Stochastic Processes
Prof. Dr. S. Dharmaraja
Department of Mathematics
Indian Institute of Technology, Delhi

Module - 7
Brownian Motions and its Applications
Lecture - 2
Processes Derived from Brownian Motion

This is a stochastic processes module 7 Brownian motion and its applications. Lecture 2 processes derived from Brownian motion. In the lecture 1, we have discussed the definition and properties of Brownian motion. We started with the random walk, then the sample path of the random walk, then we have given few properties of random walk, followed by that we have made the derivation of Brownian motion through the random walk.

Then, we have discuss the sample path of the Brownian motion followed by that, we have discussed few important properties, such as strict sense stationary increment, independent increment property, now here differentiable property, self similar property, Markov property, twin Gaussian process and the Brownian motion. Also, we have discussed the Kolmogrov equation and we have given the connection with the heat equation.

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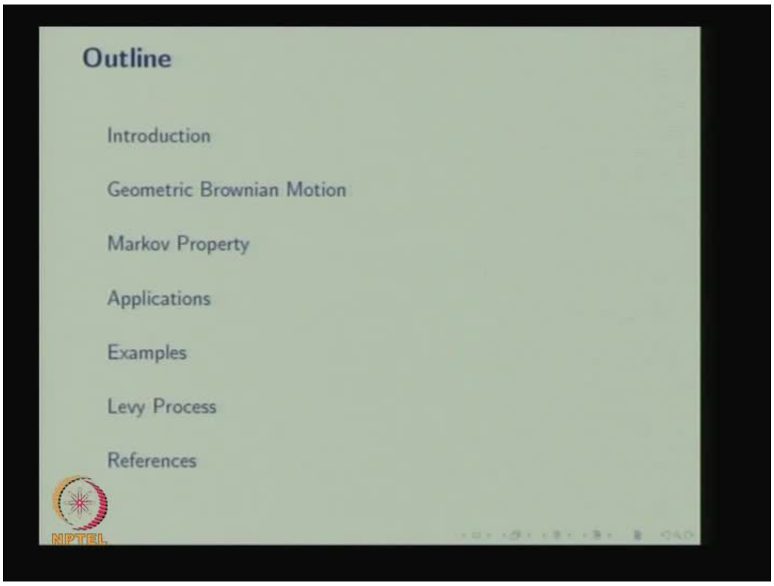

Then we have discussed the joint distribution of Wiener process also. Finally, we have discussed the martingale property of Brownian motion. So, with that the lecture 1 of module 7 was completed. In the lecture 2, we are going to cover the definition of a geometric Brownian motion, then few properties also going to be discussed. Followed by that, we are going to discuss the applications of geometric Brownian motion, also few examples of geometric Brownian motion. Finally, the process derived from Brownian motion that is a levy process also going to be discussed.

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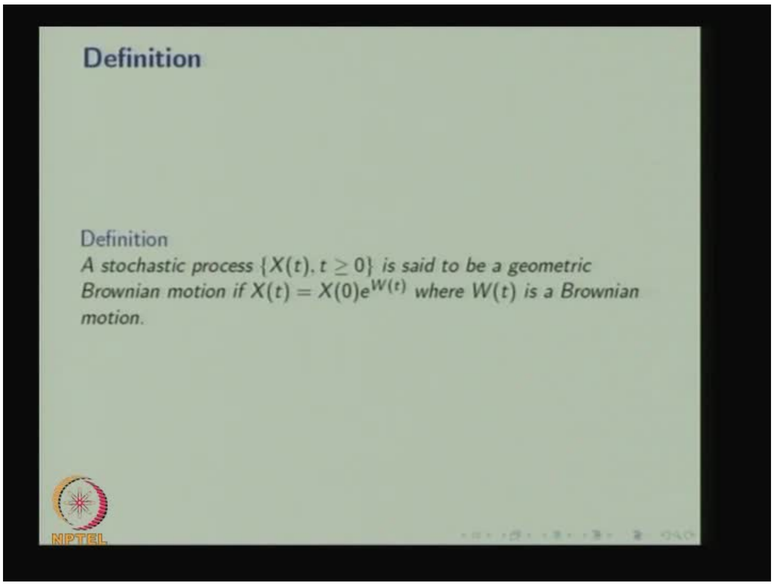

The definition of a geometric Brownian motion, a stochastic process X t is said to be a stochastic process X of t, that t varies from 0 to infinity is said to be geometric Brownian motion. If X of t is of the form X naught exponential of W, where W t is the Brownian motion that means whenever you have a Brownian motion, then if you make a another stochastic process as the function of Brownian motion, but is of the form X of t is equal to X of 0 e power W t. Then the X of t is the stochastic process is said to be a geometric Brownian motion.

You know the range of W t that is minus infinity to infinity. Since X of t is of the form X of 0 e power W t. Therefore, the range of X t will be 0 to infinity range, the range of of X t is 0 to infinity. Therefore, you can use this as the model for the stock price at any time t, like that you can go for modeling any pricing of any security or derivatives at time t. So, the X t can be directly used in the application of finance.

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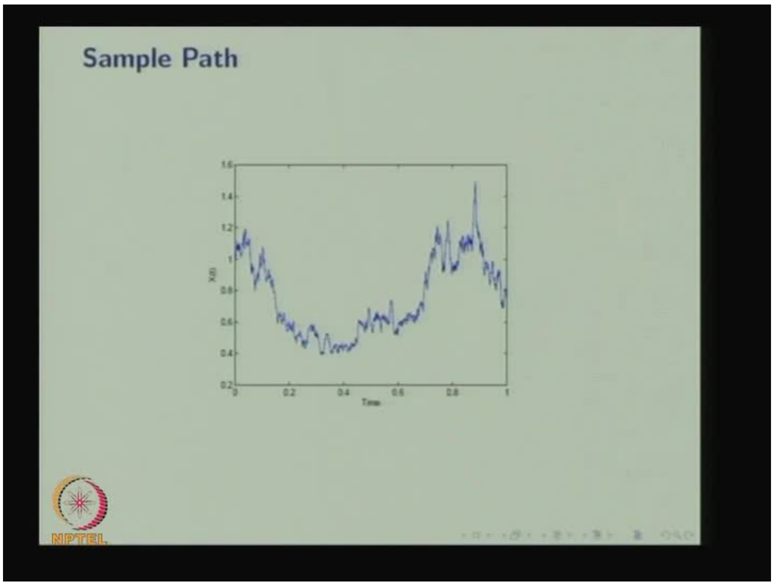

We can see the sample path over the time the X t, here the range is from 0 to infinity. So, you can see the continuous functions. You know that the sample path of W t is a continuous function and x t that is a geometric Brownian motion. This is also a continuous function because this is of the form X naught times e power W of t.

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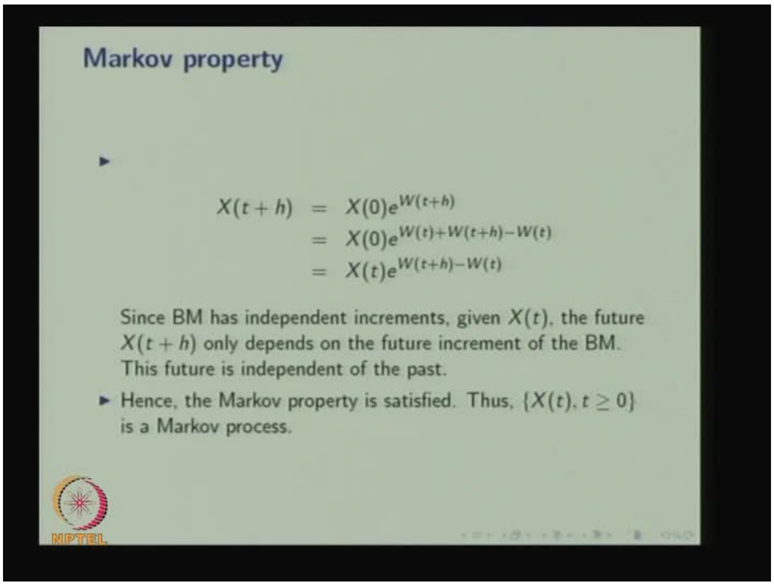

Now, we are going to discuss the the very important property that is the Markov property of geometric Brownian motion. The geometric Brownian motion, motion is of the form X of t is equal to X naught e power W t. Therefore, for any h greater than 0, you can write X of t plus h is same as X naught e power W of t plus h. I am going to verify the Markov property, therefore what I am going to do, I am going to add W t and subtract W t in the exponential term.

Therefore, the next step will be X naught e power W t plus W of t minus h minus W t. Already we know that X of t is equal to X naught e power W t, therefore X naught e power W t can be replaced by X of t. Hence, X of t plus 1 is same as X of t multiplied by e power W t plus h respectively; that means the stochastic process at the time point t plus h is same as at the time point t multiplied by exponential of the increment between the time points t to t plus h in W.

You already know that the Brownian motion or Wiener process satisfies the Markov property and also the increments are independent along with increments are stationary. Here we are going to use independent increments are independent, that means since the Brownian motion has independent increments given X t the future X of t plus h only depends on the future increments of the Brownian motion, given X t the X of t plus 1 t plus h it depends only on W of t plus h minus W of t. But since W is a Brownian motion, the Brownian motion increments are independent therefore, W of t plus h minus W t is independent of 0 to W t.

So, this feature is independent of the the past information. Here the past information is the from 0 to small time. Hence, the Markov property is satisfied because the future depends only on the present not the past. Therefore, the Markov property is satisfied hence, the W the X t is the Markov process. So, this is the above result is valid for all h greater than 0, therefore the Markov property is satisfied in the sense the future depends only on the present not on the past or it is independent of the past information. Hence, the Markov property is satisfied for satisfied by X t. Hence, the X t is the Markov process. So, the Brownian motion is also a Markov process, geometric Brownian. Brownian motion is a Markov process, also geometric Brownian motion is also a process.

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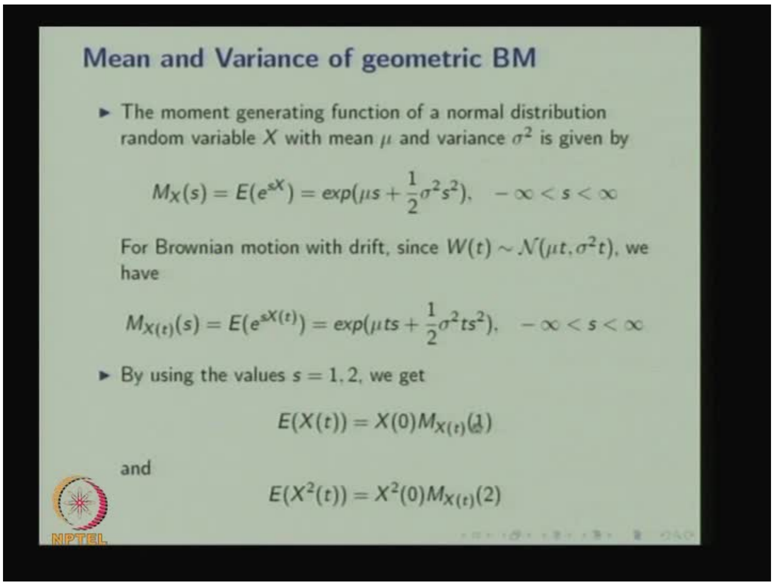

We are going to find out, what is the first and the second order moment of geometric Brownian motion. We can use the moment generating function concept. So, here the moment generating function of a normal distribution random variable X, with the mean mu the variance is sigma square is given by for any random variable, which is normally distributed in the mean mu. Variance sigma square the moment generating function which is M X of s function of s. That is nothing but expectation of e power s times X. So, this expectation is a exist, thus the the random variable X is normally distributed. So, the moment generating function exist because the moments of all order is exist.

So, that is same as the, you can do it separately this calculation. So, here I am just using the result of moment generating function of the distribution. That is same as exponential of see it is a function of s. Therefore, it is mu times s plus half sigma square s square, where s can take the value from minus infinity to infinity. Why you are using the moment generating function? Because here the geometric Brownian motion and the Brownian motion is connected in the form of X of t is equal to X of 0 e power of W t, where W t is normally distributed with the mean 0 and the variance p.

Hence, I am using the moment generating function of X of t as a function of X, that is expectation of e power s times X t. That is same as, that is same as, since W t is normally distributed, since W t is normally distributed with mu times t sigma square t for as normal random distribution, the mu is 0 and variance is 1. So, here you can make out that is same as exponential of mu t s plus half sigma square t s square, where replacing by using the same the above logic m get is of this 1. So, there is a mistake, it is it is moment generating function of w t and e of s times w t thus the x is normally distributed here, it should not be X t, it should be W t here, also it should be W t.

Therefore, moment generating function of W t is same as expectation of e power s times W t. Therefore, this is now using s is equal to 1 and 2. Now, you are, now you are finding the mean and variance of geometric Brownian function because the mean of X t is X naught times the moment generating function of X of t at 1. So, we are using the moment generating function, therefore we can get the mean of the geometric Brownian function. Similarly, finding the second order moment you can get the variance of geometric Brownian function.

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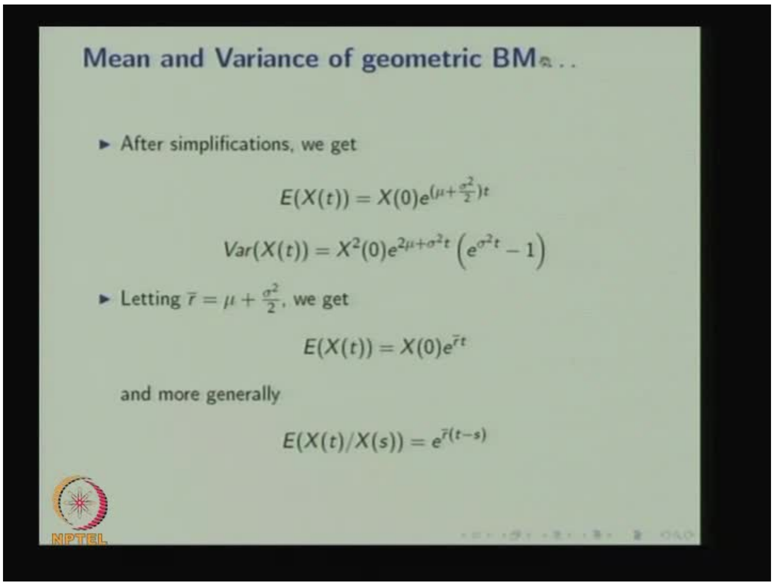

After simplification, that means first we have to use what is the moment generating function of W t and you have to use the form X is equal to X naught e power W t. Using that you can get the and after doing some simplification, you will get mean and variance of the geometric Brownian function. That is expectation of X t is X naught times e power mu plus sigma square by 2 times t. First we can get the expectation of second order moment about the (( )) and the variance is equal to expectation of X square t minus expectation of X the whole square. Then you can get the variance of X t. So, the variance of X t is of the form X naught the whole square e power 2 mu plus sigma square t multiplied by e power sigma square t minus 1.

By substituting mu plus sigma square by 2 as a r cap or r bar, we will get expectation of X of t is equal to X naught times e power r cap t. More generally, you can go for expectation of X t divided X of s, that is same as e power r bar t minus s. So, in this way we are finding mean and variance of geometric Brownian function, using the moment generating function of normal distributed random variable as a application because the geometric Brownian motion is a non-negative, because of the form X t is equal to X naught e power W t W t has the range minus infinity to infinity the e power W t.

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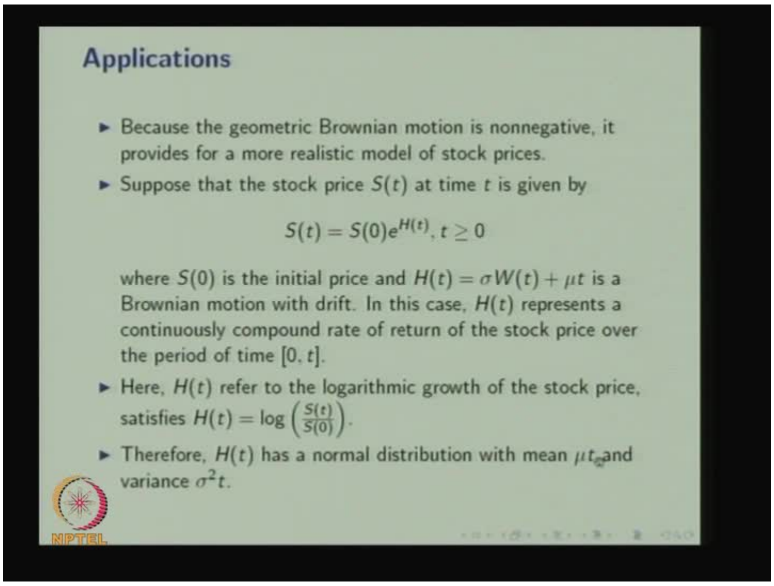

Therefore, the range of X t is 0 to infinity, so the geometric Brownian motion is non-negative random variable for fixed t. It provides more realistic models for stock prices. Whereas, one cannot use the Brownian motion to model the stock prices. Suppose, the stock price s t at time t is given by s t is equal to s naught e power h t, where t is greater than or equal to 0 where s naught is the initial price and h t is of the form mu times t plus sigma times W.

We are going to little generalized one, the geometric Brownian motion is of the form X t is equal to X naught e power W t. Whereas, here we are considering more general setup s of t is equal to s of 0 e power h t, where h t is having a mu t term along with the sigma times w to make mu is equal to 0 sigma is equal to sigma square is equal to 1. You will get the standard geometric Brownian motion.

Whereas, here this is the the h t is the Brownian motion with drift mu. In this case the h t represents the continuously compound rate of return of the stock price over the period of time 0. W t is the standard Brownian motion, the h t is equal to mu t plus sigma times W t is the Brownian motion with drift. It represents a continuous compound rate of return of the stock price over the period of time 0 to t. Here, h t represents the logarithmic growth of the stock price because it satisfies h of t is equal to log of s of t by s of 0 s of t is equal to s of 0 e power h t.

So, you can divide s of 0. So, s of t by s of 0 is equal to e power h t take logarithm on both side. Therefore, log of s t by s 0 is same as (( )). Hence, it refers with the logarithmic growth of the stock price (( )). Since, it is s of t is equal to s of 0 e power h t and h t is sigma times W t plus mu t, therefore h t you know that W t is normally distributed with the mean 0 variance t is standard Brownian motion. Therefore, h t also normally distributed random variable for fixed t.

Keep the mean we can find out the mean of h t. The mean of h t this mean is 0. Therefore, mean of h t is mu t and you have to find out the variance the variance will be 0 here. Here the variance will be sigma square and variance of W t is t, therefore the variance of h t is sigma square. So, from this equation s of t is equal to sigma times W t plus mu t by finding mean. Variance you can find out h of t is normally distributed with the mean mu t mu times t and the variance sigma square.

Already we have made this substitution mu plus of time of sigma square, that will make it as a r cap. Hence, the expectation of s of t will be s of 0 e power r bar not only that we can find out the variance. The way we have derived mean and variance of geometric Brownian motion, we can find out the mean and variance of s of t. Also observe that the expected stock price the expectation of s the expectation of s of t, we can observe that the expected stock price depends not only on the drift mu of h of t, but also the volatilities because the expectation of s of t is equal to s of 0 times e power r bar t, where r bar is mu plus half sigma square.

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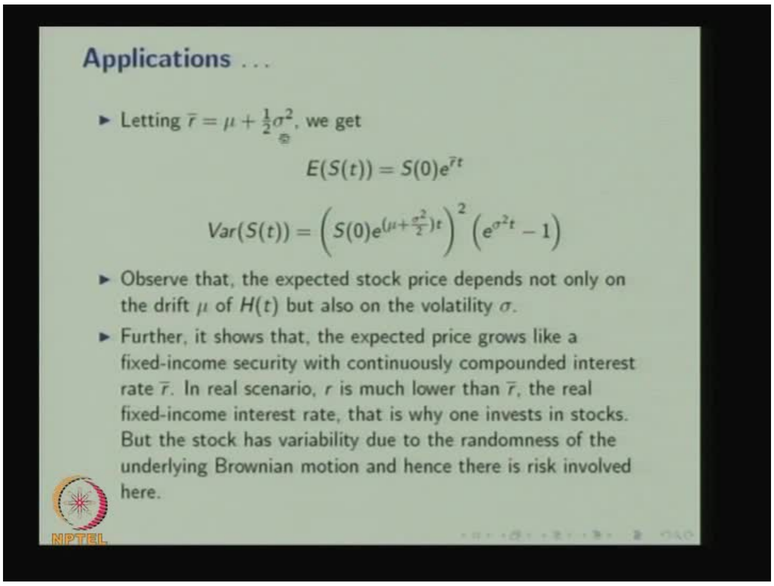

So, this r bar depends on the mu as well as sigma square. Hence, the expected stock price depends not only the drift, but also the volatility the drift of h of t, but also the volatility sigma (( )). Further it shows that the expected price grows like a fixed income security with the continuously compounded interest rate r bar s of t is the stock price at time t. The expected stock price at time t is s of 0 e power exponential of r bar t. So, it it it grows like a fixed income security with continuously compound interest rate r r bar. That is a in real scenario r is much lower than r bar, where r is the real fixed income interest rate. That is why, that is why one invests in stocks.

But, even though there is a risk attached with that in the in the fixed income scenario. The interest rate is r, there is no risk. Whereas, when you invest in stock the average or expected price growth in the form of s of t e power r bar t, where r is much lower than r bar, but there is a risk that is the difference between the fixed income scenario with the investing in stocks. But the stock has variability due to the randomness of the underlying Brownian motion and hence there is a risk involved.

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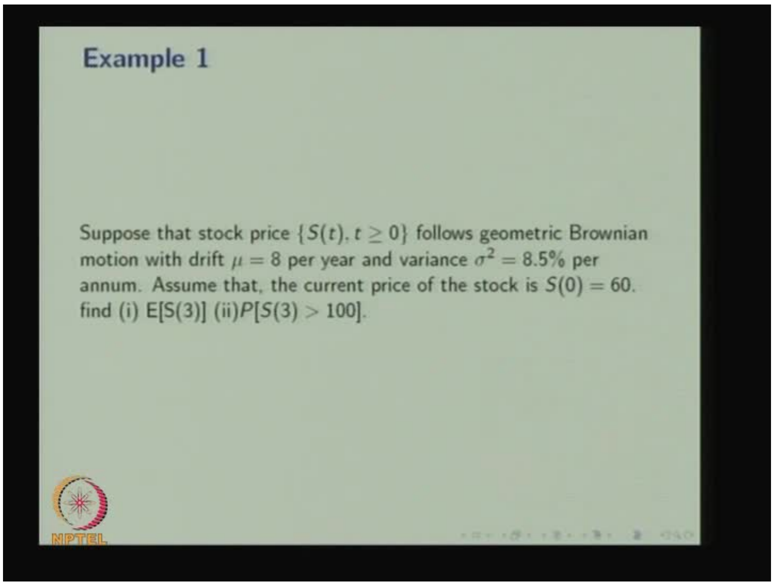

So, we have taken a one application of a geometric Brownian motion. Through that we have explained, we have derived mean and variance of the stock price. Through that we are discussing the, the risk over the investments in (( )). This is the very simple example, suppose the stock price s of t follows the geometric Brownian motion with the drift some value, here we make it 8 per year as well as variance sigma square is the 8.5 percentage per annum. Assume that the current price of the stock s naught is 60.

The questions are, what is the expected stock price at time 0.3, 3 years? Similarly, what is the probability, that the stock price at the time 0.3 is will be greater than 100 (( )). For instance you know the s of t, you can find out the probability because s of t is of the form X naught times e power h t (( )). So, we have to use lognormal concept to find out the probability and the expectation of s of t. We have already given the formula for, formula for expectation of s of t. You can use that to get the values.

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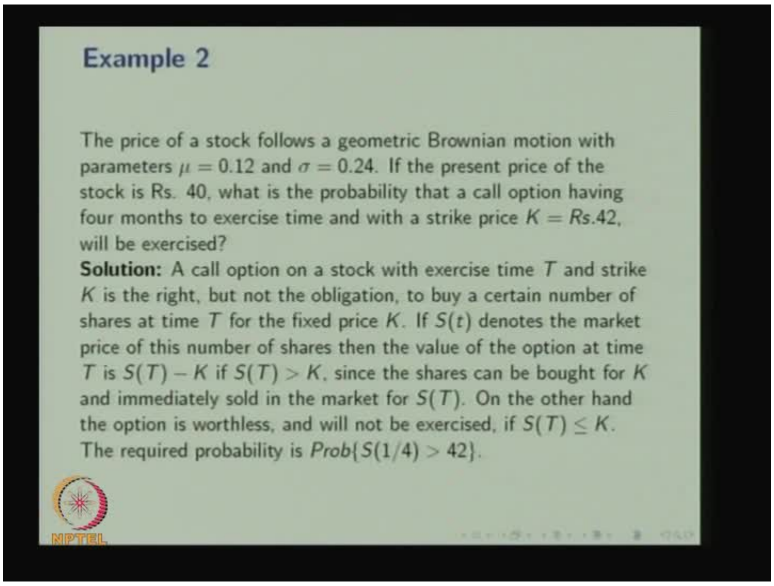

Next, the second example, the price of a stock follows a geometric Brownian motion with the parameters mu is equal to 0.12 and sigma is equal to 0.24. If the present price of the stock is Rupees 40. What is the probability, that a call option having 4 months to exercise time and with a strike price K is equal to rupees 42 will be exercised. So, a call option on a stock with exercise time t and strike price K is in the right, but not the obligation to buy a certain number of the shares at time capital T for the fixed price K, because the call option is the right, but not the obligation to buy. That is the meaning of call option.

If S of T denotes the market price of this number of shares, then the value of the option at time T is the value of the option S of T minus K if S of T is greater than K. That is if the value of the price is greater than K, then the value of the call option is S of T minus K. Since, the share can be bought for a K and immediately sold in the market for S of T. On the other hand the option is worthless and will not be exercised if S of T is less than or equal to K. The call option is the right, but not the obligation to buy. Hence, if S of T is greater than K, then the value will be S of T minus K, the value will be 0 or the option is worthless and will not be exercised if a S of T, S of T is less than or equal to capital K.

Hence, the question is what is the probability that the call option having a 4 months to exercise time. Exercise time with a strike price K will be exercise, that means the required probability is what is the probability that S of one-fourth is greater than 42 because K is 42 and the time is 4 months. Whereas, the all other values are in years, you see that the parameters are in years. Therefore, 4 months will be one – fourth, sorry 4 months will be treated as a… Here it is given for…

So, the probability of S of one - fourth is greater than 42 will be the required probability. We are not computing actual probability, thus we can use earlier concept because S of T follows a lognormal distribution. So, using a the lognormal distribution, one can find the probability of a S of T is greater than 42.

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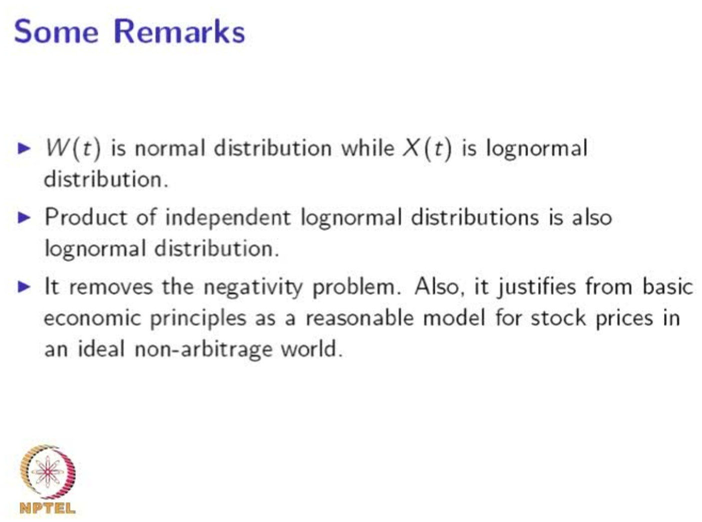

As a remark, we know that W t is the Brownian motion and the increments are independent as well as increments are stationary. W t is normally distributed, if the mean 0 and variance t, if it is a standard one. If it is not a standard one, then the W t is normally distributed with a mean mu times t and variance is sigma square. While the X of t, here it is a the X of t is the lognormal distribution. Because it is X of t is X of 0 e power W t. Hence, X of t is the lognormal distribution.

So, that is the relationship between the Brownian motion and the geometric Brownian motion. The Brownian motion each random variable is normally distributed, whereas in the geometric Brownian motion each random variable is lognormal distribution. Therefore, we can use a central limit theorem as the sum of random variable will tends to again normal distribution, if they are independent. But here since X of t is a lognormal distribution, you can use those properties with the product that is the second one.

The product of independent lognormal distribution is also lognormal distribution. The way we use the sum of independent normal distribution is a normal distribution. Normal distribution and the lognormal distributions are connected in the form of X of t is equal to X of 0 e power W t. Hence, the product of independent lognormal distributions, will also a lognormal distribution. Hence, suppose you have a independent geometric Brownian motion, if you make a product then that is also a geometric Brownian motion.

The third remarks, it removes the negativity problem. That means the Brownian motion has the range minus infinity to infinity. Since, we made X of t is equal to X of 0 e power W t, the range of X t 0 to infinity it removes the negativity problem. Also it justifies from the basic economic principles as the reasonable model for stock prices in a real non-arbitrage world. Hence, it hence it removes the non negativity problem or it takes the non-negative values. It justifies from the basic economic principles as the reasonable model for stock prices in an ideal non-arbitrage world.

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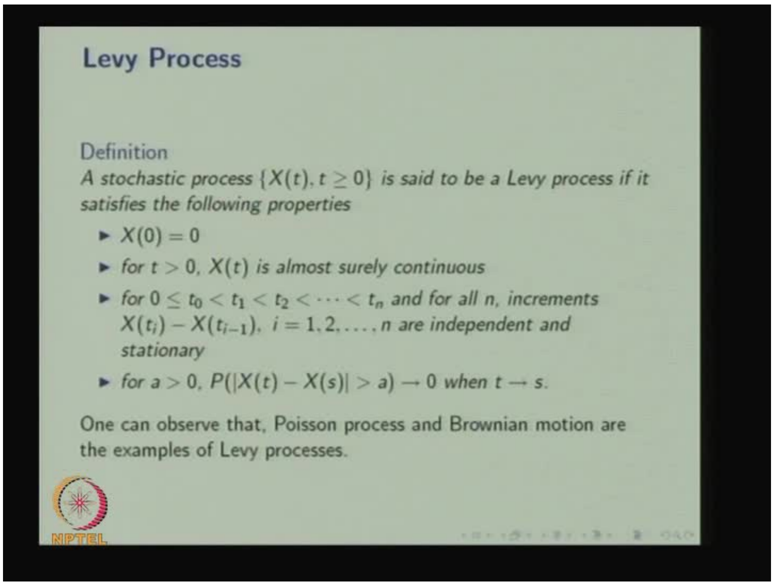

Finally, we are presenting a process derived from the Brownian motion. This is not a derived one, but this is a very generalized stochastic process comparing with Brownian motion and geometric Brownian motion. This has a lot of applications in finance. The definition a stochastic process, a stochastic process X of t is said to be a Levy process, if it satisfies the following properties. The first property X of 0 is equal to 0. The same property we have used, when we define a Poisson process, when we define Brownian motion also.

Here also the Levy process, it satisfies the property X of 0 is equal to 0. The second property for t greater than 0, X t is almost surely continuous. So, if you see the sample path of X of t it is almost surely continuous. The third property for 0 less than or equal to t naught less than t 1 less than t 2 and so on till less than t n for all n the increments. That is x of t i minus x of t i minus 1 for i varies from 1 to n, the increments are independent as well as stationary. This is true for all arbitrary n time works for all n.

The fourth condition for a for a greater than 0, the probability of the in absolute of X of t minus X of s greater than a, will tends to 0 as t tends to s. So, if this three properties are satisfied by any stochastic process, then that stochastic process is said to be a Levy process. Any stochastic process satisfying these four properties will be call it as a Levy process. One can observe the Poisson process and the Brownian motion are the examples of Levy process. The Poisson process is the continuous time, discrete state stochastic process.

Whereas, the Brownian motion is the continuous time, continuous state stochastic process. The Brownian motion, the sample path are right continuous. It was you see the sample path the inter arrival of events are independent and they are exponentially distributed with the parameter lambda. Therefore, whenever some event occurs the sample path that increment with the unit. It says the same value till the next event occurs. So, in the Levy process, the property is for t greater than 0 X of t is almost surely continuous. So, that include the sample path is a continuous as well as the sample path is the right continuous.

Hence, the property we discuss in the Poisson process for the sample path as well as the property discuss in the Brownian motion for the sample path both are satisfied by say X of t is almost surely continuous. The first property is used in both Poisson process and the Brownian motion because suppose n of t is the Poisson process, then n of 0 is equal to 0. The Brownian motion is the W t, then W 0 is equal to 0. So, the first property is same for Poisson process and the Brownian motion.

Whereas the second property, the sample path of Poisson process are right continuous functions, whereas, the sample path of Brownian motion are continuous functions. The third property is the same in both Poisson process and Brownian motion, because both the places we have the third property and increments are independent as well as stationary for all n. We would not discuss, what is the distribution of the increments in the third property? We discuss only the increments are independent and stationary.

Whereas, in the Poisson process definition, we give the fourth property as the increments are Poisson distributed random variable with the parameter lambda times t minus s, where W t minus W s is the increment for s is less than… Whereas, in the Brownian motion we say, we apply the fourth property as the W t minus W s is the increment, which is normally distributed, which is normally distributed random variable with parameters mu times t minus s.

Variance is sigma square t minus s for the standard Brownian motion the mu is equal to 0 the sigma square is equal to 1. Whereas, here for a greater than 0, the probability of in absolute X of t minus X of s greater than a, will tends to 0 when t tends to s. So, this property is valid both for Poisson distributed random variable in the Poisson process as well as the normal distributed random variable in the Brownian motion. Hence, Poisson process and Brownian motion are the examples of Levy process. So, this is the more generalized stochastic process, when we are comparing with the Brownian motion and the geometric Brownian motion.

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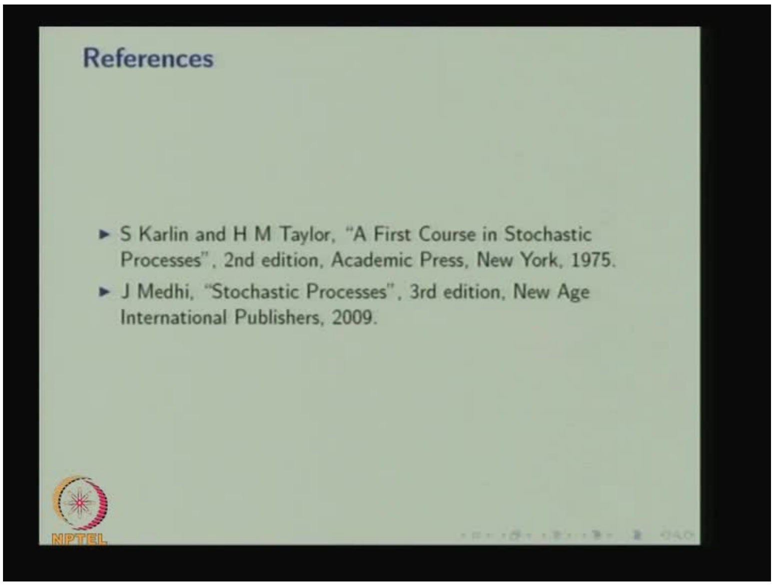

So in this lecture, this lecture, we have covered the definition of geometric Brownian motion and the important property that is the Markov property of geometric Brownian motion. Then we have discuss applications of geometric Brownian motion. How one can use the geometric Brownian motion to model the stock price and also you have given two examples. Finally, we have discussed the Levy process, the definition as well as the standard examples of Levy process. So, with this we are completing the lecture 2 processes derived from Brownian motion. Here is the, here is the list of references.