Stochastic Processes
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Module - 7
Brownian Motion and its Applications
Lecture - 3
Stochastic Differential Equations

This is a stochastic processes module seven Brownian motion and its applications. This is lecture three, stochastic differential equations. In the lecture one, we have discussed the Brownian motion definition and its properties; and in the lecture two, we have discussed process derived from a Brownian motions. In particular, we have discussed geometric Brownian motion, and then the levy process and few applications also.

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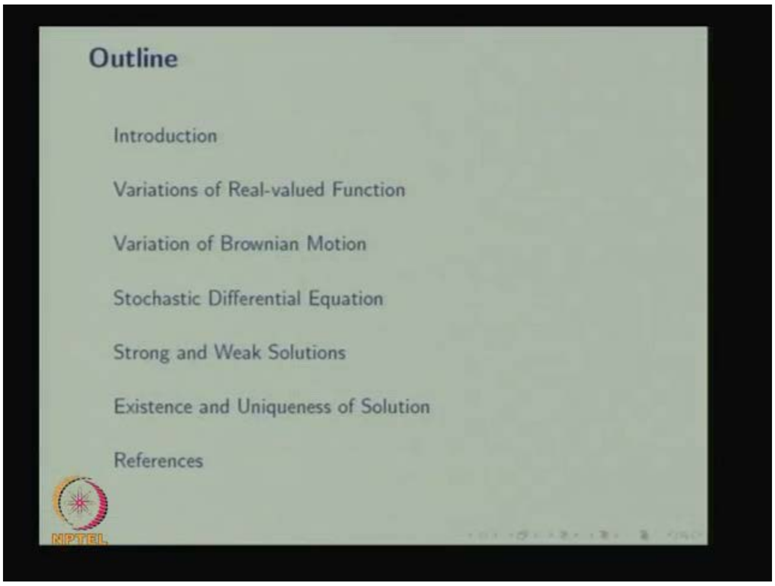

In this lecture, we are going to discuss stochastic differential equations. We are going to start with the motivation behind the stochastic differential equation, then we are going to discuss the variations of real valued functions followed by the Brownian motions, then we are going to give the definition of stochastic differential equations, then we are going to discuss the strong and weak solutions and also we are going to discuss, the existence and uniqueness of solution.

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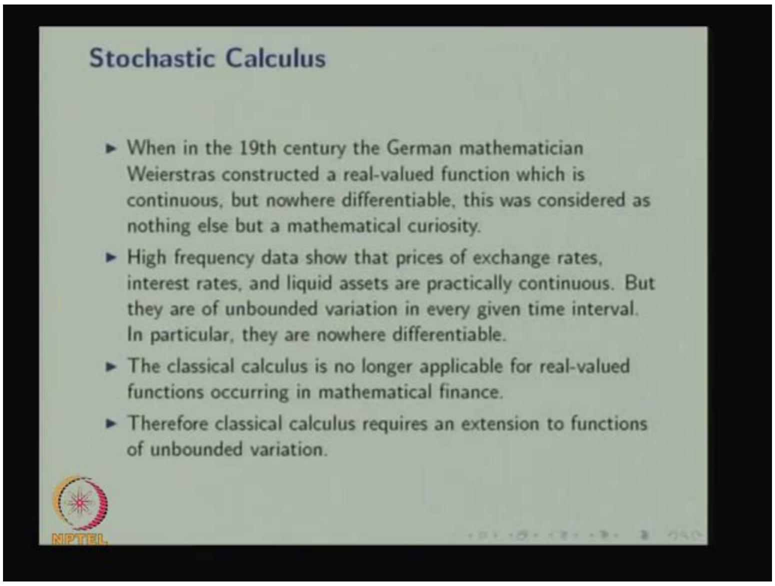

The motivation behind the stochastic differential equation. And in the nineteenth century, the German mathematician Weierstras constructed a real-valued function which is continuous, but nowhere differential, this was considered as a nothing else but a mathematical curiosity. High frequency data show that prices of exchange rates, interest rates, and the liquid assets are practically continuous. But they are of unbounded variation in every given time interval. In particular, they are nowhere differentiable. The classical calculus is no longer applicable for real-valued functions occurring in mathematical finance. Therefore the classical calculus requires an extension to functions of unbounded variation.

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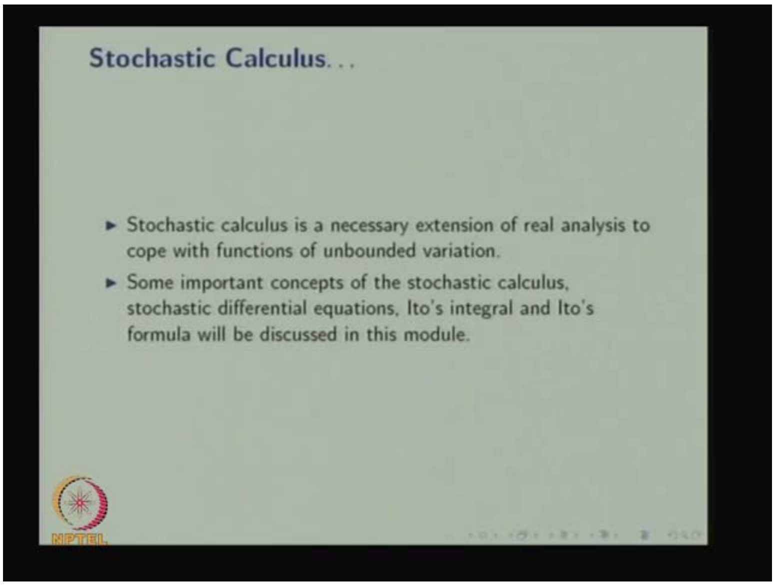

Stochastic calculus is a necessary extension of real analysis to cope up with the functions of unbounded variation. The Brownian motion, which is the stochastic process is the sample path is continuous, but the sample path is a nowhere differential and also it is a unbounded variation. Some important concepts of the stochastic calculus, stochastic differential equations, Ito integral and Ito's formula will be discussed in this module.

Ito integral is nothing but stochastic integral equations and also, we are going to discuss the Ito formula to solve the stochastic integral equation or stochastic differential equations. So, in this model we are going to discuss the stochastic differential equations. In the next lecture, we are going to discuss the stochastic integral equations and in the lecture five, we are going to discuss the Ito formulas and some important stochastic differential equations and their solutions.

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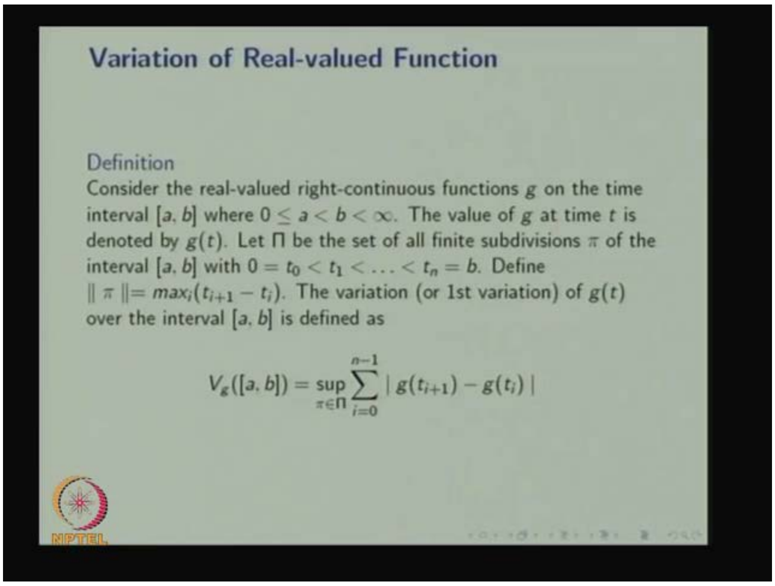

First we are going to discuss, the variations of real value function our interest is to study the variation of Brownian motion, but before that we will discuss the variation of real valued function and followed by that, we are going to discuss the variations of Brownian motion the first variation is defined as follows.

Consider the real valued right continuous functions g on the time interval a to b. The value of g at time t, is denoted by g of t. Let i be the set of all finite subdivisions pi of the interval a to b with 0 equal to t naught, which is less then t 1 which is less then and so on. Which is less then t n where t n is equal to small b, define the norm of pi that is maximum of i the length of the interval that is t i plus one minus t I, the variation or the first variation.

The first order variation of g of t, over the interval a to b is defined as the variation of g in the interval or over the interval a to b is nothing but supremum of pi belonging to pi. The summation running from i 0 to n minus one, the absolute of g of t plus i t plus t of i plus one minus g of t i. So, the modules of this difference of the value evaluated at t i plus one and t i, the difference of the function g take absolute then find the summation, then find the supremum that will be call it as a variation of g of t.

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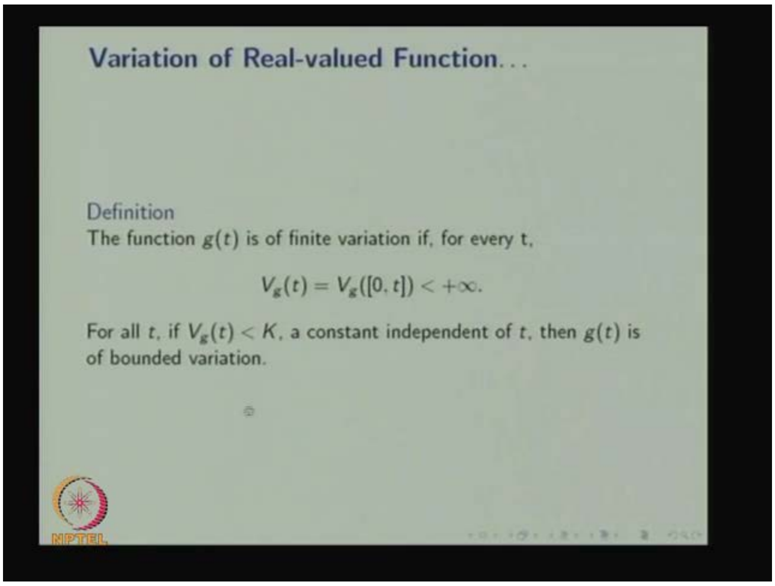

The function g t is of finite variation if, for every t, if V g of t that is nothing but the V g of between the intervals 0 to t is a finite, whenever the interval 0 to t. The first variation of the function g is a finite one, then we say the function g t is a finite variation for every t for all t, if V g of t is bounded by a constant K, which is independent of t, then we say the function g of t is of bounded variation, this is for the first order variation for the any real valued function g, alternatively a function g is said to have a bounded variation of an closed interval, its total variation is finite.

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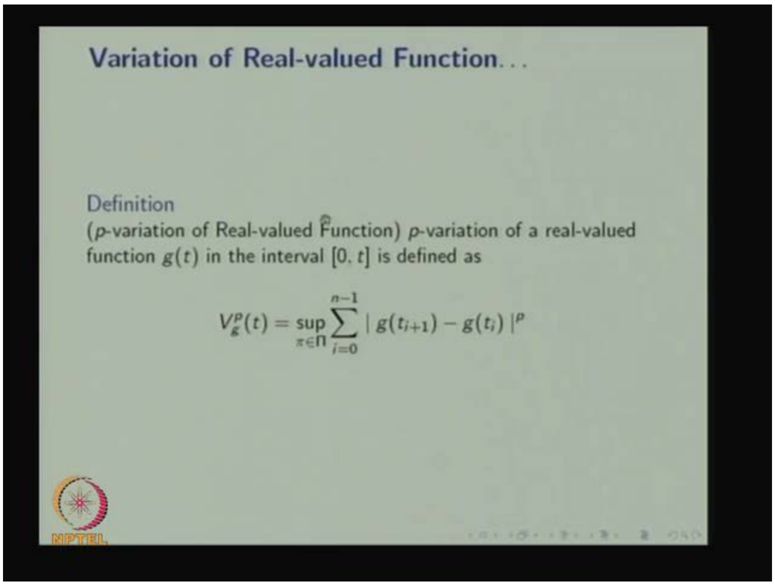

The same way, we are going to define the p-variation or pth order variation of real valued function, the same interval 0 to t that is nothing but the only difference is absolute power. So, if p is equal to one then it is a first order variation, if it is p is equal to two then it is second order variation and so on.

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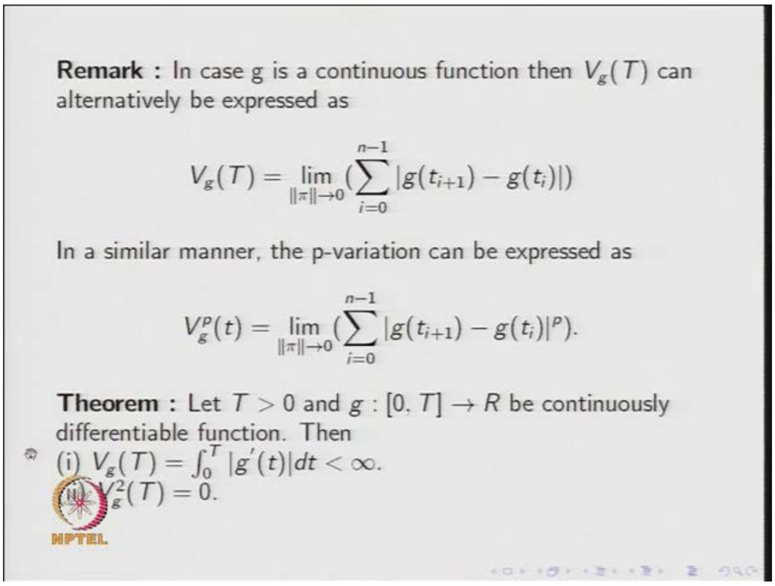

Remark in case g is a continuous function then V g of T can be alternatively expressed as V g of T is a limit norm pi tends to 0 summation, i is equal to 0 to n minus one absolute of g of t plus t of i plus one minus g of t i, where here pi is a arbitrary partitioned of the interval 0 to capital T and the norm pi is a maximum of t k plus one minus t k, where k is lies between 0 to n minus one. In a similar way the p-variation can be expressed as written here, the V g of p variation of t that is return as the limit norm pi tends to 0.

The similar expression, the absolute power p, if g is a continuous function then supreme is replaced by limit later on this video lecture, you will see how to calculate the second order variation, which is known as the quadratic variation for Brownian motion. In finding the quadratic variation of Brownian motion g of t will be replaced by w of t. And the limit will be taken in the sense of limit of sequence of random variables. Now, we have this theorem, if g is continuously differentiable function from the interval 0 to capital t, then the first order variation is integral of modulus of g dash of t with the limits 0 to capital T and second order variation is 0.

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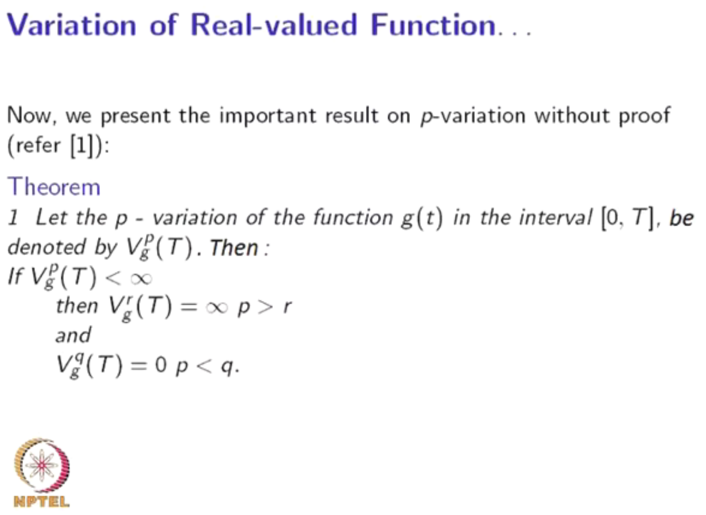

Now, we present the important result on p-variation without proof ,as a theorem. Theorem one let, let the p-variation of the function g of t in the interval 0 to capital T where, T is the positive real number denoted by V suffix g superscript p of T, if pth order variation is finite then, all the earlier order variation is going to be infinite and all the further order qth order variation from the p that will be 0. Whenever the pth order variation is a finite then all the earlier order variation from the pth order p, that will be infinity that means, it is unbounded and for variation of qth order will be 0 were q is where p is less then q. So, these are very important result and using this result, we are going to discuss the variations of Brownian motion also.

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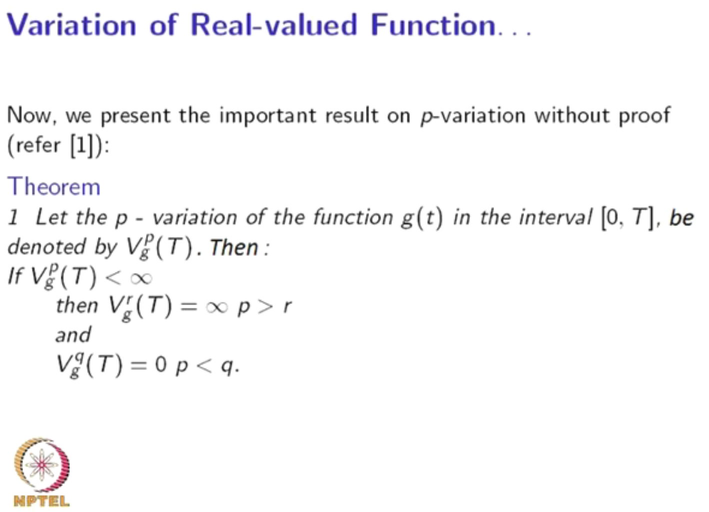

Examples for this can be found in the problem sheet.

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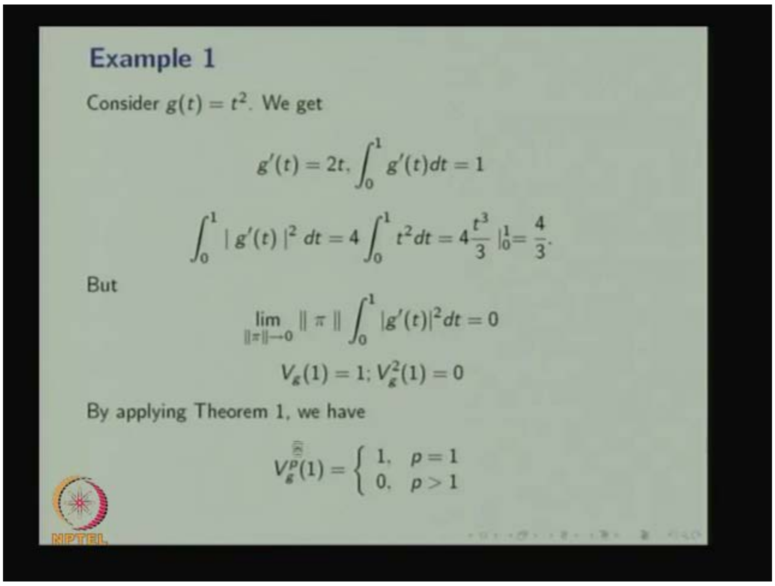

As example to understand the application of the previous theorem and the results, we have these consider the function g of t that is t square is a polynomial of order two. So, you can find the derivative and you can find the derivative absolute whole square therefore, if you find limit pi tends to 0 of pi times, this one is equal to 0 therefore, you will get the first order variation is one and the second order variation will be 0, in the interval 0 to one or at the time at the t equal to one by applying theorem one.

We get whenever p equal to one it is value is one and for all the further order, that will be 0 this is for the function, which is a polynomial of degree two therefore, you are getting p is equal to one and the first order variation is equal to one and the further orders second, third, forth and so on. The variation will be 0 for the second order degree polynomial.

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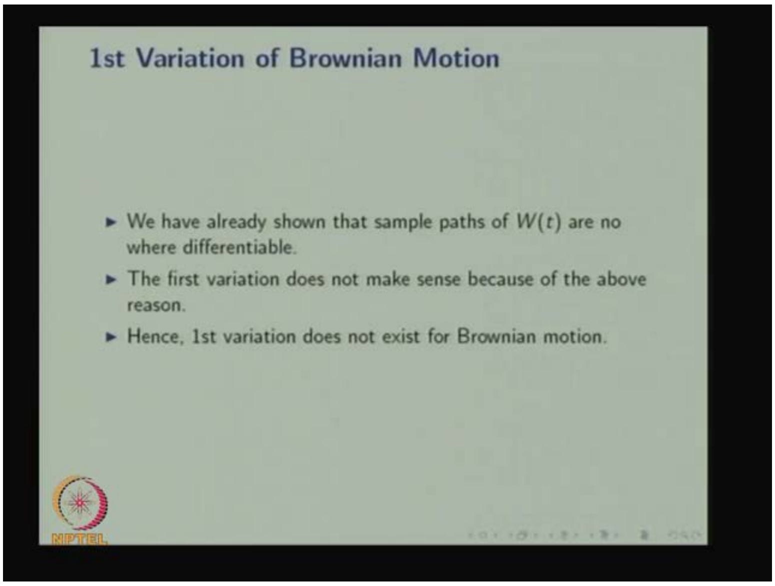

Now, we are going to discuss the first variation of Brownian motion. We have already shown that the sample path of W t are no where differentiable therefore, the first order variation does not make sense because of the above reason because the derivate, it is no where differentiable therefore, you cannot get the first order variation. Hence, the first order variation of the Brownian motion does not exists.

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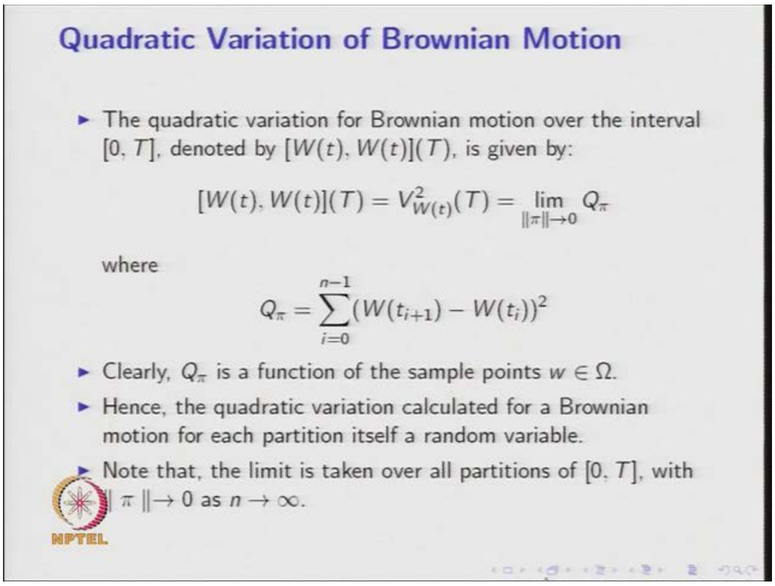

Now, we are moving into the quadratic variation of Brownian motion. The quadratic variation of Brownian motion over the interval 0 to T, where T is positive real number denoted by the notation W t comma W t of T that is given by v suffix t, that is a wrong notation its a v suffix w t is is superscript two of T. That is nothing but limit pi tends to 0 of two q pi, where q pi is defined summation i is equal to 0 to n minus one.

The difference of W's and the time point t i to t i plus one the whole square, clearly because you are making a difference of W's. so, the Q pi is a function of the sample points of w belonging to omega. And also hence the quadratic variation calculated for the Brownian motion for each partition itself a random variable, because this is the random variable the difference is the random variable, the summation will be a sum of random variables is a random variable therefore, the Q pi is the random variable and you are finding limit pi tends to norm of pi tends to 0 of Q pi.

That is nothing but note that this limit is taken over all partitions of 0 to pi, with norm of pi tends to 0 as n tends to 0 as n tends to infinity norm of pi is defined as the maximum of pi of the length of the interval t i plus one minus t i therefore, norm of pi tends to 0 means, you are finding the limit is taken over. All partitions of 0 to pi 0 to T. So, we have to find out what is the limit norm pi tends to 0 of this random variable for every n, this will be a random variable. So, we have to find out the limit taken over all partitions of 0 to T, with the norm of pi tends to 0 as n tends to infinity.

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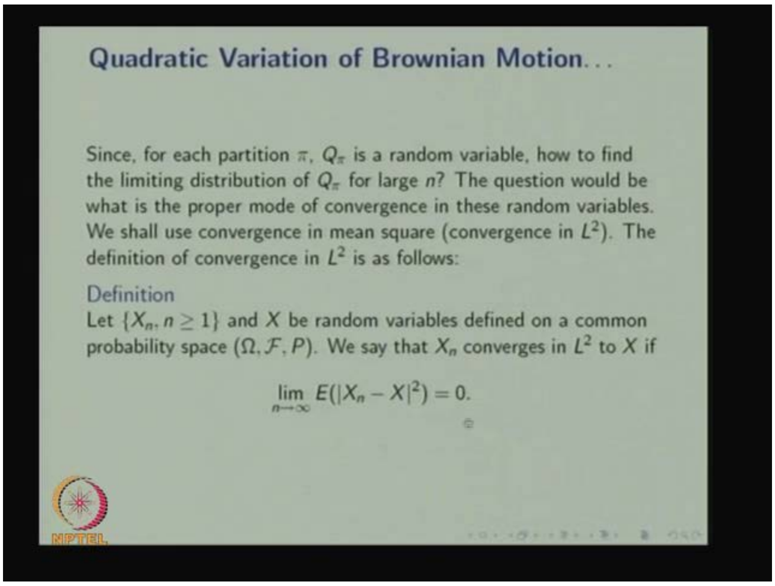

Since, for each pi for each partition pi the Q pi is a random variable, how to find the limiting distribution of Q pi for large n? The question would be what is the proper mode of convergence, in these random variables. We shall use the convergence in mean square that is convergence in L 2 to find the limit of norm pi tends to 0 q of pi as a n tends to infinity.

So, for that we are going to we are going to define the convergence in L 2, let pi n let X of n, n equal to n is greater than or equal to n and X be a random variables defined on a common probability space omega F, P. We say that X n converges convergence in L 2 to the random variable X, if limit n tends to infinity expectation of the absolute of X n minus X whole square is equal to 0.

So, if this condition is satisfied and this is the sequence of random variable and this is random variable. Both are defined in the same probability space omega F, P then we say the sequence X n convergence to the random variable X in L 2. So, thus same approach we are going to use find out the limiting distribution of the random variables Q of pi for a large n or n as n tends to infinity.

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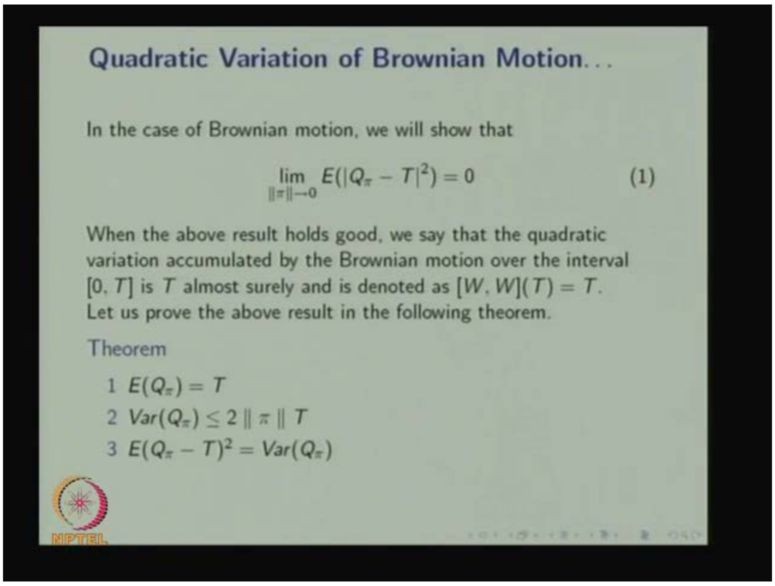

In this case of Brownian motion, we will show that limit norm pi tends to 0 expectation of absolute of pi q sorry Q pi minus T whole square is equal to 0. That means, the sequence of random variable Q of pi as a n tends to infinity converges to the random variable, which is constant to capital T in L 2. If this condition is since, this condition is satisfied, when the above results holds good, we say that the quadratic variation accumulates, accumulated by the Brownian motion over the interval 0 comma capital T is capital T in mean square and is it is denoted W of W of the interval 0 to capital T that is capital T.

So, to prove the sequence of random variable Q pi converges to the random variable T as n tends to infinity in L 2, we will prove it an three stages, the first stage we will find out, will prove that expectation of Q of pi is equal to capital T then, will prove the variance of Q of pi that is less then or equal to two times norm of pi of t. Therefore, you can prove the final result expectation of Q pi minus T whole square that is nothing but variance of Q pi because the expectation of Q pi is T therefore, expectation of Q pi minus T whole square is variance of Q pi as n tends to infinity, the Q pi will converges to the random variable t in L 2.

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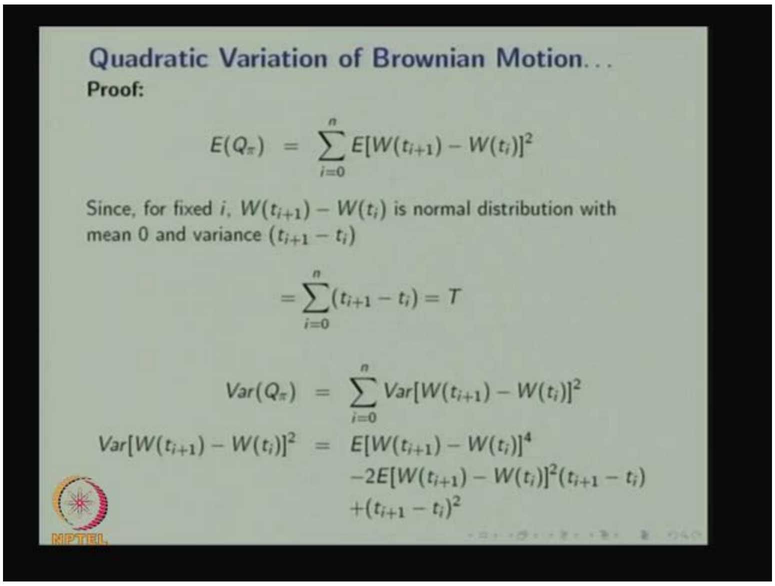

The proof as follows first we will find out the expectation of E Q pi that is nothing but the summation of i is equal to 0 to n minus one expectation of W of t i plus one minus W of t i the whole square. Since for fixed i the difference of the W's, W is normally distributed random variable with the mean 0 and the variance is nothing but the length of the interval therefore, the expectation of difference of random variable whole square is nothing but the variance therefore, for fixed i that is nothing but the t of i plus one minus t i. The summation is varies form i is equal to 0 to n minus one therefore, you will get capital T. So, the first part is a proved.

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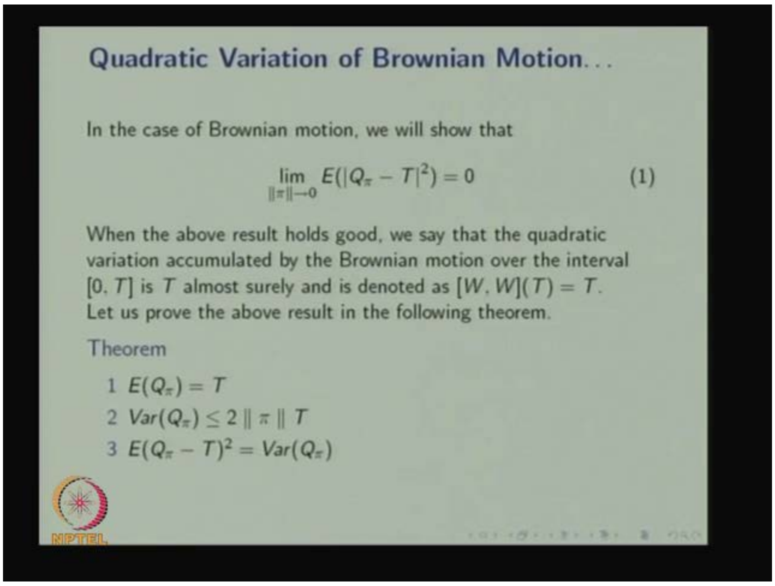

That is expectation of Q pi is equal to t.

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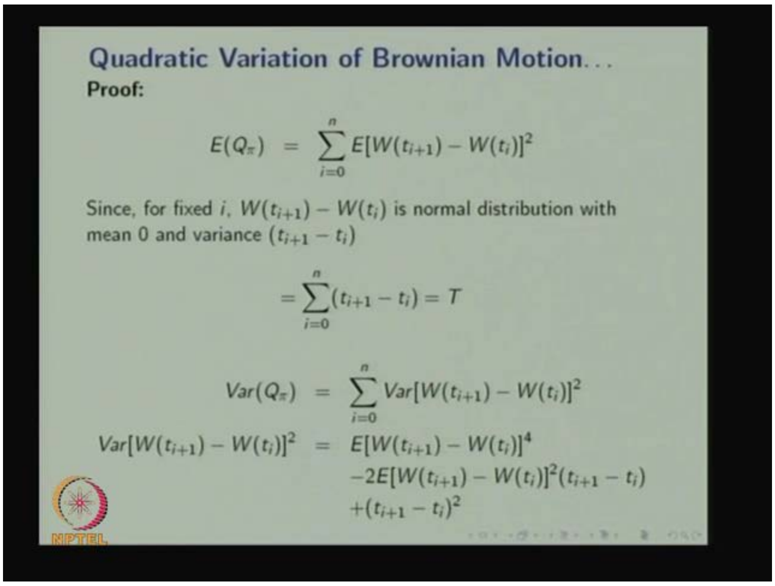

Now, will find out the variance of E Q pi. That is less than or equal to we have to prove that second part variance of Q pi is less than or equal to two times norm of pi multiplied by T. Third part is Q t's so, the variance of Q pi is nothing but summation i is equal to 0 to n minus one variance of the difference of random variable whole square but variance of difference of random variable whole square, that is nothing but the expectation of difference of random variable, whole power four minus two times expectation of difference of the random variable, whole square multiplied by t i plus one minus t i plus t i plus one minus t i whole square or fixed i.

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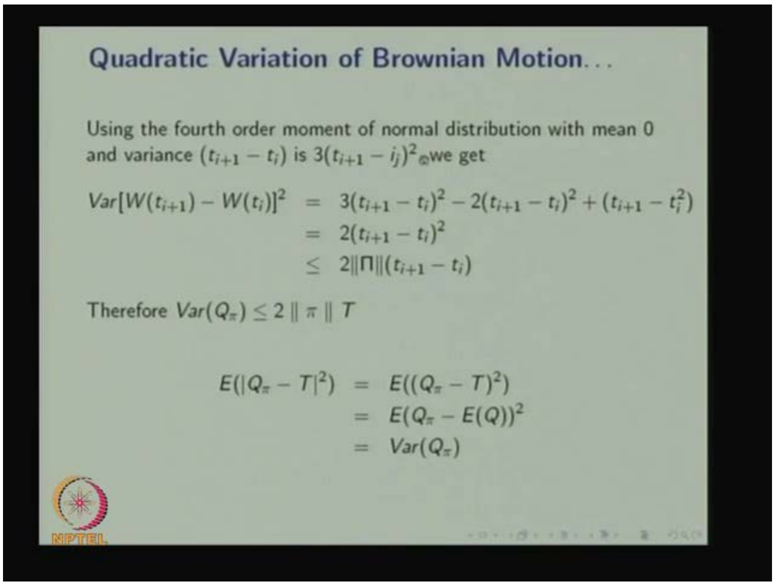

Using the forth order moment of normal distributed random variable, with the mean 0 and the variance t i plus one minus t minus one. The first term in the right hand side, the first term in the right hand side expectation of difference of the random variable power four, that is forth order moment about the forth order moment that is nothing but three times t i plus on minus, t i whole square.

Therefore, the right hand side variance of the difference of the random variable whole square that is nothing but three times, the difference the time difference whole square minus two times, the time difference whole square plus the plus time difference whole square therefore, this is nothing but two times difference whole square. The two times time difference whole square is nothing but that is less than or equal to two times norm of pi multiplied by the time difference. Therefore the variance of Q pi is less than or equal to two times norm of pi times t therefore, since you know that the expectation of Q pi is equal to T therefore, expectation of norm of Q pi minus T whole square that is nothing but variance of Q pi.

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Quadratic Variation of Brownian Motion...

Then, we get

$$[W, W](T) = \lim_{\|\pi\| \to 0} Q_\pi = T$$

since

$$\lim_{\|\Pi\| \to 0} E((Q_\pi - T)^2) = 0$$

Hence $[W, W](T) = T$. Also, for $0<T_1<T_2$, $[W, W](T_2) - [W, W](T_1) = T_2 - T_1$, i.e., the Brownian motion accumulates T_2-T_1 units of quadratic variation over the interval T_1 to T_2.

Informally we can write as $dW(t)\, dW(t)=dt$ and this dt is infact $1.dt$.

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Therefore, the second order variation of the Brownian motion W t between the, over the interval 0 to capital T is nothing but limit norm of pi tends to 0 Q pi, that is same as T. Since, limit norm pi tends to 0, the expectation of Q pi minus T, whole square is equal to 0. So, the conclusion is the second order are the quadratic variation of Brownian motion is capital T between the interval 0 to capital T.

This means, it accumulates unit quadratic variation per unit also for 0 less than T 1 less than T 2. The quadratic variation till T 2, the quadratic variation, sorry the quadratic variation till T 2 minus quadratic variation till T 1, that is same as T 2 minus T 1. That is the Brownian motion accumulates T 2 minus T 1 units of quadratic variation, over the interval T1 to T 2. Since, this is true for every interval, we refer that the Brownian motion accumulates quadratic variation at the rate one per unit this last statement, we write informally as d W t, d W t is equal to d t and this d t is in fact one times d t.

In other words, the above phenomena can be represented in differential form as a differential of W t multiplied with the differential of W t, this is a quadratic variation that is nothing but the differential of T that is a meaning of the Brownian motion, accumulates unit quadratic variation per unit time.

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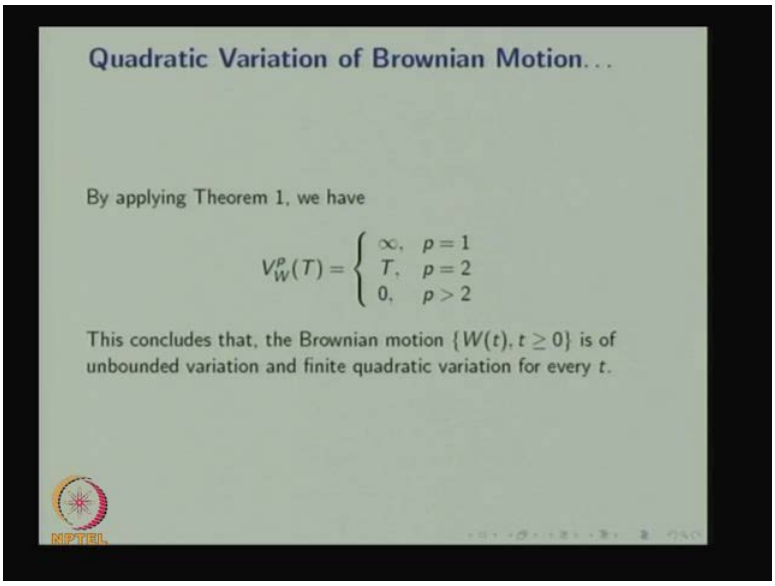

Now, we are applying the same theorem, which we discuss for the variation of real valued function. We have a the pth order variance of a Brownian motion between the interval 0 to capital T that value, will be does not exist for the first order variation. Therefore p is equal to one, it is infinity for the quadratic variation. It is capital T it is a bounded variation bounded a quadratic variation, where as the first order is a unbounded variation for p greater than two it is 0.

So, the example we have taken is g of t is equal to t square for that the first order variance is finite and the further variations are 0 whereas, for the Brownian motion the first order variation is a infinity that is unbounded variation. And the second order variation is a finite value that is capital T, the further variations are 0 this concludes that the Brownian motion is of unbounded variation because of p equal to one, the variation is infinity and finite quadratic variation because p is equal to two value will be capital T for every t.

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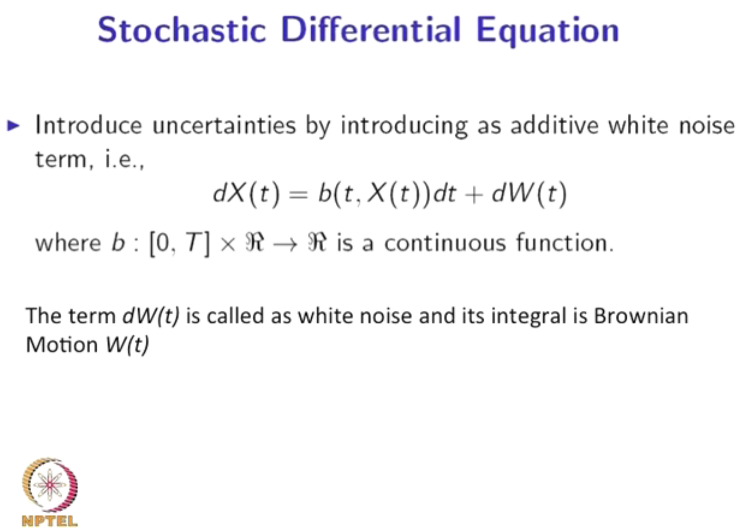
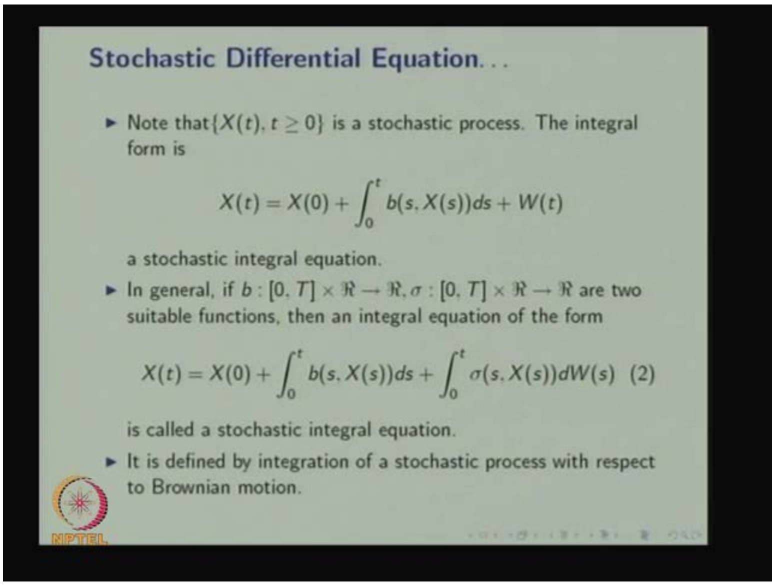


Now, we are moving into the Stochastic Differential Equations. Introduce uncertainties by introducing as additive, white noise term that is d X t is equal to b of t comma X t d t plus d W t, where b is the real valued continuous function from 0 comma T cross R. The term d W t is called as white noise and its integral is Brownian motion W t here, the above equation is also known as Stochastic Differential Equation or SDE, the meaning of which would be more clear after the introduction of stochastic interpreted concept.



Note that X of t is a stochastic processes, stochastic process the integral form of the differential equation is X of stochastic differential equation is X of t is equal to X of 0 plus integration 0 to t of b of s of X of s d s plus W t is a stochastic integral equation. In general, if b and sigma are the two suitable functions, then the integral equation of the form X of t is equal to X of 0 plus integration 0 to t function, b integration with respect to s plus 0 to t sigma s of x of s d W s.

In the equation two the first equation the first integral is different from the second integral. The second integral is integration with respect to the Brownian motion sample path W s, this integral equation is defined by the integration of stochastic process with respect to the Brownian motion.

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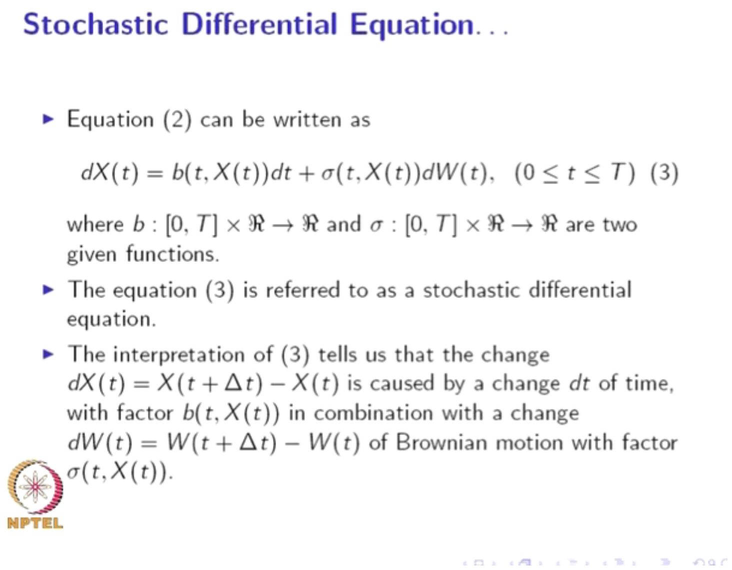

So, the equation two there is nothing but this equation, equation two can be written as d X t is equal to b of t comma X t d t plus sigma t of X t d W t, where t is lies between 0 to capital T. Where b and sigma are two given functions, this the equation three is referred to as a stochastic differential equations. The interpretation of equation three tells us that, the change that is d of X t that is nothing but the X of t plus delta t minus x t is caused by a change d t of time with the factor b of t comma X t.

In combination with change d W t that is nothing but W of t plus delta t minus W t of Brownian motion with the factor sigma of t comma X t. The Brownian motion is adapted to the natural filtration. So, the unknown in the sigma as well as b and increment of Brownian motion therefore, this equation is called stochastic differential equation.

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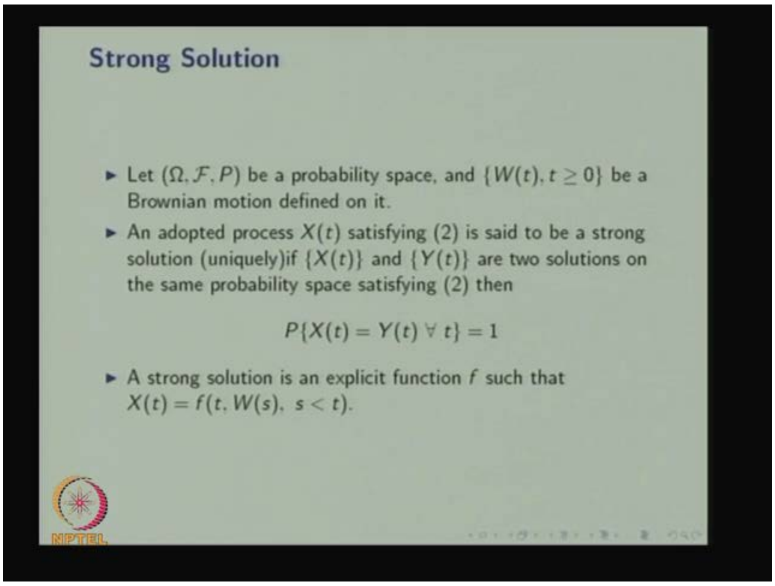

Now, we are going to discuss the there are two types of solutions for the stochastic differential equation, the first type is called Strong Solution. The second type is called weak solution. So, we are going to discuss the strong solution, first let sigma sorry let omega F, P be the probability space and W t be a Brownian motion defined on it. An adopted process X of t satisfying, the equation two that is stochastic differential equation is said to be strong solution uniquely, if X of t and W t are the two solutions, on the same probability space satisfying the stochastic differential equation two then, the probability of X of t is equal to Y of t for all t that will be one.

Then X t is called a strong solution and it is also a unique solution that means, if you have a another solution Y of t then probability of X of t is equal to Y of t for all t will be one. In general a strong solution is a explicit function f such that, X of t is a function of t comma W s where s is less than t. One can write the solution in an explicit function F of t, with the Brownian motion then the solution called strong solution.

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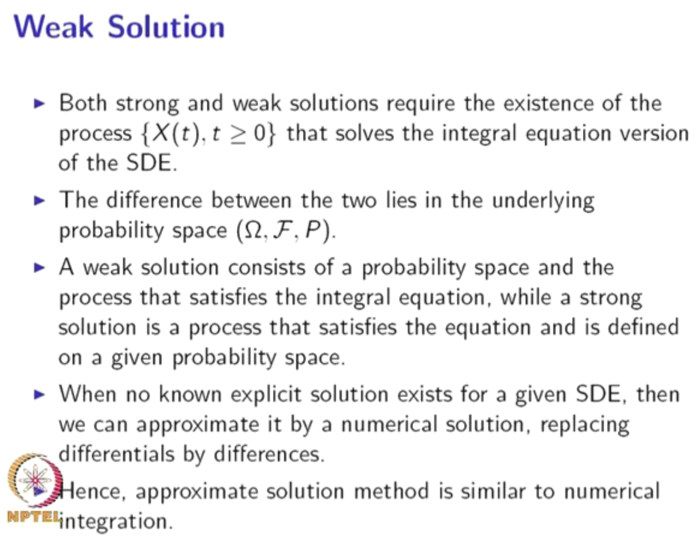

Now, we are going to discuss what is the weak solution of stochastic differential equation. Weak solution both strong and weak solutions require, the existence of the process X t, that solves the integral equation version of the SDE. The difference between the two lies in the underlying probability space. A weak solution consists of a probability space and the process that satisfies the integral equation.

While a strong solution is a process that satisfies the equation and is defined on a Given probability space. When no explicit solution exists for a given SDE, then we can approximate it by the numerical solution replacing differentials by differences hence, approximate solution method is similar to the numerical integration. So, if this we have discussed the strong solution and the weak solution of a stochastic differential equations.

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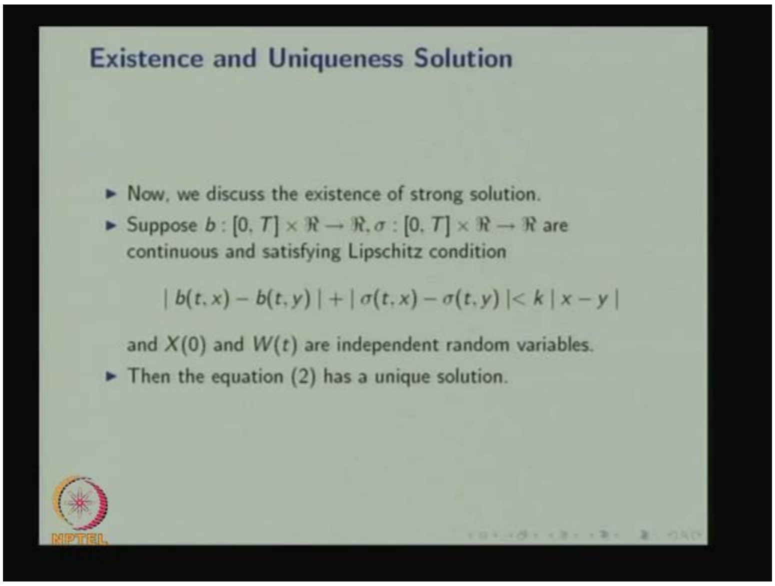

This course, we are interested to find the strong solution not the weak solution. When the above results hold good, we say that the quadratic variation accumulates, accumulated by the Brownian motion over the interval 0 comma capital T is a capital T in mean square and is equation two can be written as d x t is equal to b of t comma x t d t plus sigma t of x t d w t, where t is lies between 0 to capital T where b and sigma are two given functions .

Now, we discuss the simple examples for the stochastic differential equation. Consider stochastic differential equation d x t is equal to x t d w t, with a x of 0 is equal to one here b of t comma x is equal to 0 and sigma of t comma x is equal to x. You can verify the Lipschitz condition for this b is equal to 0 and sigma is equal to x. Hence, the strong solution exist, obtaining the strong solution will be explaining the further lectures.

We will see one more example for the stochastic differential equation, here S of t be the stock price at time t, the corresponding stochastic differential equation for this example is d of S t is equal to mu times S of d t plus sigma S of t d w t with the S of 0 is known. Here mu is a constant growth rate of the stock and sigma is a volatility, when you compare with the standard stochastic differential equation, we get b of t comma x is equal to mu of x and sigma t comma x is same as sigma x. Since, mu and sigma are constants with Lipschitz condition is satisfied. Hence, a strong solution exist and this example also, how to find the solution that will be discuss in the further lectures.

Now, we are going to discuss the existence and uniqueness solution that is basically strong solution. Now, we discuss the existence of strong solution suppose b is a continuous function similarly, sigma is continuous function satisfying Lipchitz condition, the absolute of difference of b of t comma x minus b of t comma y plus in the absolute sigma t comma x minus sigma of t comma y, if this summation is less than k times absolute of x minus y.

Where k is the positive constant and also the initial distribution X naught and W t are independent random variables, then we can say the solution is going to exist that that will be unique also. So, whenever the Lipchitz conditions satisfied with two continuous function b and sigma for a positive constant k along with X of 0 and W of t or a independent random variables. If both conditions are satisfied by any stochastic differential equation, then we can conclude it as the unique and it as the, it as the existence of strong solution as well as it will be unique.

This is similar to existence and uniqueness solution of ODE, the only difference is it does not have the term the sigma term. It as only the first which is less than k times absolute of x minus y that is Lipchitz condition for ODE. So, here also the same thing along with the continuous function sigma, if this condition is satisfied along with this condition X (0) and W (t) are independent random variables, then the given SDE as unique have the existence of strong solution and that will be unique.

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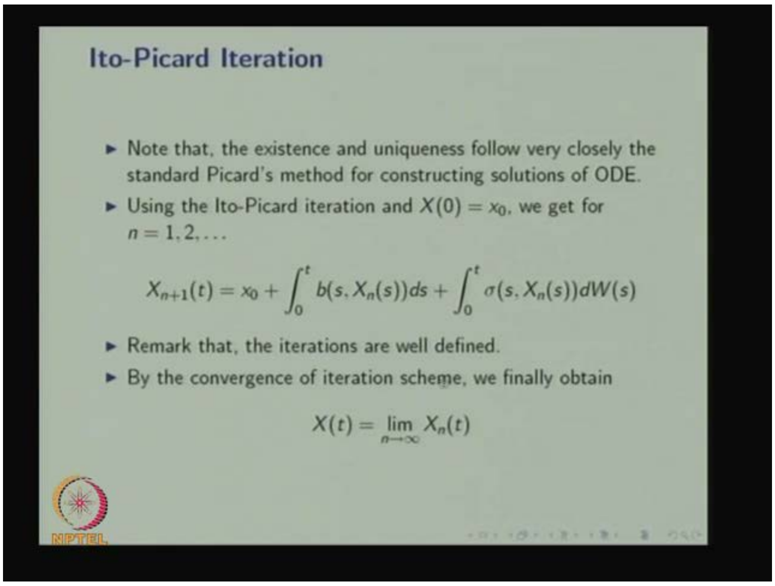

Note that, the existence and uniqueness follow very closely the standard Picard's method for constructing solutions of ODE. You know the Picard iteration for ODE ordinary differential equation and this iteration is called a Ito-Picard iteration. So, using Ito-Picard iteration X naught is equal to X (0), we get for n is equal to one, two, three, x n one plus n of t will be x naught plus the integration plus the another integration that means, with the initial value X naught. We can find for n is equal to one then, find for for n is equal to 0, you will find X 1 of t first using x naught, then for n is equal to one, you will get X of two, X suffix two of t and recursively, you can get the X n plus one of t for every n as n tends to infinity you can get the X of t.

So, remark that the iterations are well defined because it satisfies the Lipchitz condition as well as X naught and W t are independent random variables, the solution is going to be exist as well as it will be unique and this iterations are well defined by the convergence of iteration scheme we finally, obtain X of t is limit n tends to infinity X n of t for every n it is a random variable. So, this random variable converges to the random variable X of t. So, this we are showing through the Ito-Picard iteration and this Ito-Picard iteration is similar to the Picard iteration of ordinary differential equation.

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Example 2

Consider the stochastic differential equation

$$dX(t) = X(t)dW(t), \text{ with } X(0) = 1.$$

Here, $b(t,x) = 0$ and $\sigma(t,x) = x$. Hence, Lipschitz codition is satisfied.
Hence, the strong solution exists. Obtaining the strong solution will be explained in further lectures.

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Now, we discuss the simple examples for the stochastic differential equation, consider the stochastic differential equation, d X (t) is equal to X (t) d W (t) with the X of 0 is equal to 1. Here b of t comma x is equal to 0 and sigma of t comma x is equal to x, you can verify the Lipchitz condition for this b is equal to 0 and sigma is equal to x. Hence, the strong solution exist. Obtaining the strong solution will be explained in the further lectures.

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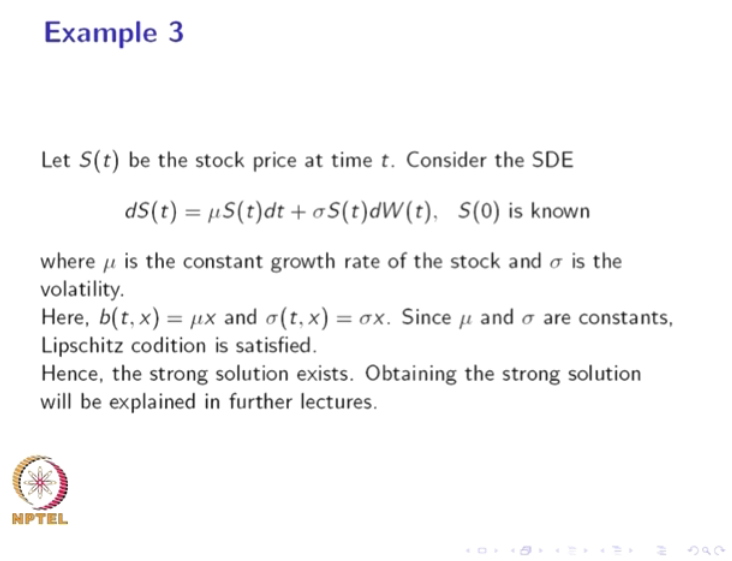

We will see one more example for the stochastic differential equation, here S of t be the stock price at time t. The corresponding. Stochastic differential equation for this example is the d of S(t) is equal to mu times S (t) of d t plus sigma S of t d W (t) with the S of 0 is known. Here mu is the constant growth rate of the stock and sigma is the volatility. When you compare with standard stochastic differential equation, we get b of t comma x is equal to mu of x and sigma of t comma x is same as sigma x. Since, mu and sigma are constants Lipschitz condition is satisfied. Hence, the strong solution exists and this example also how to find the solution, that will be discussed in the further lectures.

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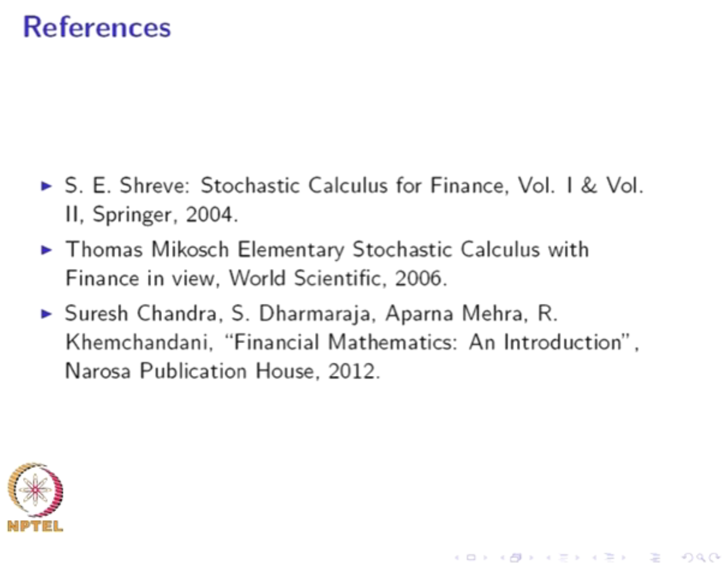

Here is the list of books for the reference, this lecture.

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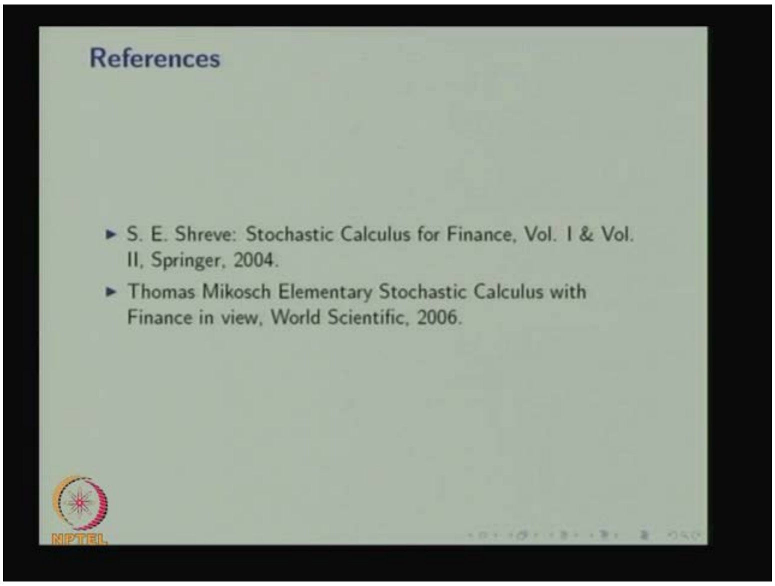

We have discussed stochastic differential equation for that, we have discussed the variations of the real valued function, starting with the first order variation, pth order variation, then followed by that we have discussed the variations of Brownian motion starting with the first order variation, quadratic variation and pth order variation also.

Then we have discussed the stochastic differential equation by adding a white noise term, in the ordinary differential equations then we have discussed, the equivalent stochastic integral equations. And also, we have discussed strong and weak solutions; and finally, we have a given existence of, existence as well as the uniqueness of strong solution and finally, we have discussed Ito-Picard iteration methods .