Stochastic Processes
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Module - 1
Probability Theory Refresher
Lecture - 3
Problems in Random Variables and Distributions

This is stochastic processes module 1 probability theory refresher, lecture 3 problems in random variables and distributions.

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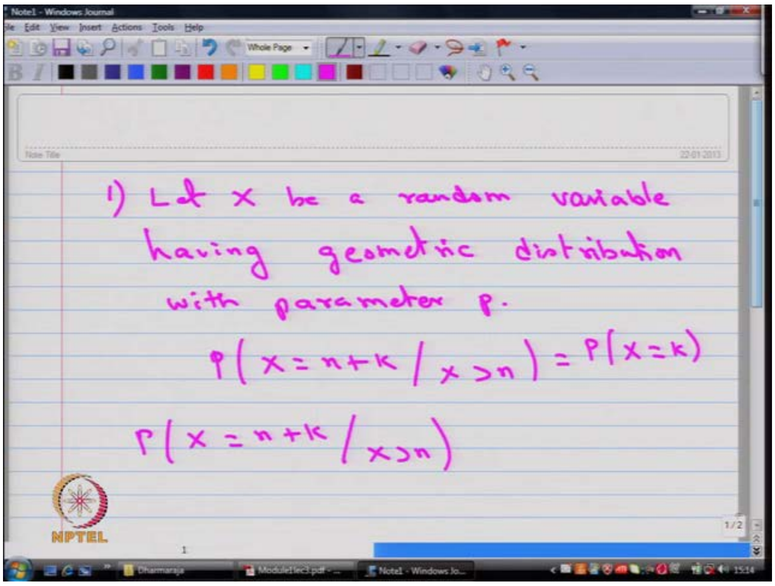

Let as a first problem, let X be a random variable having geometric distribution with the parameter P. Our interest is to find, our interest is to prove that probability of X is equal to n plus k, given X takes the value greater than n. That is same as the probability that X takes the value k for every integers n and k. We can prove this result by starting from the left hand side that is probability of X takes the value n plus k given x greater than n.

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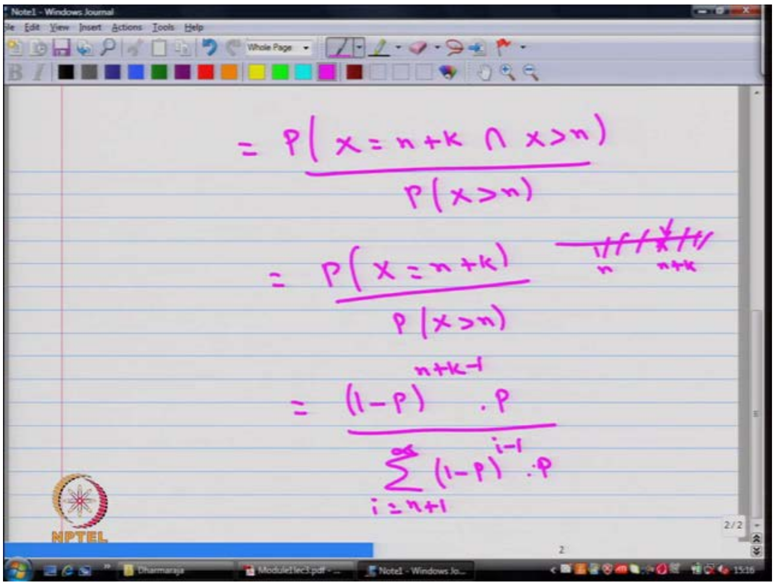

By definition this is same as probability of X is equal to n plus k intersection X greater than n divided by probability of X greater than n. That is same as. That is same as the numerator X is equal to greater than n means, all possible values n is equal to n plus k, that means the intersection is going to be probability of X takes the value n plus k, whereas the denominator is probability of X is greater than n.

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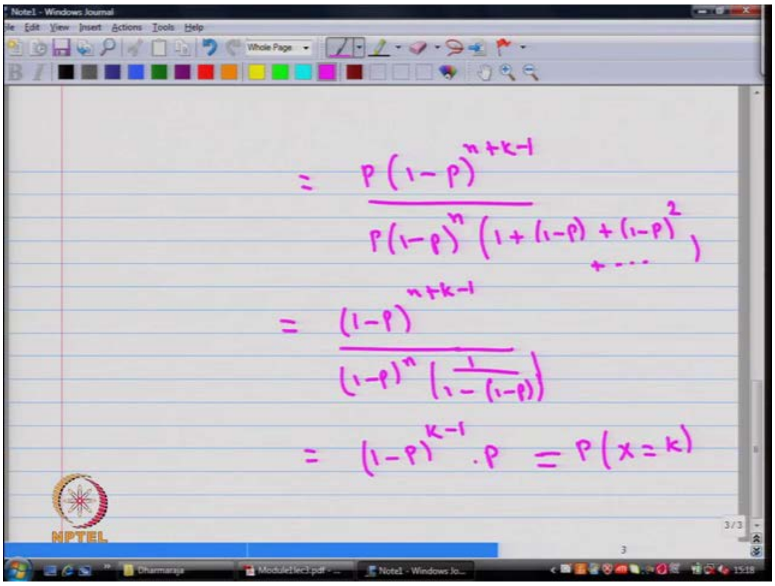

That is same as since X is a geometric distribution with the parameter P, the probability of X equal to n plus k that is nothing but 1 minus P time P power n plus k minus 1 into P.

Whereas, the denominator probability of X is greater n that means summation i is equal to n plus 1 to infinity, 1 minus P power i minus 1 multiplied by P.

That is same as numerator you can keep it as it is, whereas the denominator since the summation i is equal to n plus k to infinity, you can take P times 1 minus P power n common outside. The remaining terms are 1 plus 1 minus P the third term will be 1 minus P whole square and so on.

Therefore, you can still simplify you will get 1 minus P power n plus k minus 1 divided by 1 minus P power n, keep it as it is. This series will have the value 1 minus 1 minus P. Therefore, if you further simplify you will get 1 minus P power k minus 1 multiplied by P, that is nothing but probability of X is equal to k. So this result says the probability of X equal to n plus k given X is greater than n, that is same as probability of X is equal to k for all n and k. This is a important property of geometric distribution, this property is called a memory less property

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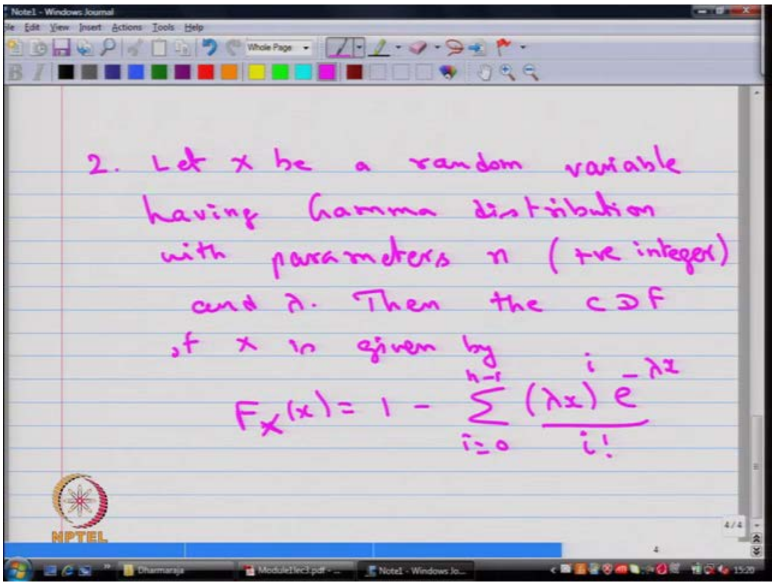

We move into the next problem let X be a random variable having gamma distribution with the parameter n. You assume that n is a positive integer and the other parameter is lambda, then the cumulative distribution function CDF of X is given by capital F of x for the random variable X. That is 1 minus summation i is equal to 0 to n minus 1 lambda x power I, e power minus lambda times x divided by i factorial. So, whenever X is a gamma distribution with the parameters n and lambda then the CDF can be written in this form.

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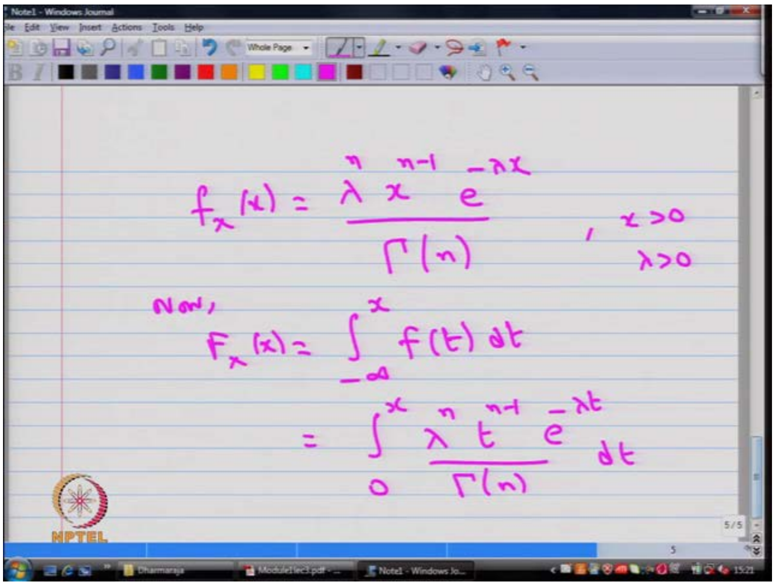

We know that the probability density function of the gamma distribution is lambda power n, x power n minus 1, e power minus lambda x divided by gamma of n. Since n is a positive integer gamma of n is a n minus 1 factorial. Now, we can find out the CDF of this random variable, that is nothing but minus infinity to x, the probability density function.

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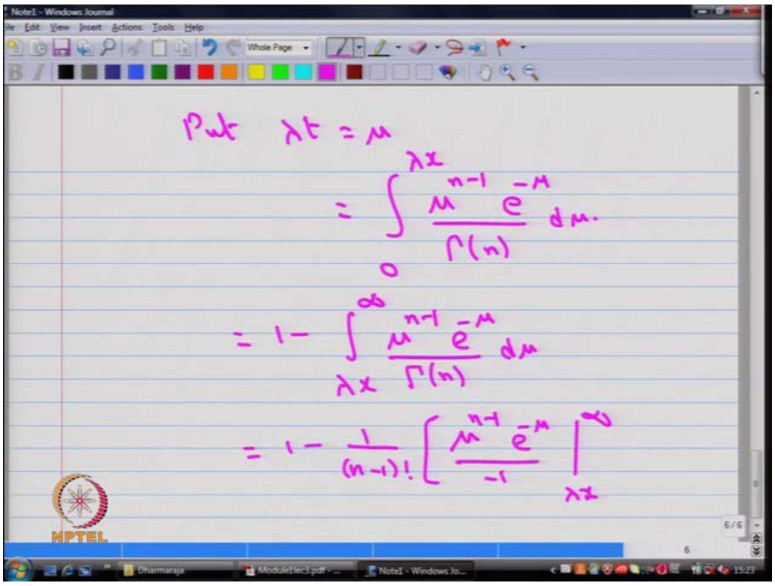

That is same as since the F of x is this is valid for x is greater than 0 and lambda is greater than 0. So, this integration is valid from 0 to x lambda power n, t power n minus 1, e power minus lambda times t divided by gamma of n, d t. Now, you have to integrate this one and get the expression for the CDF of the random variable x. So, what you can do, make a substitution lambda times t that is same as you make it as some mu. Therefore, this integration becomes the integration from 0 to lambda times x, mu power n minus 1, e power minus mu divided by gamma of n into d of mu. That is same as 1 minus integration goes from lambda x to infinity, mu power n minus 1, e power minus mu divided by gamma of n d mu.

That is same as 1 minus, since n is a positive integer gamma of n is n minus 1 factorial. So you can take it outside, you can do this integration by parts. So you will get mu power n minus 1, e power minus mu divided by minus 1 between the limits lambda x to infinity minus minus integration from lambda x to infinity, n minus 1 times mu of n minus 2, e power minus mu divided by minus 1 d mu. So the whole thing is multiplied by n minus 1 factorial

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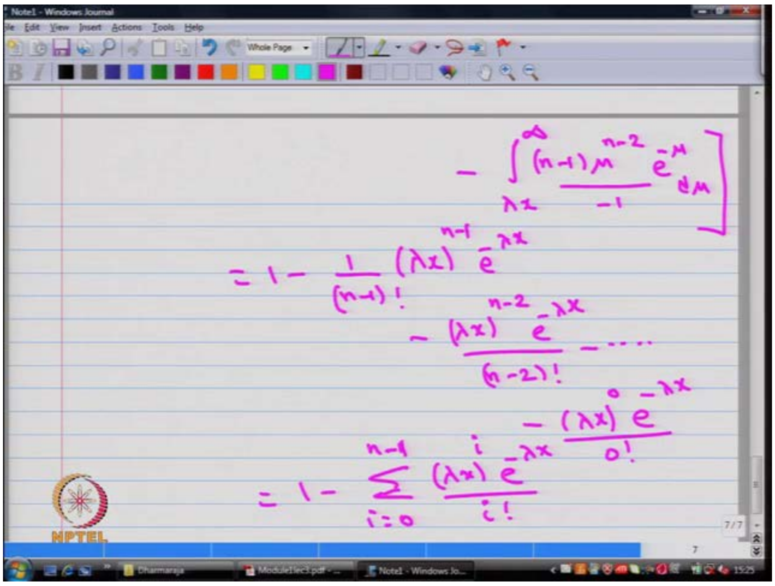

Now, you can integrate the second term again by integration by parts and when you substitute the limits for mu is infinity and as well as mu is equal to lambda x and subsequently if you do the integration by parts you will get, 1 minus n minus 1 factorial lambda x power n minus 1, e power minus lambda x, then the next term will be minus lambda x power n minus 2, e power minus lambda x by n minus 2 factorial, similarly the other terms. The last term will be by doing integration by parts again and again, the last term you will get minus of lambda x power 0, e power minus lambda x by 0 factorial.

This you can write it in the form 1 minus summation i is equal to 0 to n minus 1 lambda x power i, e power minus lambda x by i factorial. So, here we are finding the CDF of the gamma distribution when the one of the integer is a positive integer, one of the parameters is a positive integer. This result will be useful in finding the total times finding the queuing system, that will be discussed in the later modules.

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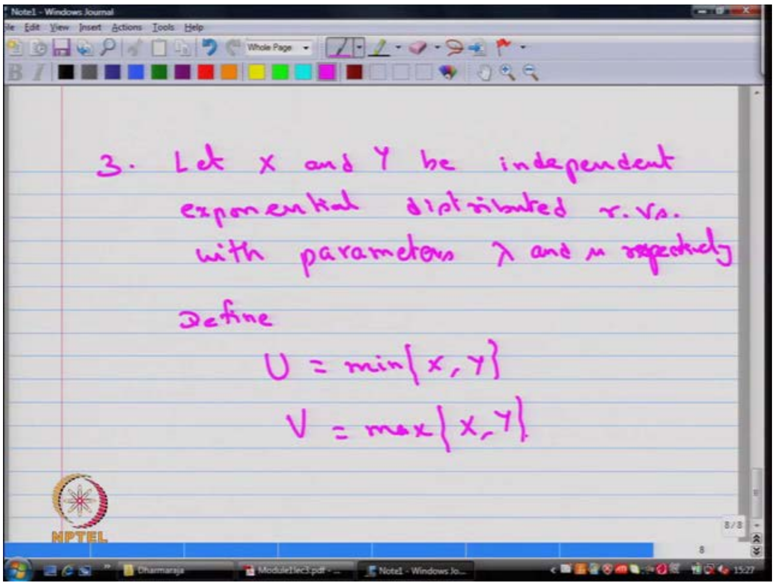

As a third example, we will discuss this one. Let X and Y be independent exponential distributed random variables with parameters lambda and mu respectively. Define capital U that is nothing but minimum of X comma Y and V is nothing but maximum of the random variables X comma Y.

The third random variable capital N that is defined as, it takes a value 0 if X is less than or equal to Y it takes the value 1, if X is greater than Y. So, using the two random variables X and Y we have defined three random variables U, V and N. Our interest is to find find the probability of capital N takes the value 0 and what is the probability of capital N takes the value 1. Fine, the second we are interested to find out the probability of N takes the value 0 and capital U takes the value greater than some t, where t is greater than 0.

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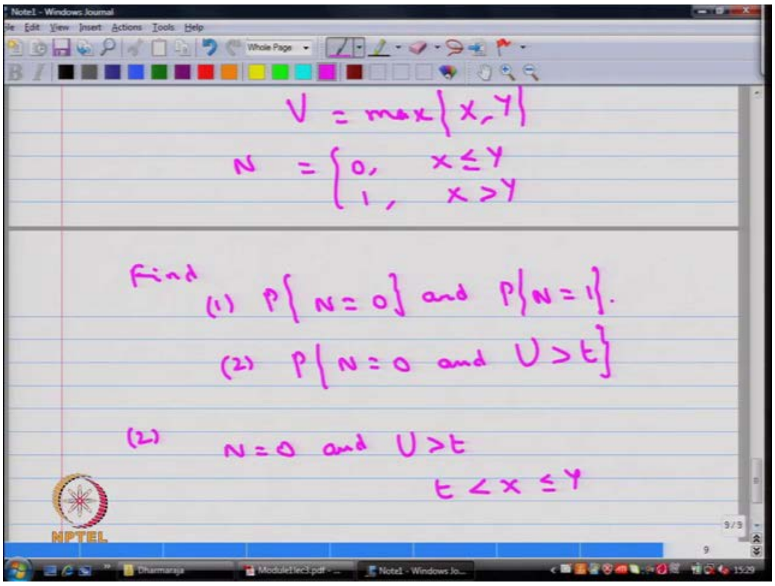

So, let us go for finding the second one first, then we will find out the probability of X equal N equal to 0 and the probability of N is equal to 1. So let us start with the two first. The event N is equal to 0 and capital U takes a value greater than t, that is exactly the event of a t less than capital X, less than or equal to capital Y. The event N is equal to 0 and capital U greater than t, that is same as t less than X less than or equal to Y.

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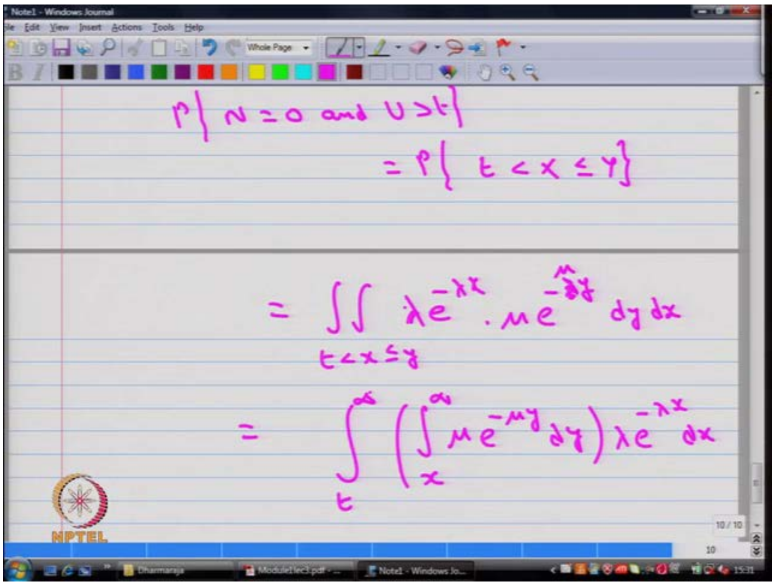

Therefore, the probability of finding N takes the value 0 and U takes the value greater than t, that is same as probability of X takes the value t, t less than X less than or equal to

Y. That is same as, that is same as the double integration with t less than X less than or equal to Y of the joint probability density function of X and Y. Since X and Y are independent random variable, the joint probability density function is product of marginal probability, sorry this is e power minus mu y, d y d x. So, the probability of t less than X less than or equal to Y, that is same as the double integration t less than X less than or equal to Y of integrant is lambda times e power minus lambda X, mu times e power minus mu y d y dx.

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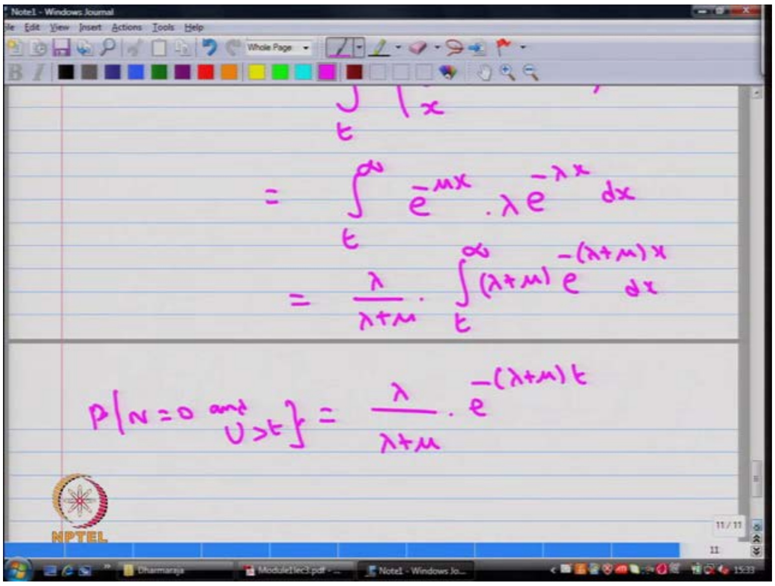

That is same as the inner integral becomes x to infinity, mu times e power minus mu y, d y then lambda times e power minus lambda x, integration with respect to x between the limits t to infinity. That is same as now the inner integration you can integrate and you can substitute the limit x and infinity, if you simplify you will get t to infinity, e power minus mu x, then multiplied by lambda times e power minus lambda x, d x. If you give the integration the interior integration you will get e power minus mu x, then the remaining things are as it is. That is same as you can keep lambda by lambda plus mu outside this becomes integration from t to infinity of lambda plus mu times e power minus lambda plus mu x, d x.

You know how to do the integration for this, if you simplify you will get the answer that is lambda divided by lambda plus mu times e power minus lambda plus mu times t. So, this is a result for probability of N takes a value 0 and U takes a value greater than t, were t is greater than 0.

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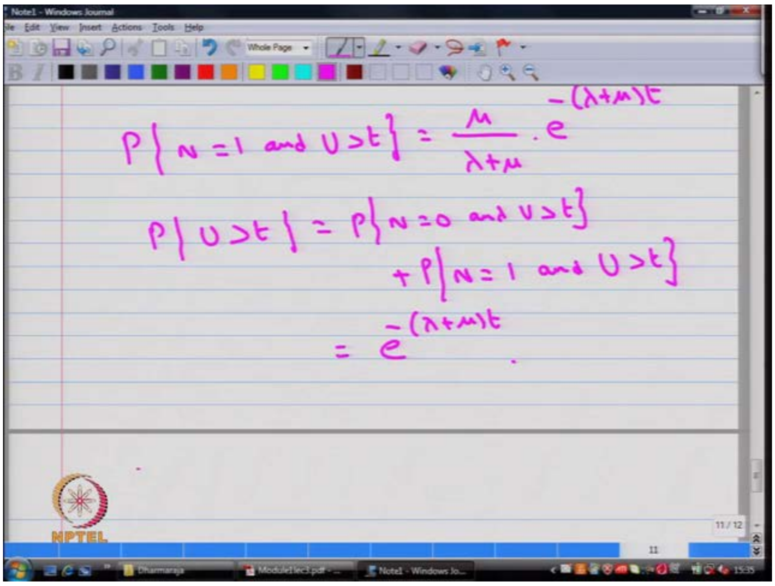

Similarly, you can work out the probability of N takes the value 1 and mu takes a value greater than t. That will be mu divided by lambda plus mu multiplied by e power minus lambda plus mu times t. So, with this part second part is over. Now, we have found the probability of N is equal to 0 and U greater than t, also we will got the probability of N is equal to 1 and U greater than t. Now, we will go for finding the first result, but before that we can find what is a probability of U greater than t also. That is nothing but finding out the probability of N is equal to 0 and U greater than t plus the probability of N takes a value 1 and U greater than t, that is a way we can find out the probability of U greater than t.

You know the result, if you add you will get the result that is e power minus lambda plus mu times t. That is the meaning of the minimum of two random variable takes the value greater than t, that is e power minus lambda plus mu times t.

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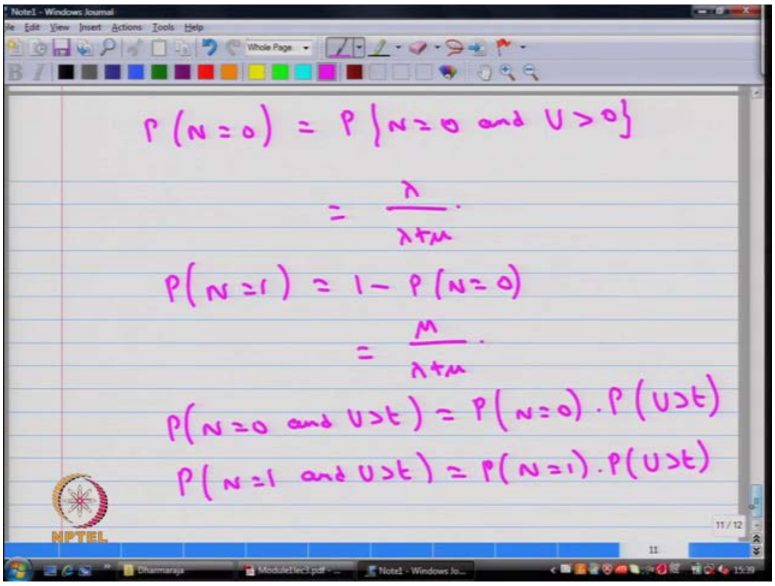

You can find out the probability of N is equal to 0 in using probability of N is equal to 0 and U is greater than 0, the t is greater than or equal to 0 therefore, probability of N is equal to 0. That you can compute from probability of N is equal to 0 and U is greater than 0. That means you substitute t equal to 0 in the previous result, you will get probability of N is equal to 0. So, that is nothing but if you recall probability of N is equal to 0 and U greater than t, that is lambda divided by lambda plus mu e power minus lambda plus mu t. Here the t can be greater than or equal to 0, so substitute t equal to 0, that will give probability of N is equal to 0.

Therefore, this will become lambda divided by lambda plus mu. Similarly, you can find out the probability of N is equal to 1 in probability of N is equal to 0 and probability of N is equal to 1 and U is greater than 0 or you can find out probability of N is equal to 1 by making one of probability of N is equal to 1 minus probability of N is equal to 0. So two ways you can find the probability of N is equal to 1, so here I am using 1 minus probability of N is equal to 0. That will be mu divided by lambda plus mu .

So, in this problem even though we asked only two things, but here we have got probability of N is equal to 0 probability of N is equal to 1, also probability of U is greater than t. If you observe, if you observe probability of N is equal to 0 and U greater than t, that is same as probability of N is equal to 0 multiplied by probability of U greater than t. Similarly, you will get the observation probability of N is equal to 1 and U greater than t, that will give probability of N is equal to 1 multiplied by probability of U greater than t. So, hence N and U are independent random variables. So, this example is useful in birth death processes therefore, I am explaining this problem as a illustrative example.

So, in this example we observe that N and U are independent random variables and also by seeing a probability of U greater than t that is e power minus lambda plus mu t, you can conclude U is exponential distribution with a parameter lambda plus mu.

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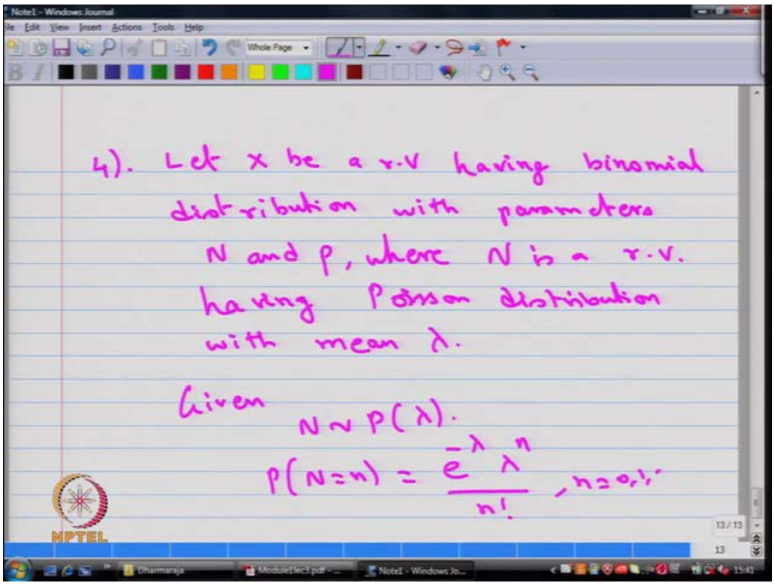

Now, we will move into the next example. Let X be a random variable having binomial distribution with parameters capital N and P where capital N is a random variable having poisson distribution with mean lambda. The question is find the marginal distribution of X or find the probability mass function of the random variable X. Given N is Poisson distribution with the parameter lambda, that means the probability mass function for the random variable N is e power minus lambda, lambda power n divided by n factorial, the possible values of n are 0, 1, 2 and so on.

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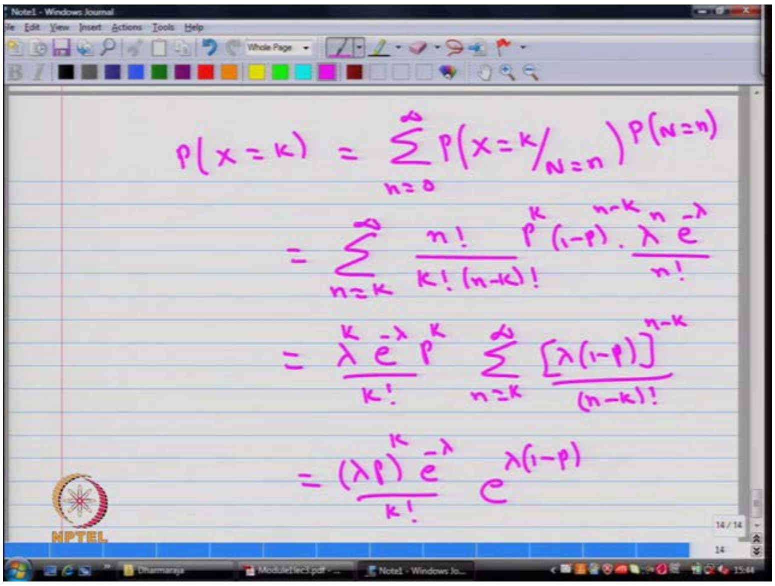

Our interest is to find out what is a probability mass function of the random variable X. That is same as n is equal to 0 to infinity, what is the conditional probability of a random variable X takes the value K, given the other random variable N takes the value small n, multiplied by probability of N takes the value small n. That is same as the n takes a value from k to infinity, n factorial divided by k factorial into n minus k factorial and P power k 1 minus P power n minus k multiplied by lambda power n, e power minus lambda divided by n factorial.

So, no need of n is equal to 0 to k minus 1 because the capital N takes a value small n therefore, the running index from k to infinity. That is same as you can take some terms outside that is lambda power K, e power minus lambda P power k divided by k factorial. The remaining terms that is n is running from k to infinity, this can be written in the form of lambda times one minus P power n minus k divided by n minus k factorial. That is same as lambda P power k multiplied by e power minus lambda by k factorial. The summation n is equal to k to infinity and so on that becomes e power lambda times one minus P.

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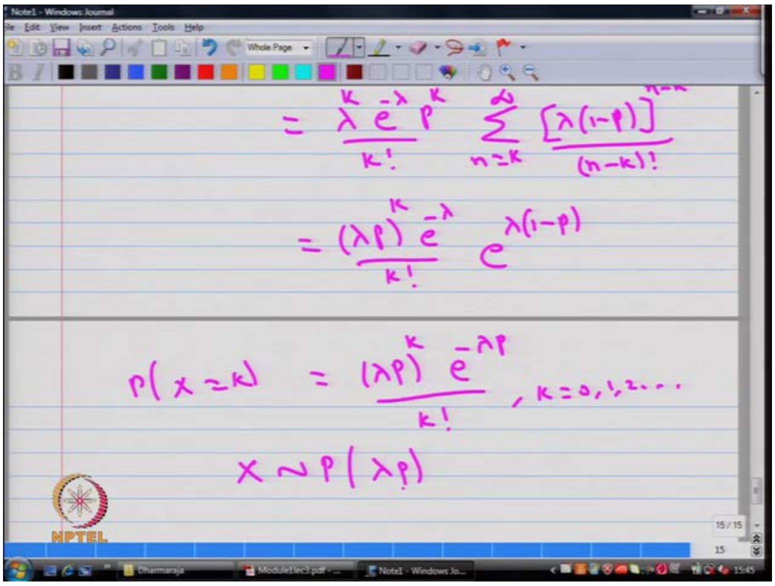

Therefore, further you can simplify therefore, the probability of X takes the value k is same as lambda P power K, e power minus lambda, lambda P divided by k factorial, where k takes the value 0, 1, 2 and so on. Hence, the conclusion is the random variable X which is Poisson distributed with the parameter lambda times P. So, this problem occurs in many situations of a stochastic modeling. Therefore, we have discussed this example in this lecture.

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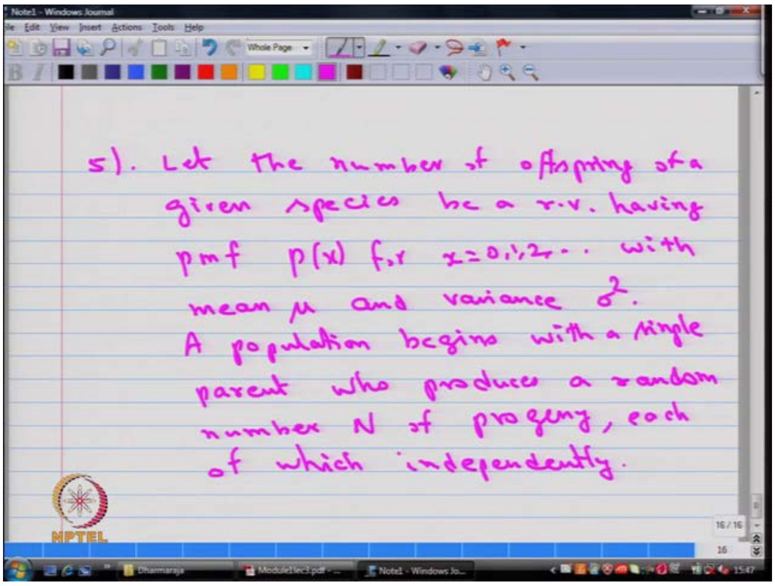

The next example, let the number of offspring of a given species be a random variable having probability mass function P of x for x equal to 0, 1, 2 and so on, with mean mu and variance sigma square. A population begins with single parent who produce who produces a random number say capital N of progeny, each of which independently produces offspring according to P of x.

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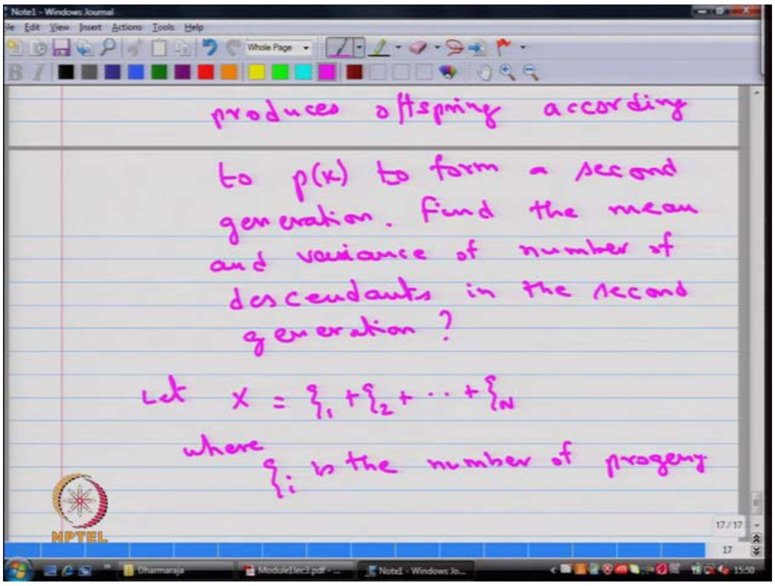

That is a probability mass function, to form to form a second generation. Find the mean and variance of number of descendants in the second generation. So, let me read out the question again, let the number offspring of a given species be a random variable having a probability mass function P of x for x is equal to 0, 1, 2 and so on, with mean mu and variance sigma square. A population begins with the single parent who produce a random number say capital N of progeny, each of which independently produces offspring according to P of x to form a second generation.

Find the mean and variance of number of descendants in the second generation. Let capital X is psi 1 plus psi 2 plus psi suffix capital N, where psi i is the number of progeny of the i th offspring of the original parent.

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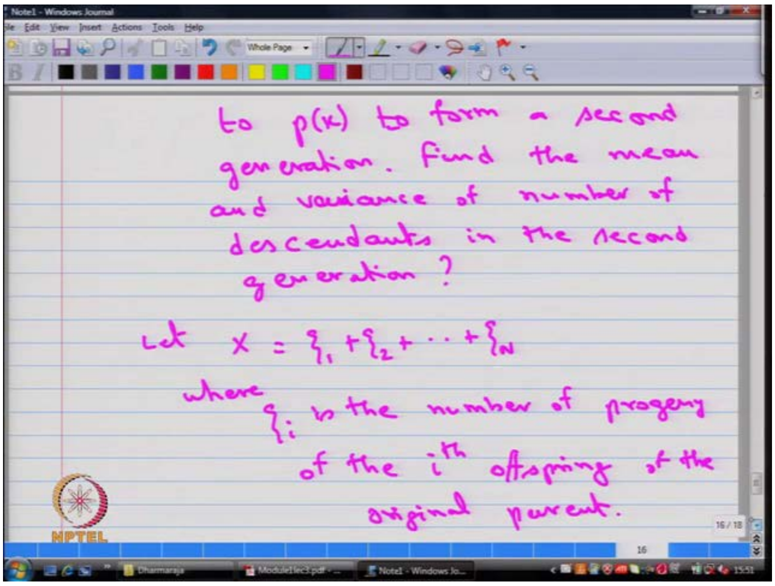

Given the P of x is nothing but the probability mass function for the random variable psi i's and also the expectation of psi i's, that is equal to mu and variance of psi i's, that is equal to sigma square. Psi i's are i i d random variables. Our interest is to find out what is a expectation of capital X, where capital X is, capital x is psi 1 plus psi 2 plus and so on till psi suffix N. Here N is also a random variable that is important.

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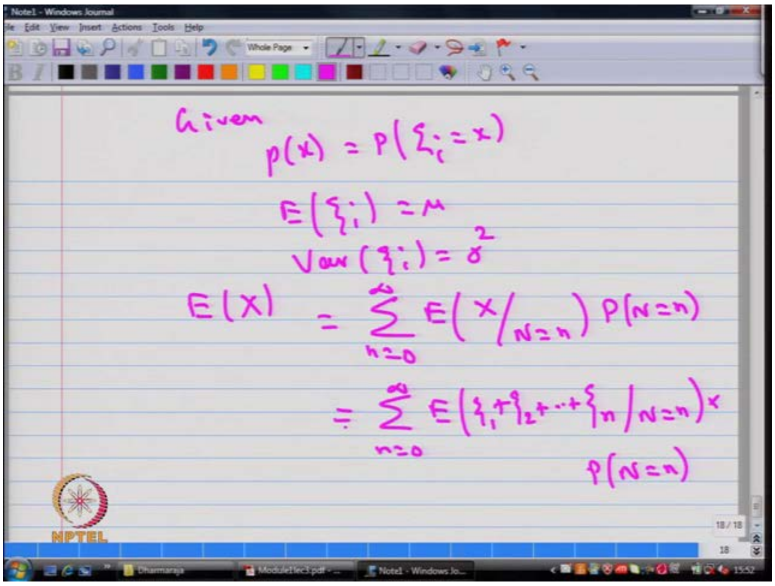

Therefore, this expectation can be computed has N is equal to 0 to infinity, the conditional expectation of X given capital N takes a value small n, multiplied by the probability mass function of capital N. That is same as summation n is equal to 0 to infinity, that is same as expectation of psi 1 plus psi 2 and so on plus psi suffix psi suffix small n, given capital N takes a value small n multiplied by the probability of N takes a value small n.

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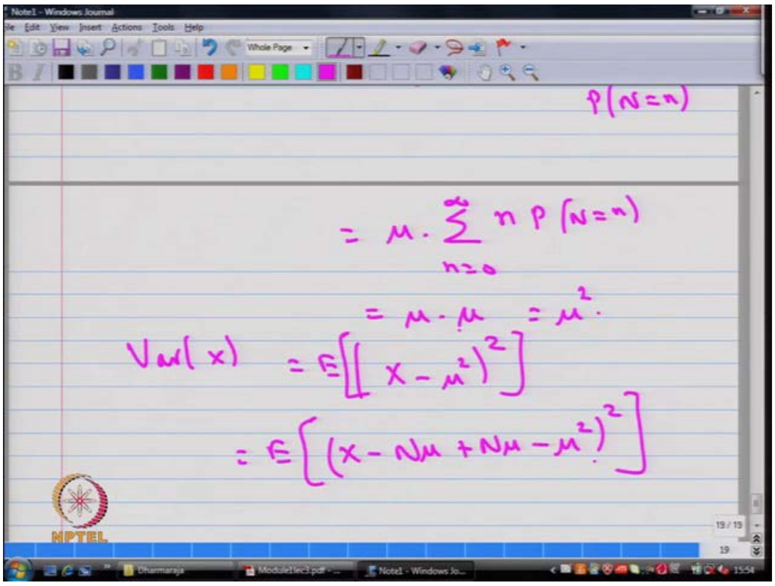

That is same as you know that expectation of psi i’s are mu. Therefore, this becomes a mu can be taken out then summation n is equal to 0 to infinity, n times the probability of capital N takes a value is n.

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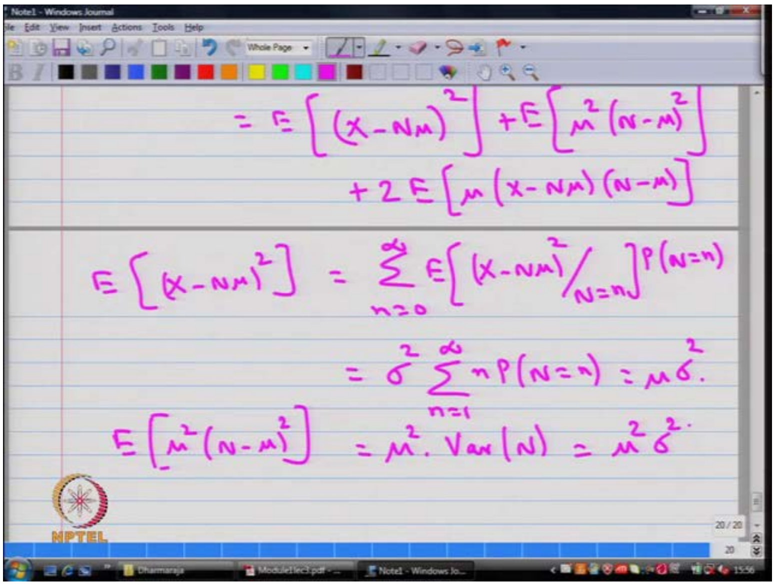

That is same as mu the expectation of the random variable capital N, that is also mu. Therefore, it is mu into mu that is equal to mu square. Now, we can compute the variance of X the same way, that is a expectation of X minus mu square because the random variable expectation is mu square, the whole square. Further this can be simplified by expectation of X minus N times mu plus n times mu minus mu square the wholes square.

One can expand therefore, you will get it is expectation of X minus N times mu the wholes square plus expectation of mu square multiplied by N minus mu the wholes square and the third term becomes 2 times expectation of mu multiplied by x minus N mu into N minus mu, by taking mu outside. Now, we can evaluate the first quantity that is expectation of X minus N mu whole square using the conditional expectation by making summation n is equal to 0 to infinity, expectation of X minus N mu whole square. Given N takes the value small n multiplied by the probability of N takes a value small n.

If you substitute the way you have done the expectation and so on finally, you will get the answer. That is sigma square summation n is equal to 1 to infinity n times probability of N is equal to n. That is same as the summation n times P N is same as the mean that is mu. Therefore, this becomes mu times sigma square, even though I have skipped one or two steps one can get a sigma square summation n times P N that is same as mu therefore, it is mu sigma square.

Similarly, you can work out the second that expectation that is expectation of mu square multiplied by N minus mu whole square. That is same as mu square is constant, the expectation of N minus mu whole square that is nothing but variance of random variable N and variance of a random variable N is sigma square therefore, it is a mu square sigma square.

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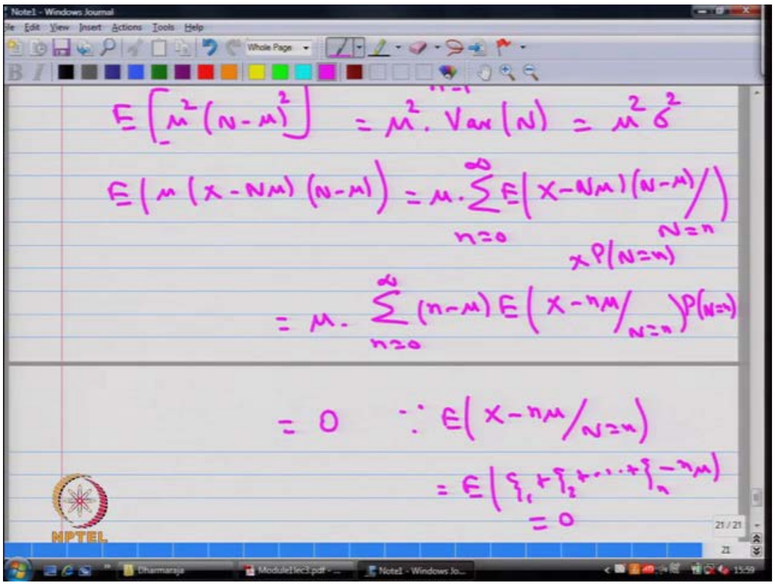

Now, we have to evaluate the third expectation. That is expectation of mu times X minus N mu multiplied by N minus mu. Here also one can use the conditional expectation that is mu times summation n is equal to 0 to infinity expectation of X minus N mu multiplied by N minus mu, condition N takes a value small n, multiplied by probability of N takes a value n. So, if you simplify you will get mu times summation n is equal to 0 to infinity n minus mu, expectation of X minus n mu given N takes a value small n, multiplied by probability of N takes a value small n.

That is same as 0 because expectation of X minus n mu, given N takes a value small n that is nothing but expectation of psi 1, psi 2 and so on plus psi n minus n mu. That expectation quantity is going to be 0 because expectation of psi i’s are going to mu and this is expectation of psi 1 plus psi 2 and so on plus psi n minus n times mu therefore, that becomes 0. Hence, the variance of X substitute all these three variance results, all these expectation results in the above this expression. Therefore, you will get variance of X is going to be mu times sigma square plus mu square sigma square and third term is going to be 0.

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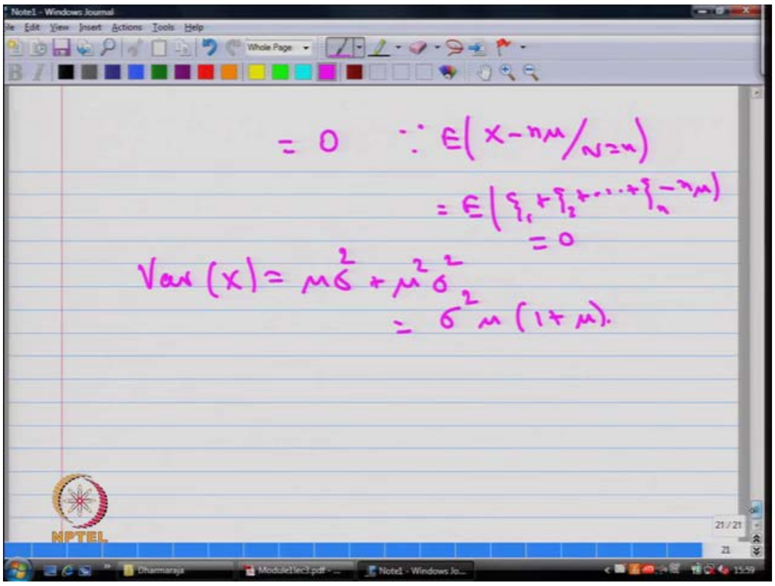

Therefore, you will get sigma square mu multiplied by 1 plus mu. This problem is useful in branching processes, therefore I discussed in this lecture. Even though there are many more problems of similar kind, we have chosen few problems for the illustrative purpose, and we are going to come across the similar problems in the course also. Therefore, I have chosen some five problems to discuss as a illustrative example.

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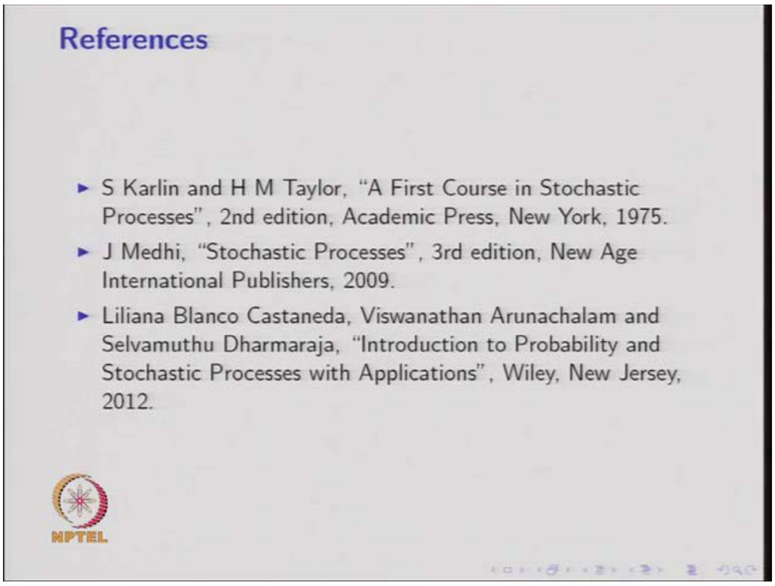

Here is the reference for this lecture.