Stochastic Processes
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Module - 6
Martingales
Lecture - 2
Definition and Simple Examples

This is a stochastic processes module 6 martingales, lecture 2 definition and simple examples.

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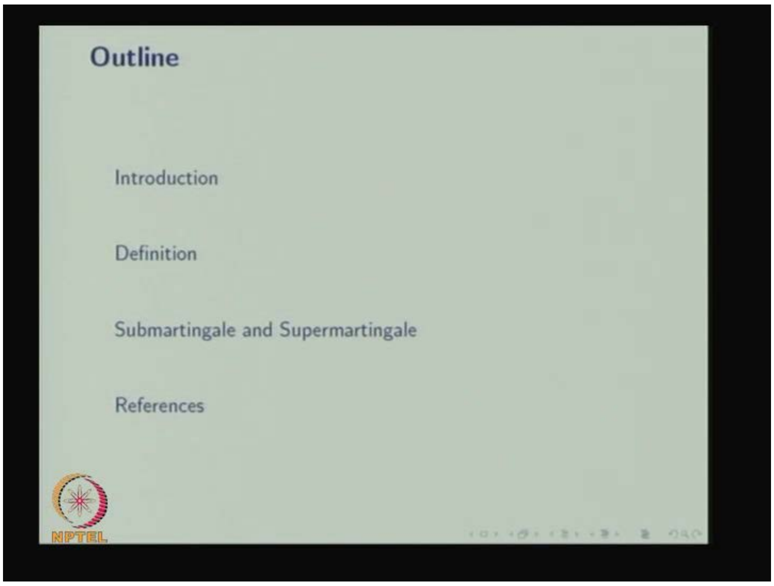

In the last lecture we have covered conditional expectation, expectation of x given y and its properties; filtration f of t t over the time, t over the 0 to infinity and its properties; then conditional expectation of a random variable x of t given that filtration f of s and its properties. In this model we will discuss an important property of stochastic process martingale.

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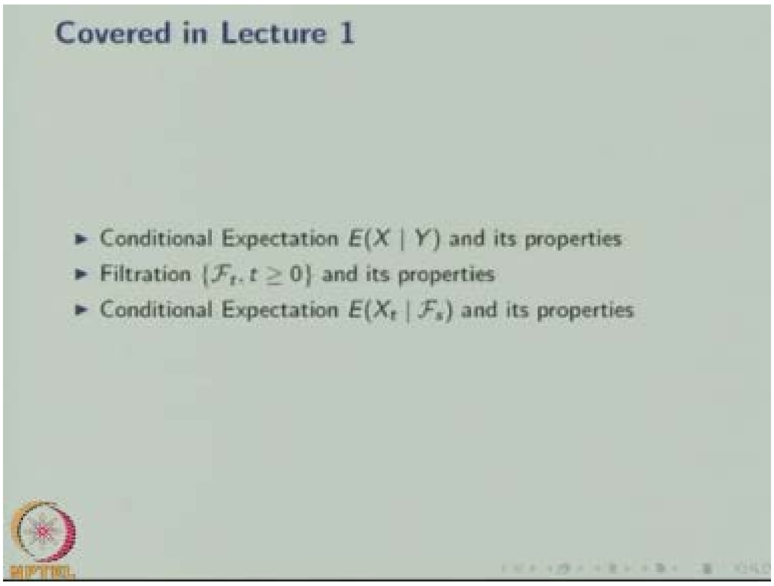

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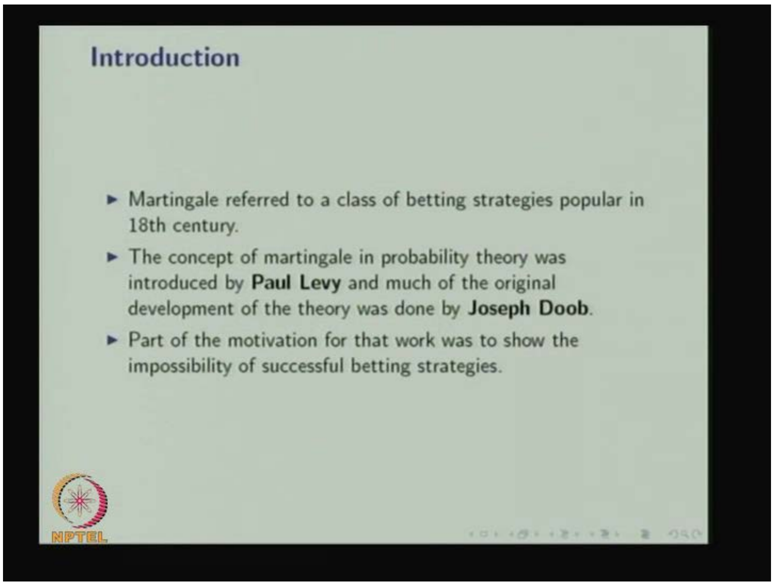

The martingale referred to a class of betting strategies popular in eighteenth century. The concept of martingale in probability theory was introduced by Paul Levy and much of the original development of the theory was done by Joseph Doob. Part of the motivation for that work was to show the impossibility of successful betting strategies.

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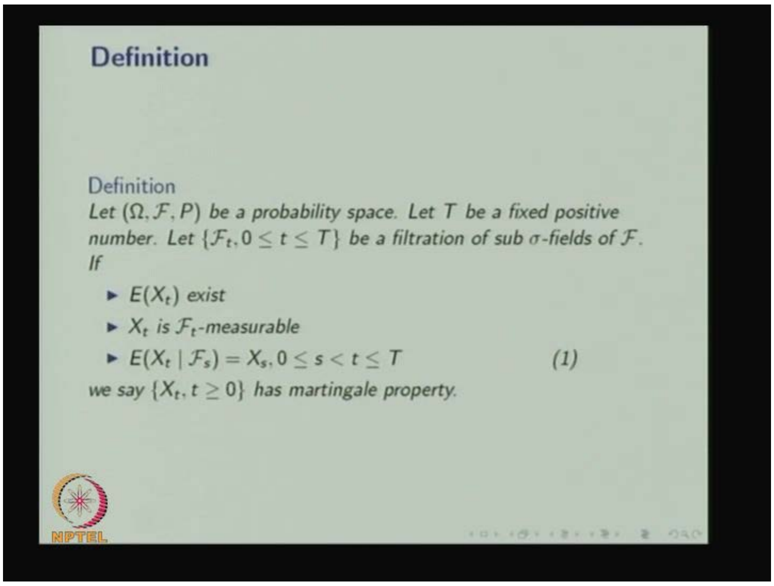

The definition of martingale is as follows: Let omega, F, P be a probability space. Let T be a fixed positive number. Let the collection of a filtration F of t where t over from 0 to capital T be a sub-sigma fields of F. So, the f is the sigma field and the filtrations are the sub-sigma fields of the F and the probability space is defined in omega F and t is the fixed positive number; using that we got a filtration of sub-sigma fields in the range 0 to capital T.

If the expectation of x of t exist for fixed t x of t is a random variable. So, if the expectation of x of t exist or in other words in the random variable is integrable; also if x of t is F t measurable. This also we discussed in the last lecture. Whenever we say the random variable is a F t measureable, that means in the sigma field generated by the random variable x of t that should be contained in F of t. If this property satisfied by the random variable for a given filtration F of t, then we say for a given sigma field, we say x of t is F t measurable. So, the second condition is x of t is F t measurable.

The third condition not only for a fixed t the expectation exists and the random variable is F t measurable. The conditional expectation that is expectation of x of t given the sigma field F of s is same as the random variable x of s. The s can take the value from 0 to small t, where t can take the value from s to capital T. If these three properties are satisfied by a collection of random variables, that is stochastic process x t, then we say the random the stochastic process has martingale property.

So in this definition, we started with the probability space and we fix some positive integer positive number, using that we create a filtration and those filtrations are nothing but sub-sigma fields of F. F is a sigma field. If you have a collection of random variables, that is a stochastic process, for fixed t it is a random variable. So, that random variable satisfies the integrable property and F t measurable property and conditional expectation over the sub-sigma field F of s; that is nothing but the filtration.

That is same as the random variable x of s. In that property satisfied for all s and t lies between the interval 0 less than or equal to s less than t less than or equal to capital T, then we say the collection of random variable x of t or the stochastic process x of t has the martingale property. So here, this stochastic process satisfies the martingale property in the interval 0 to capital T.

Because we are checking the conditional expectation in the interval 0 to capital T; therefore, this stochastic process has a martingale property in the interval 0 to capital T not 0 to infinity. If that is satisfied for all t, then we can say that random variable or that stochastic process x of t; this stochastic process x of t has a martingale property in the range 0 to infinity.

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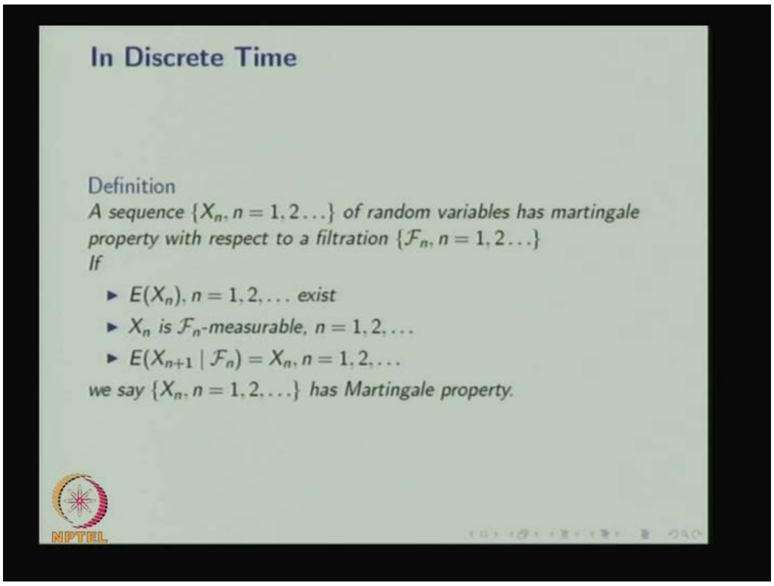

Now we present the definition of martingale in discrete time. A sequence of random variables that is X n n varies from 1, 2, and so on of random variables has a martingale property with respect to the filtration F n where n is also running from where n also takes the value from 1, 2, and so on. If for fixed n the random variable is integrable, there is expectation exist, and also each random variable X n is F n measurable.

The third condition the conditional expectation of X n plus 1 given the filtration F of n is same as X n for every n. Then we say the stochastic process has the martingale property for the collection of random variable or stochastic process has the martingale property. So, this is the definition corresponding to the discrete time; the previous definition is for continuous time.

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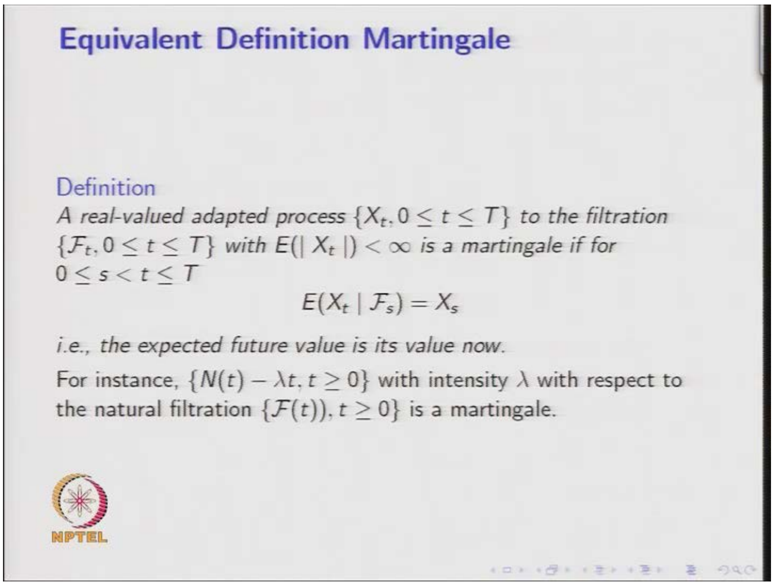

Equivalent definition of martingale is as follows: A real-valued adapted process X of t where t is lies between 0 to capital T to the filtration F of t where t is lies between 0 to capital T with expectation is finite, expectation exist is a martingale for instance N of t minus lambda t for t greater than or equal to 0 with the intensity lambda with respect to the natural filtration F of t t greater than or equal to 0 is a martingale. Here N of t is a Poisson process.

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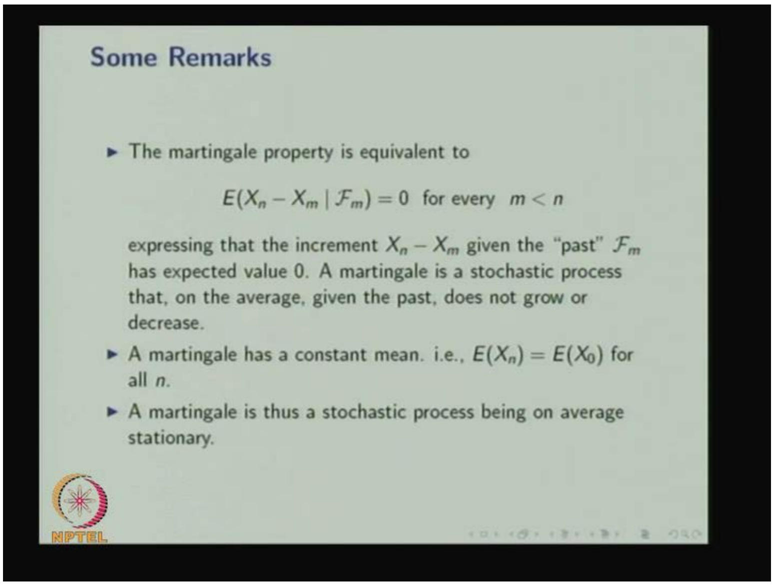

Some remarks are as follows: The martingale property is equivalent to the conditional expectation of X n minus X m given the filtration F of m is equal to 0 for every m less than n with the martingale property same as the conditional expectation with the difference of the random variable; expressing that the increments X n minus X m given the past F m has expected value 0. The martingale is a stochastic process that, on the average, given the past, does not grow or decrease, that is the meaning of the above expression.

A martingale is a stochastic process that, on the average, given the past, does not grow or decrease. If it increase or if it decrease, we have a different name for that particular property. Now we are discussing it does not grow or decrease. So that is called, that stochastic process is called martingale, A martingale has a constant mean; that means if you have a stochastic process satisfying the three conditions, hence the stochastic process has a martingale property or the stochastic process is a martingale, then the expectation is going to be constant; that is expectation of X n is same as expectation of x naught for all n.

Note that Markov property can also be given in terms of expectations. In other words, the expectation of X n is same as expectation of X m for all n. That is same as a martingale is thus a stochastic process being on average stationary; average stationary means it has the time invariant property in average or in mean usually in the stationarity property or the time invariant properties discussed in the distribution. But here in the martingale it has a expectation of X n is equal to expectation X naught for all n.

Therefore a martingale is the stochastic process being on average stationary. These above remarks are also valid for continuous time; while martingale concepts involve expectation, the Markov process concepts involve distribution. Whenever you discuss a stochastic process with a martingale property, it involves the conditional expectation; whereas whenever you discuss the stochastic process with Markov property, it involves the conditional distribution.

A Markov process need not necessarily be a martingale because the stochastic process having a Markov property; therefore, it is going to be a Markov process. A stochastic process having a martingale property; martingale concepts involve conditional expectation whereas the Markov property concept involves the distribution. Hence a Markov process need not necessarily be a martingale. The martingale has a lot of applications in branching processes and finance.

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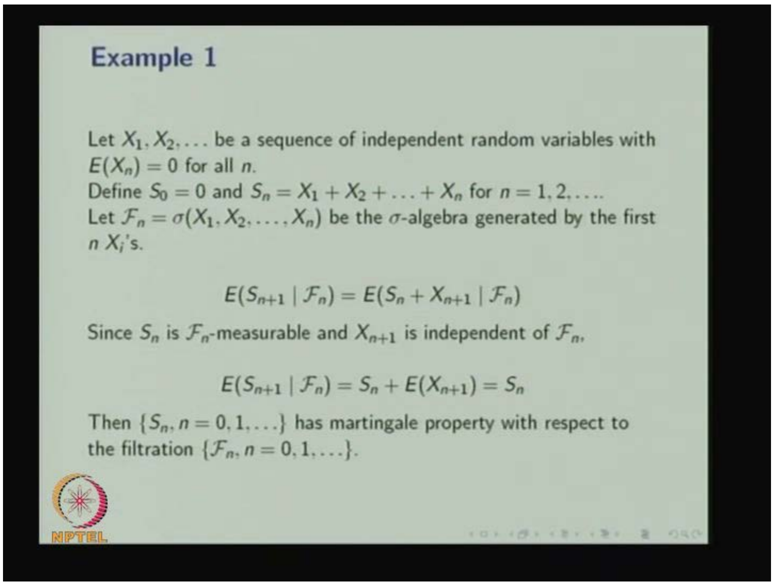

Now we are going to consider the few examples. Example 1: Let X 1, X 2 be a sequence of independent random variables with expectation of X n is equal to 0 for all n. We are not giving the distribution of the random variables; we have said it is a sequence of random variable and all the random variables are mutually independent and you know the expectation of the random variable is equal to 0 for all n. Now you are defining a new random variable S naught is equal to 0 and S n is equal to sum of first n random variables X i's for all n is equal to 1, 2, and so on.

The filtration F n is defined from the sigma field generated by the n random variables X 1 to X n with the sigma-algebra generated by the first n X i's. Now we are trying to compute what is the conditional expectation of S n plus 1 given the filtration F of n; that is same as conditional expectation of you can replace S n plus 1 by S n plus X n plus 1 because that is the way we define S n. S n plus 1 is going to be first n plus 1 random variables, the first n random variables will be S n, the last term will be X of n plus 1.

The way we created the filtration F n is nothing but the sigma field is generated by the first n random variables; therefore, the S n random variable is the F n measure. Because each X i's are F n measurable, each X n is F n measurable, the S n is nothing but the first n random variable X i's random variable summation; therefore, S n is also F n measurable, X n plus 1 is independent of F n, because F n is the information till first n X i random variables; therefore, X n plus 1 is independent of F n.

Therefore, a conditional expectation of S n plus 1 given the filtration F n the information up to n; that is nothing but since S n is a F n measurable, therefore S n is known, because you know the information till n; that means S n is also known. Therefore, S n has to be treated as a constant. So, the conditional expectation of S n given F n is going to be S n. The second term the conditional expectation of X n plus 1 given F n; since F n is independent of X n plus 1, it is not the information up to n is not going to affect the value of X n plus 1 for the random variable X n plus 1.

Therefore, it is a just instead of conditional expectation it is a expectation of X of n plus 1. But already we have made expectation of n is equal to 0; therefore, the expectation of n plus 1 it is for all n expectation of n is equal to 0. Therefore, expectation of X of n plus 1; this is also 0. Therefore, the conditional expectation is going to be S n. Since you know the information up to n, the S n is a value. So, the conditional expectation of a S n plus 1 given F n is equal to S n; this is nothing but the martingale property; the last condition of martingale property in a discrete time.

Whenever you want to conclude the given stochastic process is a martingale has a martingale property, three conditions has to be checked. The first one is expectation exist, you can find out the expectation of S n. Since expectation of X n is 0, expectation of S n is also 0. In the second condition, the S n has to be a F n measurable; that is also verified. In the third condition, the conditional expectation has to be S n; this is also verified.

So, since three conditions of the definition which we discussed for the discrete time satisfied, we conclude the given stochastic process S n, the collection of or the sequence of random variables S n has a martingale property. This martingale property is with respect to the filtration F n. Because this martingale property is with respect to this filtration, there may be a possibility this stochastic process may not have the martingale property with respect to some other filtration. So, the given stochastic process S n has a martingale property with respect to this filtration F n.

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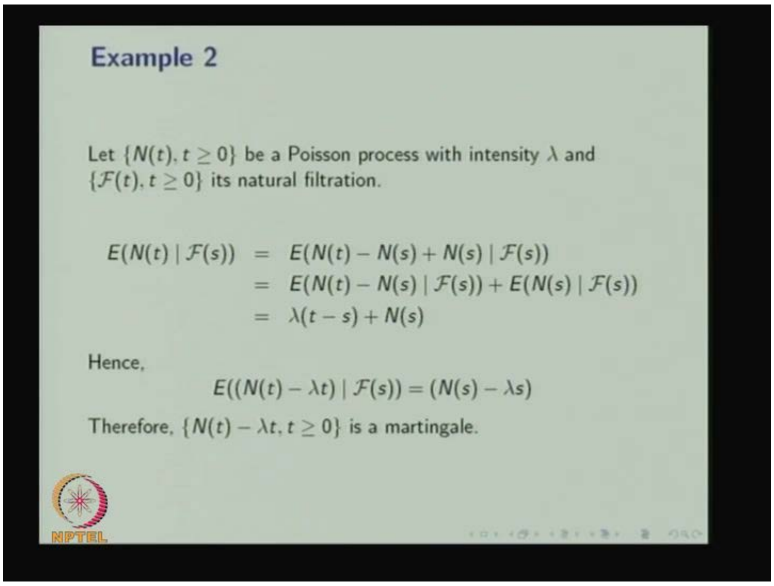

In the second example, this stochastic process is a continuous time discrete state stochastic process. So, here let N of t where t varies from 0 to infinity be a Poisson process with intensity lambda or parameter lambda and the F of t is its natural filtration. The natural filtration means it has the information till time t and each random variable N of t for fixed t is F t measurable; that is the meaning of a natural filtration.

We know that for fixed t N of t is a Poisson distributed random variable with the parameter lambda. Therefore, the mean of N of t that is same as the parameter lambda. So, the first condition is satisfied. The second condition for fixed t, N of t has to be F t measured. Since it is a natural filtration, the N of t is a F t measurable. The third condition we are going to verify. The conditional expectation of N of t given F of s where F of s is the filtration at time s or the information till s; obviously, here s is less than t. So, to find out this conditional expectation, what we do? We add and subtract N of s with the N of t; instead of conditional expectation of N of t given F of s, we will subtract N of s and add N of s.

Expectation is a linear operator; therefore, you can split these three terms into two different conditional expectations. Therefore, the first one you can keep N of t minus N of s; in the second one you can keep it separately N of s. Hence you have conditional expectation of N of t minus N of s given the filtration F of s plus conditional expectation of N of s given F of s. In the first term, this conditional expectation is nothing but you know the information till time s and we are asking the conditional expectation of N of t minus N of s given F of s; that means this t minus s and s, this is a non-overlapping intervals; s is a point, t minus s is the non-overlapping intervals.

So, this is the random variable corresponding to the non-overlapping interval with respect to s. Therefore, you know the property of Poisson process for s less than t N of t minus N of s is nothing but the increments. Thus the increments are stationary and independent. Therefore, the N of t minus N of s is independent of F of s. If it is independent, the conditional expectation is nothing but expectation of N of t minus N of s. You know the Poisson process properties. Since N of t is a Poisson process, N of t minus N of s for fixed t and s this is a Poisson distributed random variable with the mean lambda times t minus s.

Therefore this conditional expectation will be lambda times t minus s based on the N of t minus N of s is independent of F of s and for fixed s and t, N of t minus N of s is a Poisson distributed random variable with the mean lambda times t minus s. Whereas the second term, conditional expectation of N of s given F of s; that means, for information till time s, what is the expectation of N of s at the same time s. So, since you know the information till or up to the time, s N of s is constant. N of s is a constant; therefore, expectation of constant is a constant; therefore, it is a N of s. It is not a lambda time s because you know the information till time s.

Once you know the information till time s; that means you know the value of N of s also. Once you know the value of N of s; therefore, N of s is no more the random variable. So, it is a constant. So, expectation of a constant is a constant. So, it is N of s. Hence its conditional expectation of N of t given F of s is same as you can take lambda t in this side. So, conditional expectation of N of t minus lambda t given F of s is same as N of s minus lambda times s.

If you see this is an expectation of N of t minus lambda s given F of s, that is same expression. Therefore, as such expectation of N of t given F of s is not N of s. It has the some positive value t minus s is always greater than 0. Therefore, lambda times t minus s will be greater than 0. Therefore, this conditional expectation is always greater than or equal to n of s. Hence N of t is a not a martingale whereas if you treat N of t minus lambda t as a stochastic process over the t ranges from 0 to infinity, then this stochastic process is a martingale.

The N of t is not satisfying is same as N of s; the condition expectation is not equal to N of s, but it is greater than or equal to N of s. Therefore, N of t is not a martingale; whereas if you make a another stochastic process that is nothing but N of t minus lambda times t or t greater than or equal to 0, then this stochastic process satisfies the third condition and also satisfies the other two conditions. Therefore, the N of t minus lambda t is a martingale.

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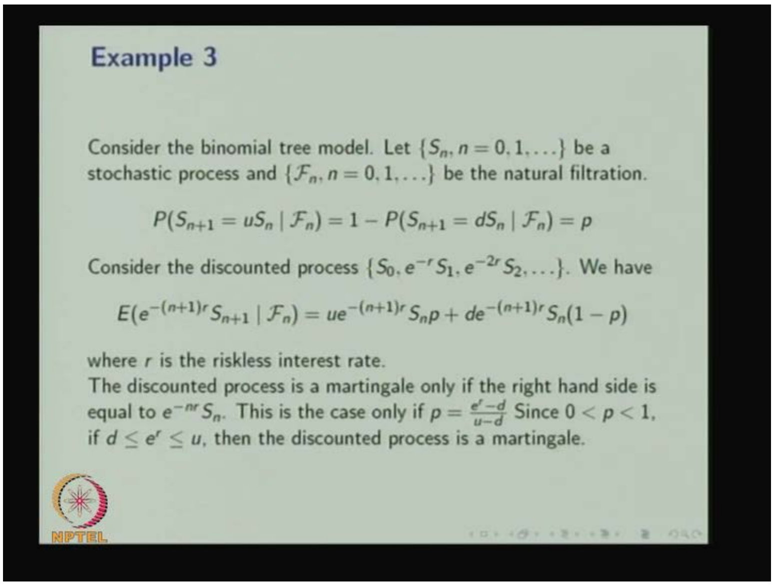

Third example: This is related to the application of finance. Consider the binomial tree model. Let S n be a stochastic process and F n be the natural filtration. Define the probability of S n plus 1 given u times S n given F n; that is same as that is P and the probability of S n plus 1 is equal to d times S n given F n is equal to 1 minus P, where u and d are the next value of S n with the probability p and q respectively. Therefore, suppose the previous the n th value was S n, then the next value will be it is decremented with d; therefore, d times S n or it would have been incremented with u. Therefore, u times S n will be the S n plus 1 th value.

Therefore this stochastic process is called the binomial tree model. Now I am considering the discounted stochastic process; that is nothing but e power minus r times basically e power r t S of t. Since it is a discrete time stochastic process, the first S 1 is multiplied by e power minus r, whereas the second random variable S 2 is multiplied by e power minus 2 times r and so on. Therefore the nth random variable S n will be e power minus n times r, where r is the riskless interest rate.

So whenever you multiplied the e power minus r times t, the corresponding stochastic process is called the discounted stochastic process. The discounted stochastic process is a martingale if only if the right hand side is equal to e power minus n times r times S n, because here it is a conditional expectation of e power minus n plus 1 times r S n plus 1 given F n. If this quantity is same as e power minus n times r multiplied by S n, then this discounted stochastic process will be a martingale or it has the martingale property.

So, this is the case only if the P value takes e power r times minus d and divided by u minus d, if the p is the probability of incremented by u. If the P is equal to e power r minus d divided by u minus d, then the discounted stochastic process is a martingale. So, that is possible because since P lies between 0 to 1, since whenever the r is the riskless interest rate e power r is also lies between d to u. Therefore the u minus d and e power r minus d; this value is going to be lies between 0 to 1. So therefore, with the proper value of r, d, and u, if the P is of this form, then the discounted stochastic process is a martingale.

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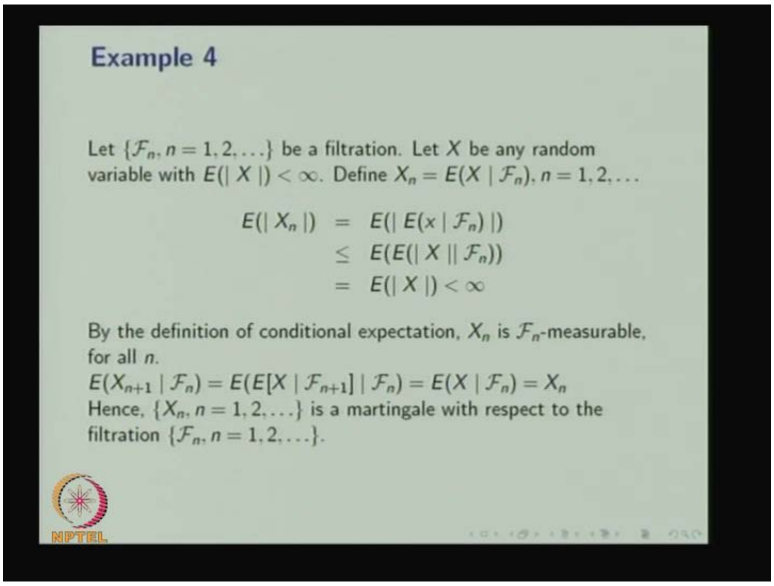

Now we are moving into the fourth example. In the fourth example, we start with the filtration F n is a sequence of sigma means and this is the filtration. Let X be any random variable with the random variable is a integrable. Define X n is a conditional expectation of X given F. So, you are defining a sequence of random variables with the help of the conditional expectation of the random variable with given the information up to n or the filtration n.

So, suppose you want to prove the sequence of random variable or the stochastic process is a martingale, then it has to satisfy the three conditions which we will discuss in the discrete time of a martingale property. So, first we are checking whether the random variable X n is integrable. So, if you want to prove the X n is integrable, then you have to prove the expectation in absolute random variable has to be a finite; that is finite, then the random variable is integrable. So, this is same as expectation of inabsolute, you can replace X n by expectation of X given F n.

You can take the absolute inside the expectation and this is nothing but it is absolute of X because that is the way we define; since X n be any random variable with the expectation is a finite and you know the definition of a expectation of expectation X given y is going to be a expectation of X. So, we are using that property. Hence expectation of expectation of X given the filtration F n is same as expectation of that random variable. So, here the random variable is absolute X and this is already proved that it is a finite;

therefore, this is also going to be finite value. Hence the expectation of absolute X n is finite, the random variable X n is integrable.

So, the first condition is verified. By the definition of conditional expectation, X n is a F n measurable. The way we have written X n is a expectation of X given F n. So, this the definition of conditional expectation; whenever you write conditional expectation of X given F n and that exist with S n; that means the random variable X n's are F n measurable for all n. So, by the definition of conditional expectation X n is F n measurable for all n. So, the second condition also satisfied.

Now, we are going to verify the third condition. The expectation of X n plus 1 given F n is same as, you can replace X n plus 1 by the definition; that is expectation of X n given F n plus 1. You are replacing X n plus 1 with the conditional expectation, given F n. You know the property of the filtration. The filtration property is a F 1 contained in F 2, F 2 is contained in F 3 and so on.

Therefore F n is contained in F n plus 1 if you have two sigma fields and one is a sub-sigma field of other one; F n is a sub-sigma field, F n is a sub-sigma fields of F n plus 1, then the conditional expectation of conditional expectation X given F n plus 1 given F n is same as conditional expectation of X given F n. We are using the conditional expectation given sigma fields with two sigma fields F n contained in F n plus 1; we are using the property.

Hence expectation of X given F n by the definition, expectation of X given F n is nothing but X; therefore, this is equal to X n. So, left hand side we started with the expectation of X n plus 1 given F n; the right hand side we land up X n. This is the third property; this is the third condition, we have defined it in the martingale property in discrete time. So, hence all the three conditions are satisfied by the sequence of random variable X n's.

Therefore the stochastic process X n is a martingale with respect to the filtration F n. Because we have used this filtration to conclude X n is a F n measurable and find out the conditional expectation is same as the X n and the random variable X n is a integrable. Hence the stochastic process is a martingale with respect to the filtration F n.

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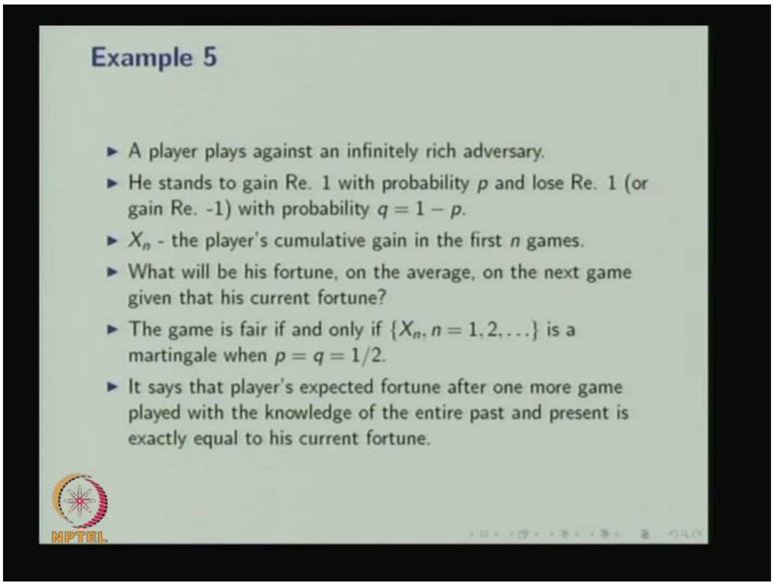

Now we present the example which was introduced in the beginning of lecture in this module. So, this is the example we have given as a motivation for the model martingale. A player plays against an infinitely rich adversary. He stands to gain rupees 1 with the probability p and lose rupees 1 with the probability q. We are defining the random variable X n; the player's cumulative gain in the first n games. What will be his fortune, on the average, on the next game given that his current fortune? We are asking the measure in a conditional expectation.

The game is fair if and only if this sequence of random variable X n is a martingale; that is for this sequence of random variables or this stochastic process will be a martingale when p and q is equal to half; that means when the game is fair; that means, whether he will gain 1 rupee or he lose 1 rupee with the equal probability 1 by 2 and 1 by 2, then the game is fair. Whenever the game is fair, the given stochastic process is a martingale.

And also whenever the given stochastic process is a martingale, in that case, the game will be a fair game; that means, the p and q will be half 1 by 2. The conclusion is, it says that the player's expected fortune after one more game played with the knowledge of entire past and present is exactly equal to his current fortune. The conditional expectation of his one more game expected fortune given the knowledge of entire past and present; that means, the filtration till time t or till time n in the discrete case; that is same as exactly equal to his current fortune.

That means it is same as expected; that is same as x suffix n for a discrete case or it is x of t for a continuous case. So, whenever the game is fair that is p and q is equal to half, then the given stochastic process X n is a martingale. And the conclusion of this problem is the players expected fortune after one more game played with the knowledge of entire past and present is exactly; exactly is important because later we are going to say more than or less than, for that we are going to name them different. So, here it is exactly equal to his current fortune.

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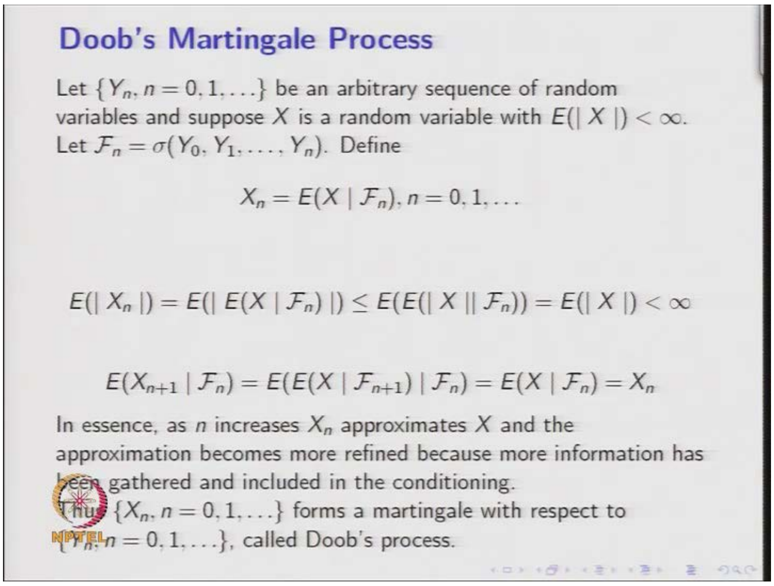

Now, we are moving into the concept called Doob's martingale process. When we say the given stochastic process is going to be a Doob's martingale process or Doob's process, see the definition: Let Y n be an arbitrary sequence of random variables and suppose X is a random variable with expectation in absolute finite. Define X n is the conditional expectation of X n given Y naught, Y 1, and so on till Y n for every X n where n is running from 0, 1, 2, and so on.

Expectation of expectation of X given F n plus 1 given F n, we know that F n is contained in F n plus 1 and using the property, this becomes expectation of X given F n; that is same as X suffix n. In essence, as n increase X n approximates X and the approximation becomes more refined because more information has been gathered and included in the conditioning. Thus X n n is equal to 0, 1, and so on forms a martingale with respect to Y suffix n where Y n for n is equal to 0, 1, 2 called the Doob's process.

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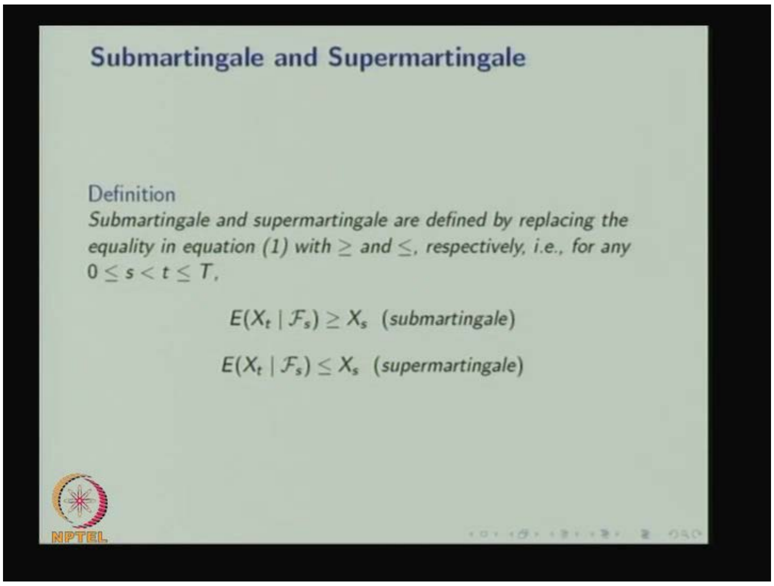

Now we define submartingale and supermartingale. Submartingale and supermartingale by replacing the equality in the equation one; the equation one in the definition of martingale whether it is a discrete time or the continuous time, if you replace equality sign with greater than or equal to or less than or equal to respectively for any 0 less than or equal to s less than t less than or equal to capital T, the third condition. The first two conditional expectations exist and then X t is F t measurable or X n is F n measurable; that is the two conditions.

So, there is no change in those two conditions; only the change in the third condition. That is by replacing the inequality in equation one of with a greater than or equal to less than or equal to, then the corresponding stochastic process will be called it as submartingale whenever it is greater than or equal to sign or it is supermartingale if conditional expectation is less than or equal to X of s. If it is equal to for all this interval, in that case it is a martingale. If this property is satisfied greater than or equal to s where s is less than t and t is less than equal to capital T, then the stochastic process is called a submartingale and the less than or equal to condition is satisfied, then the stochastic process is called a submartingale.

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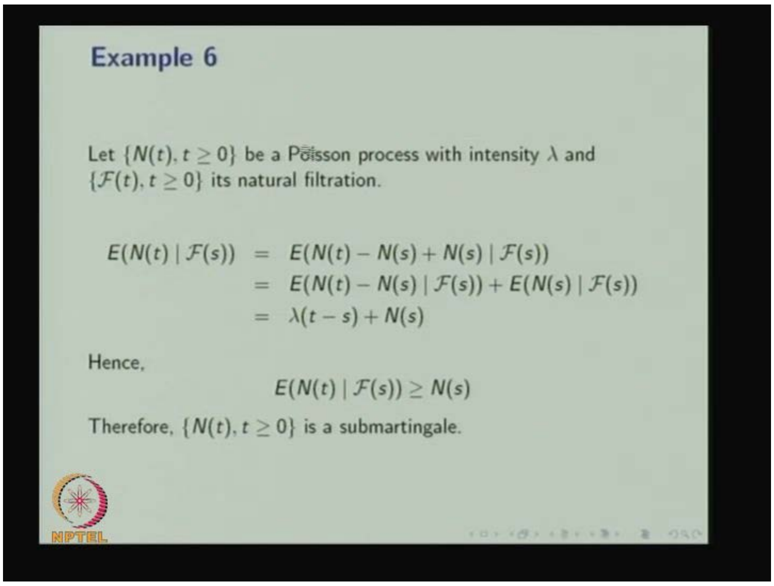

Now we present the Poisson process is a sub martingale. We take the same example N of t is a Poisson process with the intensity lambda and earlier we have proved N of t is not a martingale because conditional expectation is greater than or equal to N of s, because t minus s is greater than 0, lambda is strictly greater than 0. Therefore, this quantity is greater than or equal to N of s.

Hence conditional expectation of N of t given F of s is greater than or equal to N of s is always greater than or equal to for all s less than t, where t is less than or equal to infinity. So, this condition is satisfied for all s and t. Hence the given stochastic process; that is the Poisson process is a submartingale because of this greater than or equal to submartingale.

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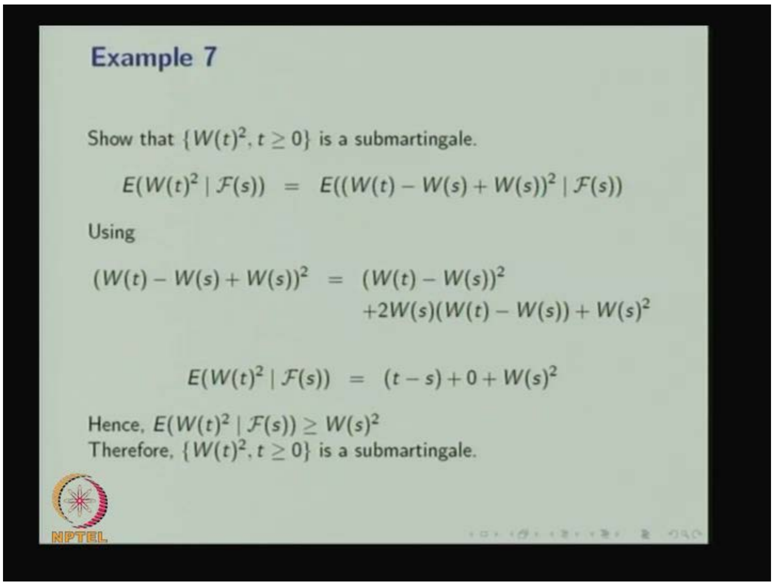

Now we present the W t square is a submartingale where W t is a Brownian motion W t is a Brownian motion, so we are going to use the property of Brownian motion; that is a one Particular important stochastic process which has a lot of applications in financial mathematics. So, here we are going to prove W t whole square is a submartingale. How? The conditional expectation of W t whole square given F of s is you add one term and subtract one term; that is plus W s minus W s inside the W t whole square.

Using the identity W t minus W s plus W s whole square will be same as W t minus W s whole square plus 2 times W s W t minus W s plus W s whole square. So, why I am doing this way of adjustment? Because I am going to use the property of non-overlapping intervals are independent. We are directly checking the conditional expectation. The first two conditions are obviously satisfied. The first one is a W t is integrable. The Brownian motion for fixed t, W t is a normally distributed with the mean zero and the variance t for a standard Brownian motion. Therefore, the mean exist. The second one W t is a F t measurable.

So, whenever we did not discuss the filtration; that means, we have a natural filtration F of t. Whenever we have a natural filtration, that means this random variable is a F t measurable, F of W of t is a F t measurable. Therefore, second condition is also satisfied and we are checking the third condition. The third condition, the right hand side W t minus W s whole square plus 2 times W s times W s minus W s plus s and so on. So, now we are applying the conditional expectation here given F of s. Therefore, the conditional expectation is a linear operator; therefore, it is a conditional expectation of this term given F of s, conditional expectation of this term given F of s plus conditional expectation of this term given F of s.

You know that F of s is nothing but the information up to time s and W t minus W s; that is also normally distributed with the mean zero and variance t minus s. I am discussing about the standard normal distribution. Later we are going to discuss the Brownian motion in detail. So, here I am going to use only the distribution and the mean of a Brownian motion as well as an independent property. So, here the W t minus W s is independent of F of s; therefore, the conditional expectation is nothing but expectation of the W t minus W s whole square. Since W t minus W s is normally distributed with the mean 0 and variance t minus s, this expectation is nothing but the t minus s.

The second term again two times conditional expectation of W s multiplied by W t minus W s given F of s and F of s is independent of W t minus W s. And you have the information till s; therefore, W s has to be treated as a constant. So, the two times W s will be treated as a constant. So, hence expectation of W t minus W s given F of s is nothing but expectation of W t minus W s because of f of s is independent of W t minus W s. And you know that expectation of W t minus W s is zero because this is a normally distributed with the mean zero variance t minus s.

The third term, the conditional expectation of W s whole square divided by F of s is nothing but W s whole square, because you know the information till time s. Therefore, W s whole square has to be treated as a constant; expectation of constant is constant. Hence expectation of W t whole square given F of s is nothing but t minus s plus 0, hence t minus s plus W s whole square. So, this is obviously greater than or equal to W s whole square. Therefore, the W t whole square is a submartingale because it satisfies the third condition with the greater than or equal to sign in the conditional expectation. Therefore, the stochastic process W t whole square is a submartingale.

Some remarks: A stochastic process necessary of a random variable X n is a submartingale with respect to the filtration F n n varies from 1, 2, and so on, if and only if minus X n is a supermartingale with respect to the same filtration F if and only f is very important.

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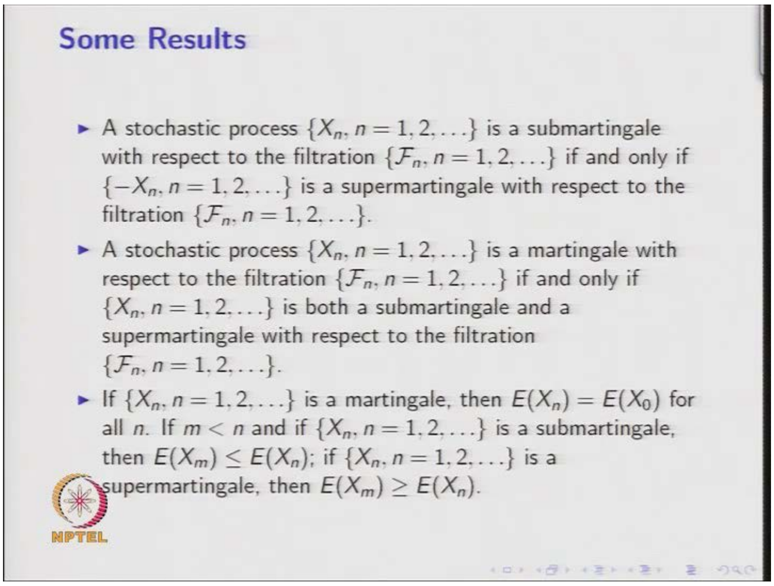

Whenever you have a submartingale you can always convert it into the supermartingale with the minus sign or if you have a supermartingale; you have a stochastic process with the supermartingale property satisfied, then you can convert this stochastic process into the submartingale stochastic process by changing the sign. Note that both the stochastic process we are discussing the martingale property submartingale or supermartingale property, with respect to the same filtration that is important. You cannot convert submartingale into supermartingale with the different filtration, not in general. If you have a same filtration, then you can transform the submartingale into supermartingale by changing the sign in both ways.

The second remark: A stochastic process X n n varies from 1, 2, and so on is a martingale with respect to the filtration F n if and only if X n is both submartingale and super martingale with respect to the same filtration. That means if you have a given stochastic process martingale, if and only if the same random variable will be treated as a submartingale as well as supermartingale. Because we are changing, we are replacing the equality sign by greater than or equal to, similarly less than or equal to; we are not replacing by strictly greater than or strictly less than. Since we have replacing the equality sign in the conditional expectation of the third condition, the definition of martingale with less than or equal to greater than or equal to. Hence if you have a martingale then the same thing will be supermartingale as well as submartingale.

Similarly if you have a stochastic process is both submartingale and supermartingale without changing the sign or without doing any change, the same stochastic process is a submartingale as well as the supermartingale, then definitely that will be a martingale. Because if it is both submartingale and supermartingale that means there is no conditional expectation with the greater than or equal to or less than or equal to; it must be equal to the conditional expectation. Hence the given stochastic process is a martingale. These above remarks are also valid for continuous time.

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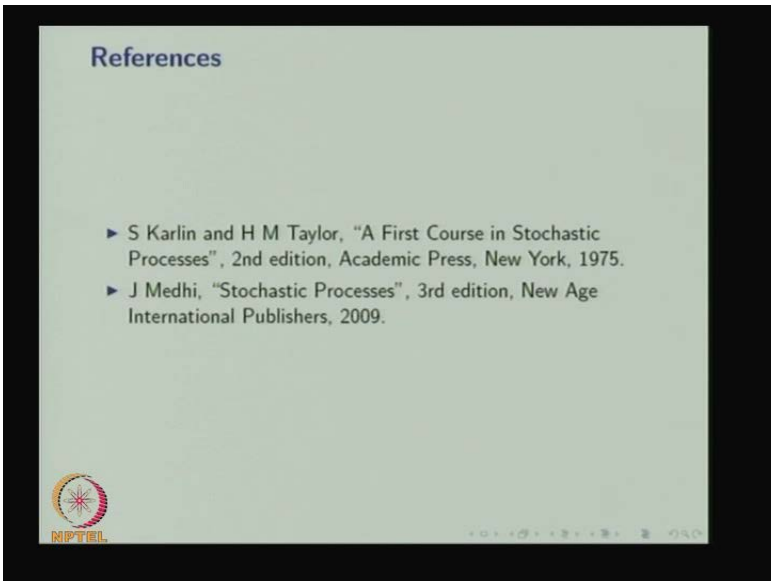

In this lecture, we have covered the definition of martingale in continuous time, the definition of martingale in the discrete time, then we have discussed few examples and followed by that we have discussed, when we can say the given stochastic process is a supermartingale or submartingale. We have given few examples for the supermartingale as well as submartingale also. And finally, we have given some remarks over martingales, submartingale and supermartingale. Here is the list of references. With this the lecture 2 of module 6 is complete.