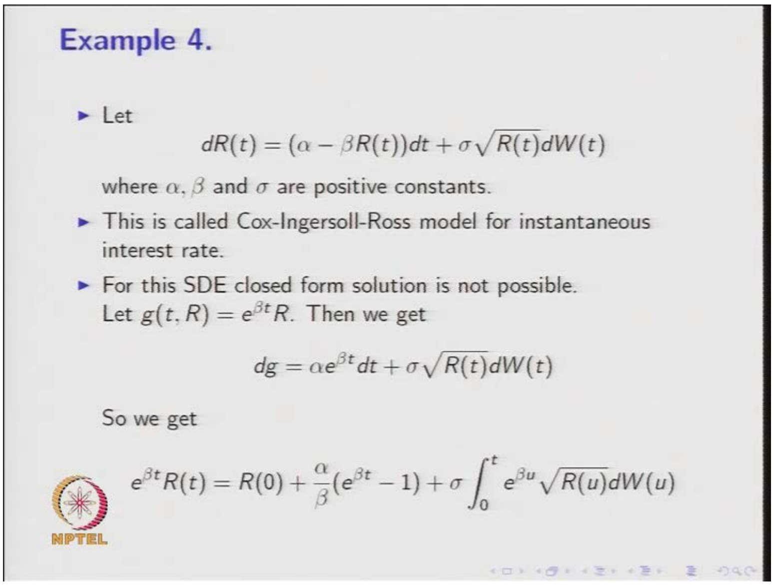

Next, we consider this example d of R(t) is equal to alpha minus beta times R(t) multiplied by dt plus sigma square root of R(t) dW(t). Where alpha comma beta and sigma are positive constants. This is called CIR model for instantaneous interest rate.

For this stochastic differential equation the close form solution is not possible. This can be proved easily by taking g of t is equal to e power beta times t multiplied by R(t). Hence, the stochastic differential equation for g(t) would be dg(t) is equal to alpha times e power beta t times d t plus 6 square sigma times square root of Rt dW(t). The stochastic integral form we have e power beta times t R(t) is equal to R(0) plus alpha divided by beta times e power beta t minus 1 plus sigma times this stochastic integral. With the integrant e power beta(u) square root of R(u) where R(u) is unknown.

In this equation the term square root of R(u) appears in the right hand side of the Ito integral. We are not able to get the close form solution or close form expression for R(t) in you are solving this stochastic integral equation. Some numerical techniques are needed for computing r(t) at any given t.

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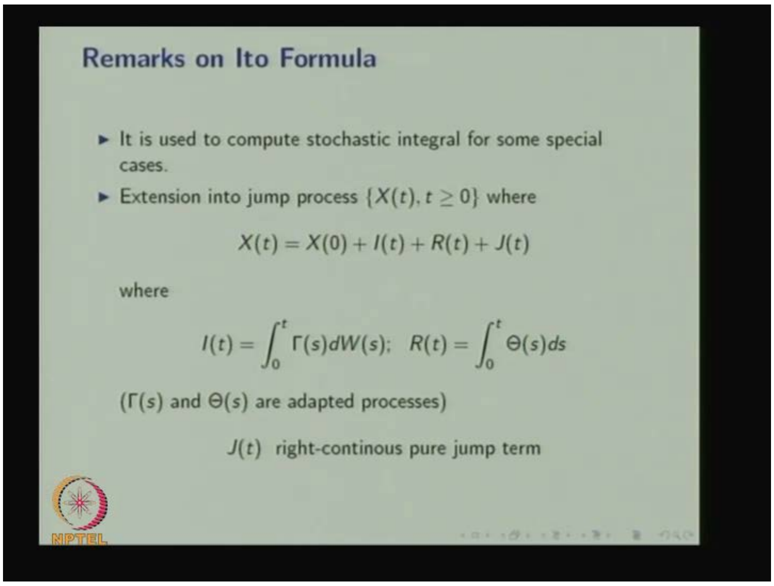

The remarks on the Ito formula: the first important remark it is used to compute stochastic integral, it is used to compute the solution of a stochastic integral for some special cases. Not for every stochastic integral equations or stochastic differential equations.

In these examples, we have used for the Ito processes or diffusion processes in general diffusion process, but this can be extended to the jump process also. The jump process can be written in the form of X(t) is equal to X(0), I(t) plus R(t), J(t). In this the J(t) terms is a continuous pure jump term. So, if you exclude like this J(t), then the x(0) plus I(t) r(t) that will give a continuous part. This gives the continuous part and this gives the right continuous pure jump term. So, the I(t) is nothing but the Ito integral whereas R(t) is a Riemann integral.

Therefore, the integration with respect to s and here integration with respect to vernier process. And, both are adapted process therefore without J(t) without J(t) X of t that has a continuous part this much. That is a Ito process including the J(t) that is a right continuous pure jump term. Therefore, this is a jump process. So, one can apply the Ito formula for the jump process also.

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Remarks on Ito Formula...

- $$\begin{aligned} f(X(t)) &= f(X(0)) + \int_0^t f'(X(s))dX^c(s) \\ &\quad + \frac{1}{2}\int_0^t f'(X(s))dX^c(s)dX^c(s) \\ &\quad + \sum_{0<s\leq t} [f(X(s)) - f(X(s-))] \end{aligned}$$
- Extension into multidimensional stochastic processes also.

So, the difference is with Ito formula for the Ito process this is a extra term. Because x of i's has a right continuous jump therefore this is going to be f(X) of s minus f(X) of s minus a left limited the point s; evaluate between the interval 0 to t.

You include this term plus include this term and that will be the Ito formula for the jump process where x of superscript c means it is a continuous part of stochastic process X(t). That means, without X(t) term this is a continuous part X superscript c(t) is nothing, but x(0) plus I of t plus R(t).

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Remarks on Ito Formula...

- $$\begin{aligned} f(X(t)) &= f(X(0)) + \int_0^t f'(X(s))dX^c(s) \\ &\quad + \frac{1}{2}\int_0^t f'(X(s))dX^c(s)dX^c(s) \\ &\quad + \sum_{0<s\leq t} [f(X(s)) - f(X(s-))] \end{aligned}$$
- Extension into multidimensional stochastic processes also.

The Ito formula can be extended to the multidimensional stochastic processes also. First we started with the Ito process then we have extended to the jump process and this can be extended Ito formula can be extended to the multidimensional stochastic processes also. Note that it is used to compute the solution for special cases of a stochastic differential equations or Ito integrals.

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Example 1. Geometric Brownian Motion

$$dS(t) = \mu S(t)dt + \sigma S(t)dW(t)$$

with $S(0) = S_0$.

$S(t)$ – Stock price at time t

μ – Constant growth rate of the stock

σ – Volatility

Assume solution $S(t) = f(t, W(t))$.Comparing the above SDE with ITO Doeblin Formula version 2, we get:

$$\frac{\partial f}{\partial x} = \sigma f, \quad \frac{\partial f}{\partial t} + \frac{1}{2}\frac{\partial^2 f}{\partial x^2} = \mu f$$

Solving this we get

$$f(t, x) = e^{(\mu - \sigma^2/2)t + \sigma x}$$

$$S(t) = f(t, W(t)) = e^{(\mu - \sigma^2/2)t + \sigma W(t)}$$

In the next lecture we are going to discuss sums important stochastic differential equations which have the applications in a mathematical finance. The first example is geometric Brownian motion. So, the underlying stochastic differential equation is this where S of t is stock price at time t and mu is a constant growth rate of a stock and sigma is the volatility. And, we can find the strong solutions of t has the function of t with W(t).

So, we can use the version 2 of Ito formula we get partial derivative of f with respect to X is same as sigma times f the partial derivative of f with respect to t plus half second order partial derivative of with respect to X that is same as mu times f. When you solve these 2 equations we get f(t) comma X is same as e power mu minus sigma square by 2 times t plus sigma of X.

Hence, the stock price at time t that S(t) is same as e power mu minus sigma square by 2 multiplied by t plus sigma times W(t); where W(t) is the Brownian motion.

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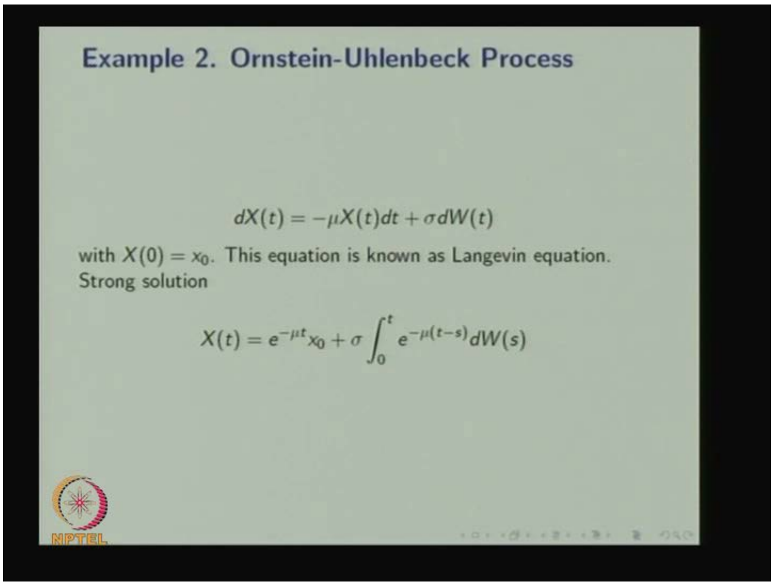

And, in the second example this is called Ornstein-Uhlenbeck process and corresponding stochastic differential equation is this. This equation is also known as the Langevin equation. And, using the Ito formula you can find the solution of X(t) also; X(t) is of this form.

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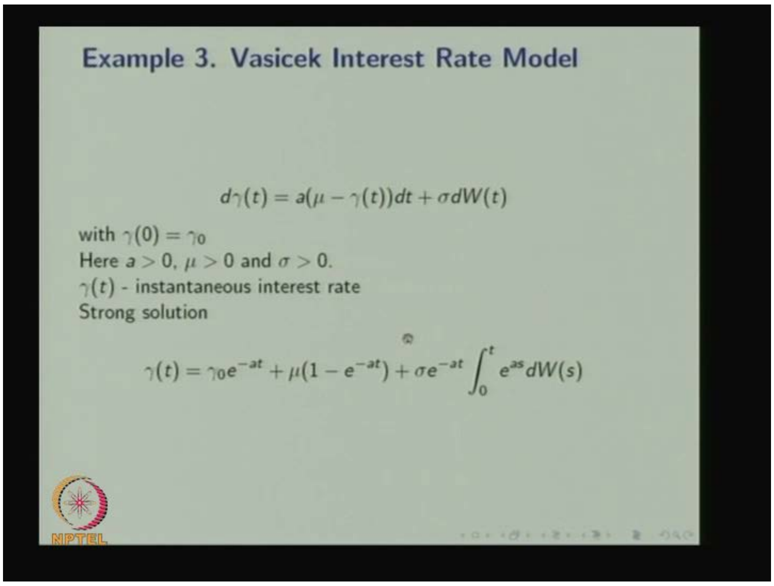

The third example this is also has application in a financial mathematics that is called Vasicek interest rate model. Where R(t) is instantaneous interest rate the corresponding stochastic differential equation is this. You can find the strong solution for this stochastic differential equation also. Both examples 2 and 3 can be solved as explain for example 1. The detailed solution can still be seen in the problem sheet.

So, these are all the three important stochastic differential equation will be discuss in the next lecture. And, we will find the solution using Ito formula version 1 or version 2.

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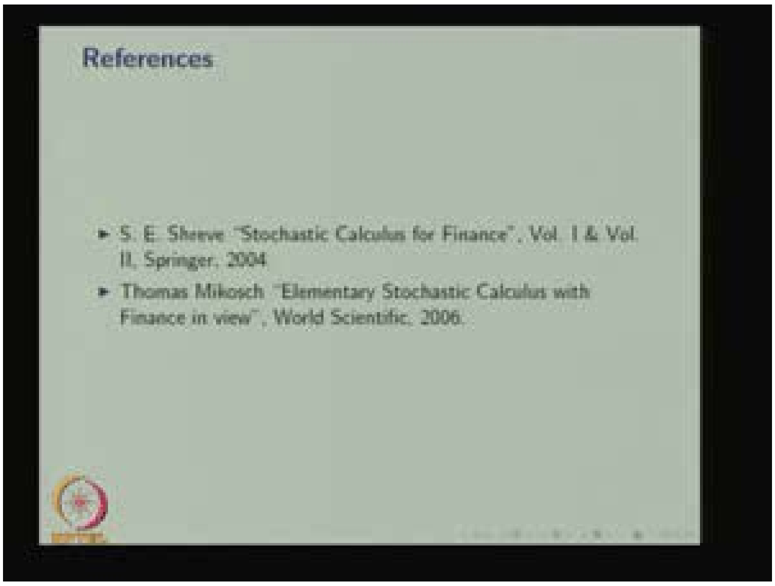

These are all the references for this lecture. So, with this we have completed starting with the Brownian motion, geometric Brownian motion and stochastic differential equation, Ito integrals. And, how to solve the stochastic differential equation using Ito formula? The next lecture, we are going to discuss some important stochastic differential equations and their solutions.