Stochastic Processes
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Module - 5
Continuous-time Markov Chain
Lecture - 6
Queuing Networks

This is stochastic process module 5 continuous Markov Chain; lecture 6 queuing networks. In the last 5 lectures, we started with the definition of continuous time Markov Chain, then Kolmogrov differential equation, Chapmann Kolmogrov equation, infinitesimal generator matrix, in the lecture 1. Lecture 2 we discuss the birth, death process. Lecture 3, we have discuss the special case of birth, death process, Poisson process. Then the fourth lecture, we have discussed the application of continuous time. Markov Chain in queuing models - that is the first special case, special Markovian queuing model that is m m 1 queue. Then in the fifth lecture, we have discussed the simple Markovian queuing models other than m m 1 queue.

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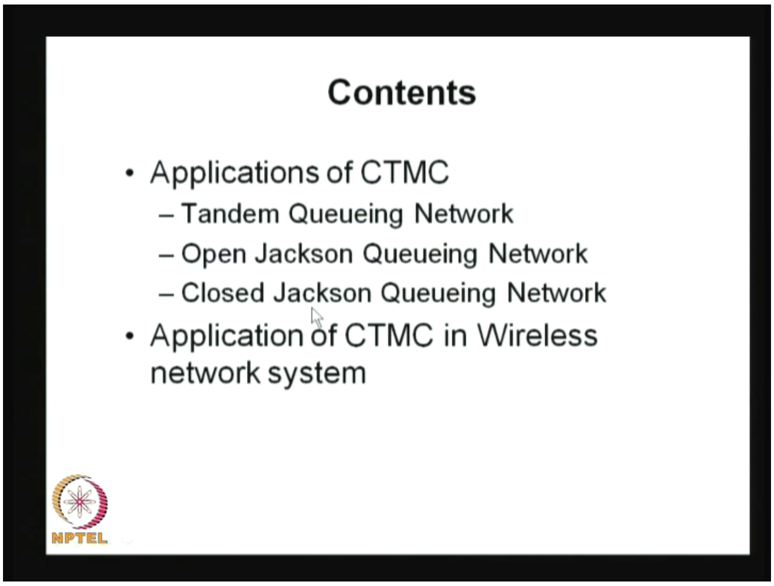

Now, we are moving into the sixth lecture, that is what the application of continuous time Markov Chain in queuing networks. Basically the queuing network is nothing but, a network consist, consisting of several interconnecting queues, either you can say network of queues or queuing network. In today’s lecture, we are going to cover the queuing network as the application of continuous time Markov Chain. In the last 2 lectures, we have discussed the application of CTMC in particular birth, death process, but now we are moving into the more general setup of CTMC, that is coming in the queuing networks. That means the underlying stochastic process of the queuing network is a more general continuous time Markov Chain whereas, all the simple Markovian queues the underlying stochastic process is the birth, death process.

The study of queuing networks is going to be a very vast area. So, we are going to cover here only very simple queuing networks. The first one is a tandem queuing network. Then we are going to discuss the open queuing network or we can say open Jackson queuing network. Also we are going to discuss the closed Jackson queuing network. Then followed by that we are going to discuss the application of CTMC in performance analysis of wireless network system. If time permits, I will do it in this lecture or I will do in the next lecture.

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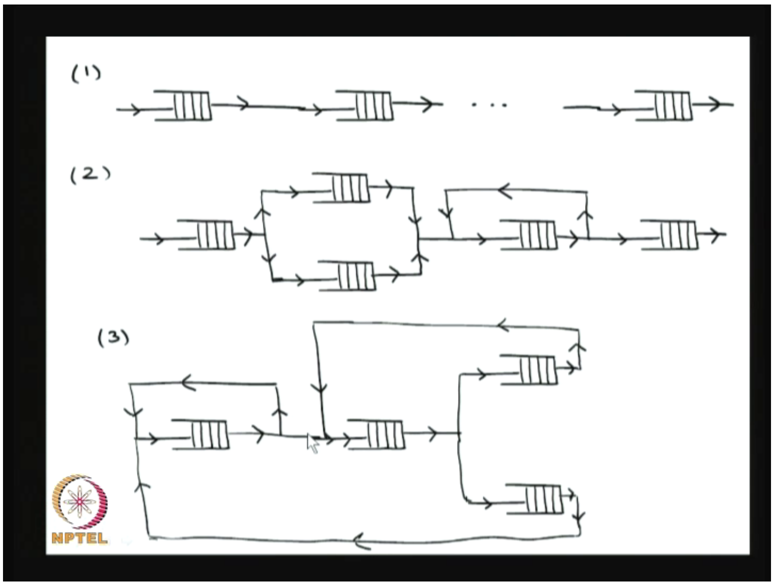

Now, we are going to discuss the queuing network. Queuing network is nothing but a network consisting of several interconnecting queues. So, we have discussed already the queues in the fourth lecture, I am giving a three different examples for the queuing network. So, you can say this as the first queue or we can say the queuing network as the node 1, which consists of separate 1 queue where the arrival comes waiting for the service.

After service is over, then it moved into the second queue and this is the separate queues in which the arrival wait for the service. After the service is over, then it move into the next queue. Like that it has many queues. So, this is the one simple example of the queuing networks. This queuing network is called tandem queuing network. Because the output of first queue is the input for the second queue and there is no feedback, the output of the second queue is the input for the third queue and so on. Like that it it is a system in which, it has some finite number of queues in series. Therefore, this is called tandem queuing network.

Second type, this also consisting of many queues interconnecting queues, whereas here the output of the first queue spltted into some probability. The summation of this probability and this probability is 1, moved into the input for the second queue as well as third queue. If you label this as the node 1, node 2, node 3, node 4, node 5, therefore this is a 5 node, 5 nodes queuing network. The output of second queue and the third queue is the input for the third queue. Whereas, after the service completion of the third queue with some probability, the jobs or customers can moving again come to the third queue. Therefore, this this types are called queues with feedback. We call it as a queues with feedback.

Then, the other proportion, it moved into the input for the fifth queue. After the service is completed, it goes away from the system. So, this type of queuing networks in which, the arrivals comes from the outside some source and it departure from the system. There is a possibility the the customer who finish the service in the fourth queue, it can depart from the system not moving into the fifth queue also. So, this is the example for the open queuing network. The third type here also we have four queues or four nodes, but there is no output. There is no output of queues from the output from the system as well as, there is no input from the outside the domain.

That means there is a constant number of customers or jobs will be moving from one node to the other nodes with this probabilities. These probabilities are called the routing probabilities, in which after the service completed in the second queue, the customers or jobs move into the third queue with some probability. It will move into the fourth queue. Therefore, this probability is called routing probability of jobs or customers moving from 1 node to the other nodes with some probability. So, this is called this type is called closed queuing network.

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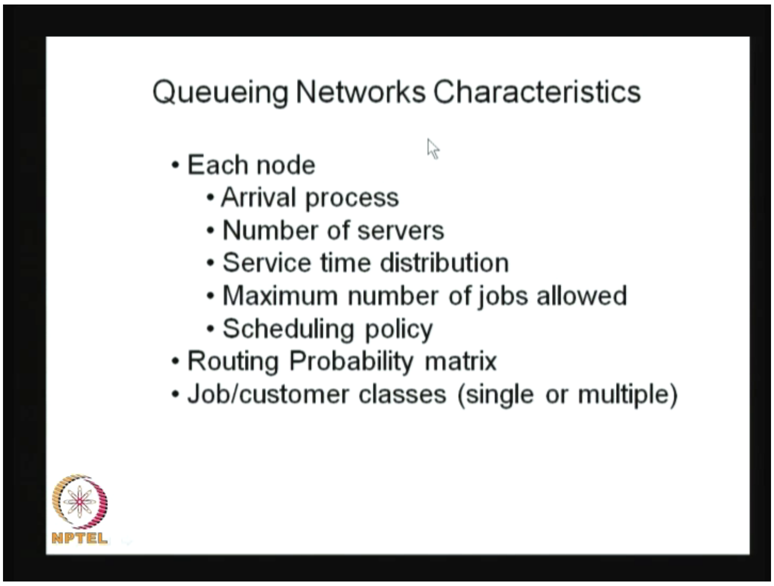

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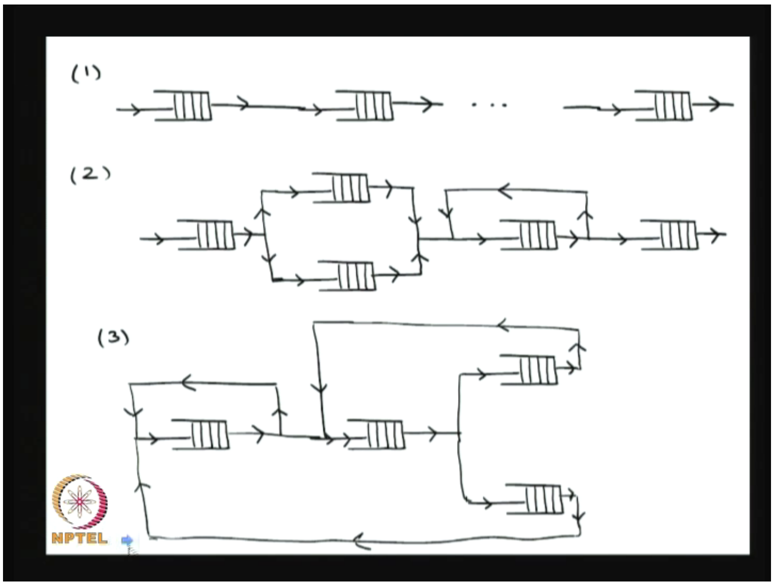

Although, I have started with some three examples and these three examples are related to the different types of queuing networks, but before I move into the types of queuing networks. Let me give the characteristics of queuing networks. So, what are all the minimum information is needed or is provided to study the queuing networks? It consists of nodes and interconnected, so the interconnected are given by the information routing probability matrix.

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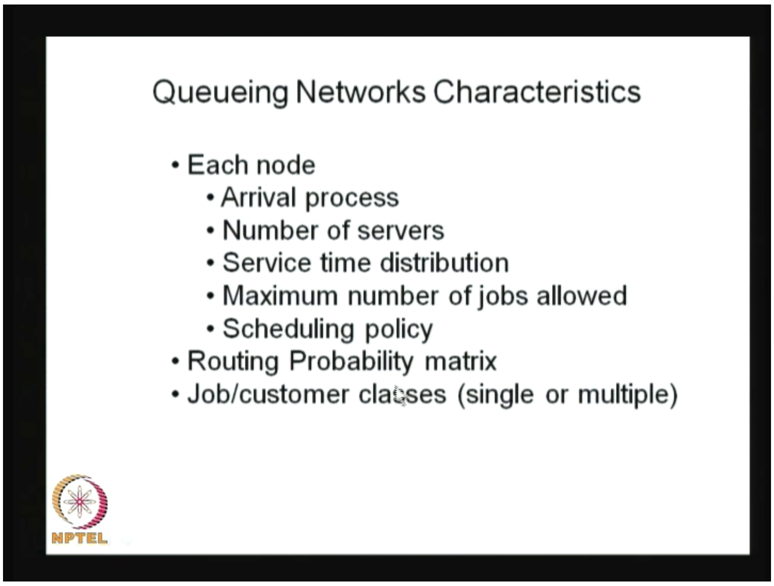

Suppose, in this example if the node 2 after the service is over, the packets or the customers or jobs are moving into the node 3. Then the routing probability is p 2 3. So, p is the matrix routing probability matrix. Second row third column element is corresponding to the routing probability of customers or jobs or packets moving from node 2 to node 3, after the service is completed. If that particular probability is 0, need routing probability matrix, that matrix order is number of nodes in the queuing networks.

The entries are the probabilities values. Therefore, it will be either it will be greater or equal to 0 and lies between 0 to 1. Obviously this routing probability matrix will be a stochastic matrix, the row sum is one all the entities are lies between 0 to 1. Now, what are all the information is needed in the each node or in the each queuing system inside the queuing networks.

You have to provide the arrival process, arrival pattern, if it is inter arrival time is a random variable, then what is the distribution of that? Then you have to provide number of servers in each node, whether it is a only one server or multi server or infinite servers. Then what is the time taken for the service, whether it is a deterministic or probabilistic? If it is probabilistic, then what is the distribution of service time for each node you have to provide? If it is one server, then what is the service time distribution? If it is more than one servers, then they are the identical servers or non-identical and what is the distribution of each servers service time.

Next, what is the capacity of the queuing system in each queues, in each the whole queuing networks? What is the maximum number of jobs or customers are allowed as a waiting space in the waiting space with their infinite capacity system or finite capacity system? If it is a finite capacity system, then if the customers are coming from the previous some other node and if the system is full, then it will be blocked.

Therefore, you should consider what is the capacity of the each queues? Also, what is a scheduling policy? In what policy the customers are getting served in each queue? So, scheduling policy whether it is a first come first served or last come first served or priority based or random order. So, you should know, what is the service what is the scheduling policy for the each node? So, if you provide these five information for each node and the routing probability matrix as well as what type of customers are coming into the whole queuing networks whether only one type of customers are moving inside, the different queues or more than one type of customers are in the system?

Our interest in this course, how the CTMC is used in the or CTMC is playing a role in a queuing network. Therefore, I am going to discuss very simple queuing network, in which you can able to get the, you can able to map the underlying stochastic process with the CTMC. Therefore, we can get some of the performance measures through the knowledge of CTMC.

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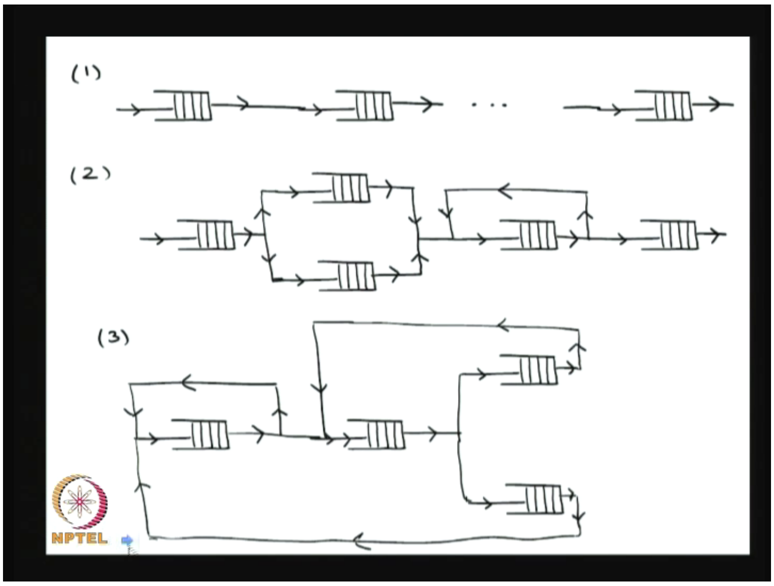

Now, I am discussing the type of queuing networks even though I have explain the type of queuing networks through these three examples, like that you can frame many more queuing networks. But in this course, I am going to cover only these three types of queuing networks. Therefore, I have given these three examples.

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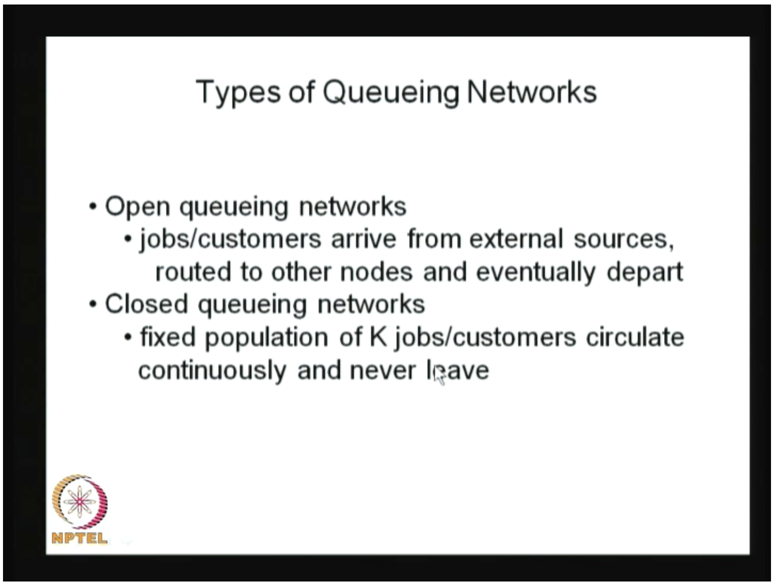

So, through these three examples, I am going to classify the queuing networks as the majorly in two parts. That is a open queuing network and a closed queuing network. What is the meaning of open queuing networks? The jobs or customers arrived from external sources routed to other nodes and eventually depart. Therefore, there is a source and there is a sink also. The external source and it has eventually depart from the queuing networks also.

Therefore, that type of queuing network is called open queuing networks. The closed queuing network has fixed number of population, some constant capital K number of jobs or customers circulate continuously and never leave. So that type of queuing network is called closed queuing network. That means always if you count how many customers in the system, system means how many customers in each node? If you sum it up, then that total number is going to be always capital K that is a population size.

Therefore, no one will leave the system and no one depart from the system also. It is a constant number of customers always there in the system at the different queues. You can always convert the closed queuing network into the open queuing network by including one source and sink you can convert the closed queuing network into open queuing network.

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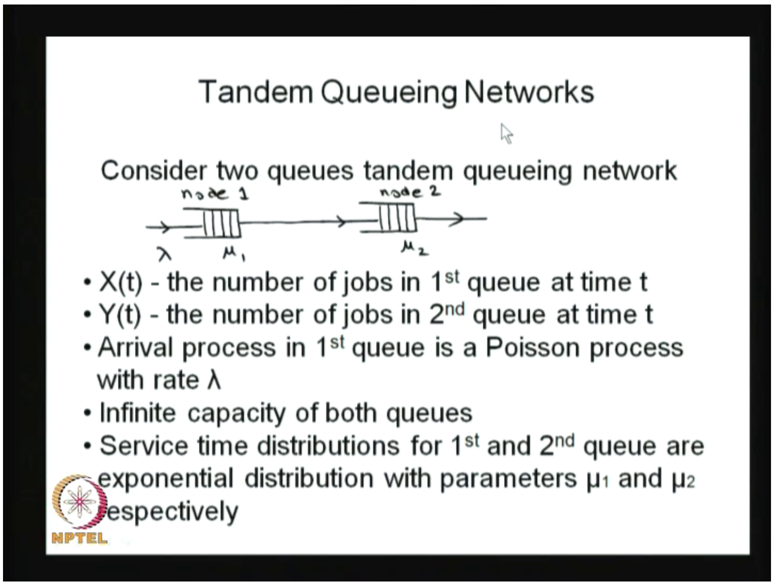

So, in general you can classify the queuing networks as the open queuing network and closed queuing network. The tandem queuing network is the special case of open queuing network, in which all the nodes or the queues are interconnected in a series. Let me start with a very simplest tandem queuing networks with 2 nodes or 2 queuing system connected in series. We are going to relate the queuing network with the stochastic process and so on. Therefore, let me start with the random variable X of t and Y of t.

X of t denotes number of jobs or customers in the node 1 at any time t node 1 or first queue both are 1 and the same Y of t the another random variable, that denotes the number of jobs in the second queue at time t. Therefore, that together X of t and Y of t, you can think of a vector z of t is equal to X of t the Y of t. That vector is a random vector at any time t and if you collect over the t, then that is going to be a stochastic process. So, this stochastic process instead of one random variable, it has two random variables X of t and Y of t. Therefore, it is a vector.

So, that is a stochastic process X of t comma Y of t for t greater than or equal to 0, that is a stochastic process. Since, X of t and Y of t are the number of jobs in the first queue and second queue. You are observing over the time, how many customers in the first node and how many customers in the second node at any time t? Therefore, this is the discrete state continuous time stochastic process. Also you have seen that infinite capacity in both the nodes, both the queues that means after the service is over in the first node.

It will be immediately arrived into the second node and if no customer in the system at the time, then he will get the service immediately. The service will be started immediately, otherwise the customer who comes after completing the service in the first node, he has to wait till his service start. So, I am making one by one assumption, so that the underlying stochastic process is going to be a continuous time Markov Chain that is my objective. So, for that now I am making a assumption.

The first assumption, the inter arrival time of a customer's entering into the first node that is exponentially distributed with the parameter lambda or the arrival process into the first queue is the Poisson process with the parameter lambda. So, the population is infinite the customers or jobs or packets, for example, packets for the telecommunication system or any communication system. So, the customers are entering into the node 1, that process is a Poisson process with a parameter lambda. Now, I am making the assumption for service, I make the assumption for the service time that is also exponentially distribution, with the parameter mu 1 for the first node and mu 2 for the second node.

Other than the inter arrivals are exponential distribution, I make the other assumptions that is service time for the first node and service time for the second node. Both are exponentially distributed with the parameters mu 1 and mu 2 respectively. So, ultimately I would not this has to behave as a m m 1 queue and this also has to behave as a m m 1 queue independently. Therefore, I make all the assumptions, the inter arrival times are independent with the service for the first node. Similarly, this one is a the arrival for the second node is independent of the service of the second node and so on.

So, in that assumption each queue in this queuing network is going to behave as a m m 1 queue and the departure of the first node, that is will be the input for the second node. Here we can use the Burke's theorem, this is m m one queue model. Therefore, using the Burke's theorem, you can conclude the departure process is also Poisson process because the arrival process is a Poisson and the service wait is mu 1 using the Burke's theorem. You can conclude the departure process is also Poisson process with the parameter of the same parameter of arrival process.

Therefore, here the arrival process for the node 2 is also Poisson process, with the same parameter. Lambda infinity is independent of the service not only that the departure process is independent of the number of customers in the system and so on. The inter arrival therefore, you will have a two independent m m 1 queue using the Burke's theorem. Now, this will separately act as a m m 1 queue because the arrival process is the Poisson with a parameter lambda. Already we made the assumptions arrivals are independent with the service. Service is exponentially distributed with the parameter mu 2 after the service is over, the customers leave the system.

Therefore, this is a separate m m 1 queue because we made a infinite capacity in both the queues. Therefore, the two queues tandem queue, the underlying stochastic process X of t comma Y of t going to be the continuous time Markov Chain continuous time Markov Chain.

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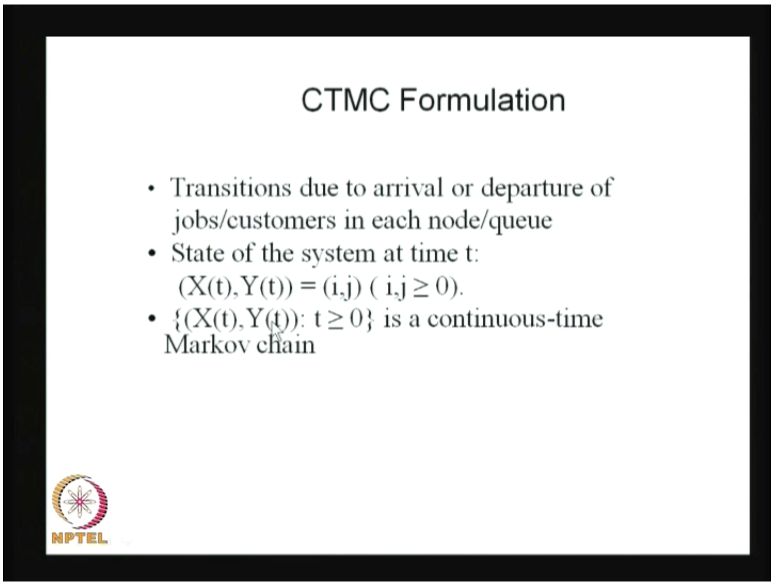

So, now I am going to formulate the CTMC from the 2 queues, tandem queuing network the transitions due to arrival or departure of jobs in each queue. Based on this, I can make a state of the system X of t comma Y of t and that is going to be a continuous time Markov Chain. This is not going to be a birth, death process because you have a two random variable X of t and Y of t. Each one independently m m 1 queue m m 1 queue is the the underlying stochastic process for the m m 1 queue, that is a birth death process whereas, here you have a together X of t comma Y of t.

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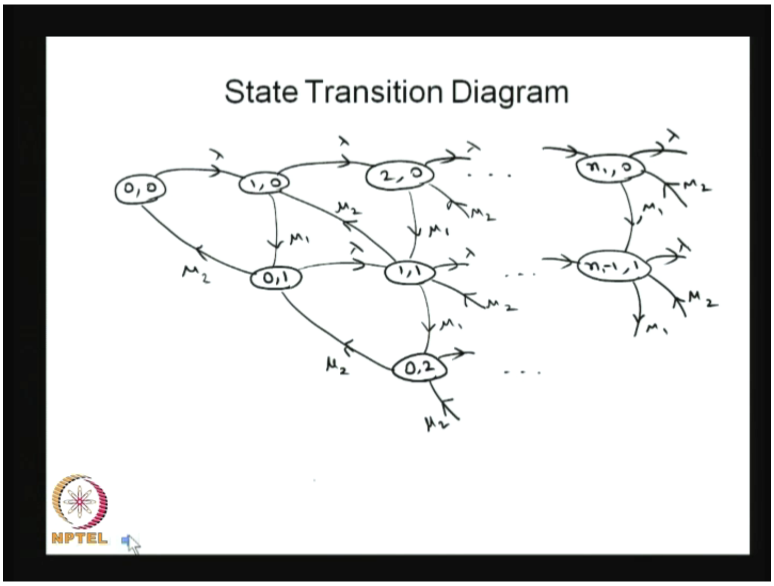

Therefore, this not going to be a birth death process, it will be a general continuous time Markov Chain. So, once you identify this as a continuous time Markov Chain because the Markov property is satisfied by the stochastic process X of t comma Y of t. We can go for drawing the state transition diagram. It is Markov in nature because the two queues act independently and are themselves m m 1 queuing system. It satisfies the Markov property. The first index is for the number of customers in the first node and second index is for the number of customers in the second node.

So, 0 comma 0 means, in both the queues no one in the system, if one arrival comes into the first node arrival cannot come into the second node only. The arrival come to the first node the inter arrival is exponential distribution. Therefore, the rate of moving from 0 comma 0 to 1 comma 0, that is lambda. Similarly, there is a possibility of one more customer entering into the system, when one customer in the first queue. So, it will be a parameter lambda. Therefore, the rate will be lambda for the arcs moving from 0 comma 0 to one comma 0 1 comma 0 to 2 comma 0 and so on.

Whereas, after one customer already in the first node, there is a possibility, the first node the first the server in the first node, who would have completed the service before the one more arrival into the first node. Therefore, the service is exponential distribution with the parameter mu 1. Therefore, the system goes from 1 0 to 0 comma 1, that means the customer who was under the service that service is over. Therefore, we moved into the second queue now. The first first node has 0 customer and the second node has 1 customer in the system and his service will start once, he enter into the second node.

This so, this rate will be mu 1. Now, the situation is the situations are either one more arrival not one more arrival. One arrival to the first node or the customer who is in the second node, he would have completed his service. So, that service is over. Therefore, that rate is mu 2, then the system goes to 0 comma 0 or it will go to 1 comma 1 with the rate lambda, sorry lambda. So, the same way one can discuss for 1 comma 1 also. That means one customer in the first node 1 customer in the second node. So, the one possibility is the first customer's service is over.

Therefore, it will be a 0 2 with the rate mu 1 or 1 more arrival takes place. Therefore, it will be a 2 comma 1 with the rate lambda or the second service would have been finished. So, that rate is mu 2. Therefore, it is 1 comma 1 to 1 comma 0. So, this are the possibilities, the system can move from 1 comma 1 to other states 0 comma 2 1 comma 2 or 1 comma 0 and so on. So, this is the way you can visualize the different transition arcs and with the corresponding rates. So, this is a state transition diagram for two queues, tandem queuing network obviously from this diagram itself, we can say that this is not a birth death process, if it is a birth death process, then it has to be a 1 dimensional and the system has to move forward one step or backward one step, no other moves.

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Stationary Distribution

Solve

$$\pi Q = 0; \sum_{(n_1, n_2) \in \Omega} \pi_{(n_1, n_2)} = 1$$

$$P(X = n_1, Y = n_2) = (1 - \rho_1)\rho_1^{n_1}(1 - \rho_2)\rho_2^{n_2}$$

$$\rho_1 = \frac{\lambda}{\mu_1} < 1; \rho_2 = \frac{\lambda}{\mu_2} < 1; n_1 \geq 0; n_2 \geq 0$$

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So, here you have a 2 dimensional Markov Chain. Therefore, this is not a birth death process, it is a continuous chain, Markov Chain. The corresponding stochastic process is a continuous time Markov Chain. Since, we land up the underlying stochastic process is a continuous time Markov Chain. Now, you can find out the stationary distribution that is our interest. Solve pi q is equal to 0 and summation of pi i is equal to 1. Now, the pi i's are not just 1 pi i. It is 2 index 2, so by solving this equation, that means you write the balance equation. Then use the summation of probability is equal to 1.

You need a condition, so as long as the rho rho 1 is a lambda divided by mu 1, that is less than 1 as well as rho 2. That is lambda divided by mu 2, that is also less than 1, then the stationary distribution exist. You can find out the stationary distribution probability as a probability, that n 1 customer in the first node and n 2 customer in the second node, that is nothing but n 1 customers in the first node. That probability distribution is 1 minus rho 1 times rho 1 power n 1. The n 2 customers in the second node, that is nothing but 1 minus rho 2 times rho 2 power n 2.

So, this is sort of product the probability of n 1 customer in the first node and probability that n 2 customer in the second node together. That is same as what is a probability, the n one customers in the first node multiplied by what is the probability that n 2 customers in the second node. This form is called product form solution. So, for a tandem queuing network the stationary distributions are of the product form solution product form this set together probability is a product of individual m m 1, sorry m m 1 infinity stationary distribution probabilities. Therefore, this is called product form.

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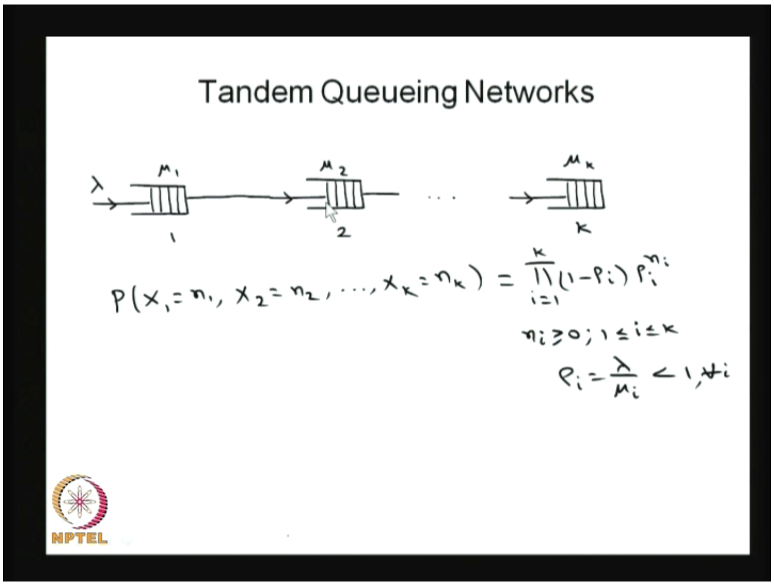

So, this exists as long as rho on1is less than 1 and rho 2 is less than 1. If it is greater than or equal to 1, then the stable, then the system is not stable equivalently stationary distribution does not exist. Now, I can extend this two queues queuing network into k queues or k nodes. So, there also my interest is to find out the stationary distribution and the probability of n one customer in the first node n 2 customer in the second node and n k customer in the and so on. Till n k in the kth node, then joint distribution is same as product of individual distribution of n i customers in ith node.

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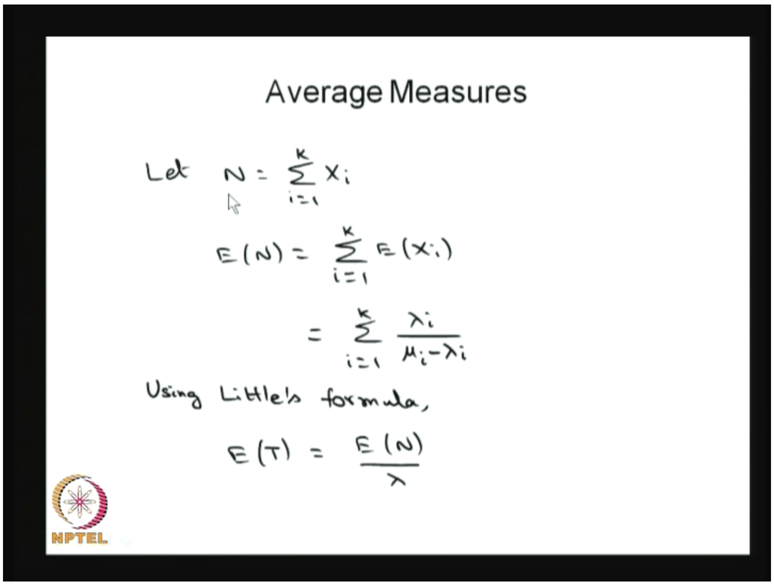

So, this is a product form product of i is equal to 1 to k and this result is nothing but stationary distribution of m m 1 infinity queue. So, this exist as long as all rho i's is less than 1. Since, the arrival is lambda mean arrival rate is lambda. Therefore, all the rho i's are lambda divided by the service rates for the corresponding queue. So, once you know the number of customers in what is a distribution of number of customers in each node for the k node, tandem queuing network you can find out the total number of customers in the whole queuing networks.

That means you have to sum it up all the average number of customers in each node, if you have sum it up that will be the total number of customers in the whole queuing network. And since this each node is going to behave like m m 1 queuing system. Therefore, you can use the average number of customers in a m m 1 queuing system. That result that is lambda i divided by mu i minus lambda I. Whereas, here all the lambda i's are lambda because it is a tandem queue and if sum it up that is going to be the total number of customers in the system.

Once you know the total number of customers in the system. System is the whole queuing network using Little's formula, you can find out the average time spent in the system. Some books they say average response time average sojourn time, so all those things are same. So, the average sojourn time using Little's formula you can get it after substituting expected number in the system divided by the arrival rate.

Because little's formula is applied to the queuing system in the in the sense the arrival rate to the system is lambda and total number of customers in the system. So, that the whole all that series, all the queues that whole tandem queue, that you treat it as the one system. It satisfies all the Little's law conditions, that is first come first served and arrival rate is lambda.

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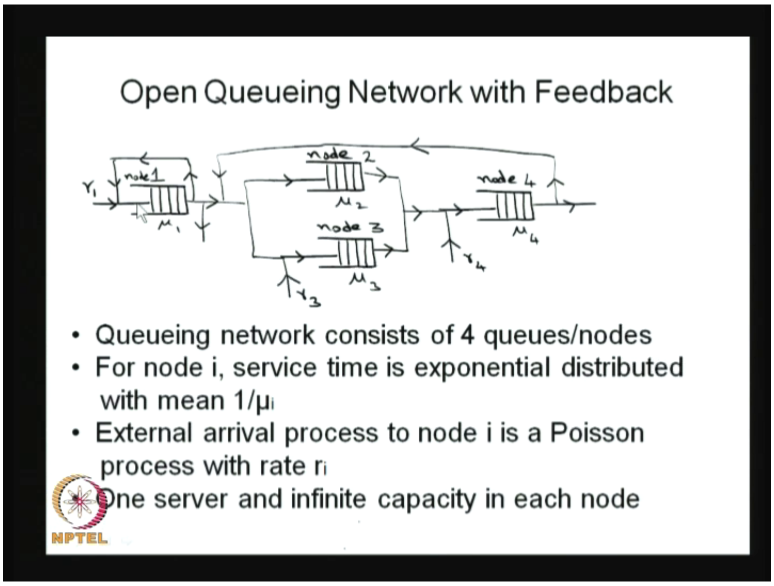

After the service is over, the system the customers leave the system and so on. So, you can apply the Little's formula and get the average time spent in the system for the whole system not for the individual queues. Now, I am moving into the open queuing network with feedback. So, I am just giving a one simple scenario of 4 nodes queuing network, which with the feedback also. This is the open queuing network and here the assumptions are for each node the service time is exponentially distributed with the mean 1 divided by mu i or the parameter mu i. Only 1 server in each node and infinite capacity only 1 server in each node and infinite capacity of the for each node also.

The external arrivals external arrival process to the node I, that is a Poisson process with the rate lambda i's, that means for the node one there is a external arrival. That is r 1, there is no external arrival to the node 2. Whereas, there is a external arrival from for the node 3. That is r 3 that arrival is a Poisson process and this arrival is also Poisson process with the parameter r 4. I have not supplied the routing probabilities, I have not supplied the routing probabilities. So, after the service is over in the first node with some probability, it moved into the again node 1 with some probability, it goes to node 3 as well as node 4 in this partition. We can multiply this and this, then with some probability it goes away from the system.

So, the summation of this plus this and this probability has to be 1. Sometimes we would not draw this arc outgoing arc. So, there are two possibilities, either this arc probability is

0 or non-zero. Whenever the summation of other than the going out if that probability is not equal to 1, that means 1 minus of summation of probability that is 1 minus of this summation of probability is greater than 0. Then that is the probability, that the system is after the service is over the customers, who finishes the service in node 1 leave the system.

Whenever, the summation of arcs from the node i to all other nodes, if that summation of probability is less than 1, then the 1 minus of that summation of probability, that is the probability in which after the service is over the customers are leave the system. So, that is a way you make out the probability for this, otherwise we need to supply the routing probability matrix. So, here I am concluding with the open queuing network with the feedback and the assumption are infinite capacity in each node and single server and arrival external arrival process is a Poisson process with the rate r i's.

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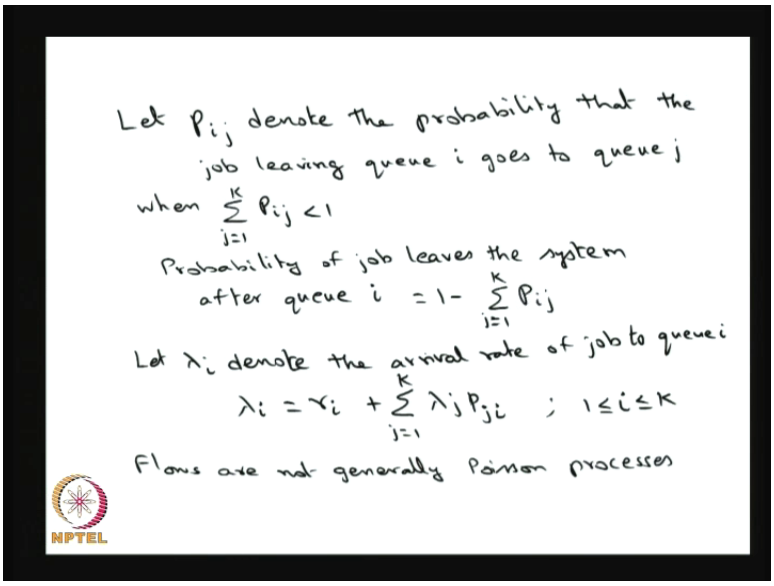

Whereas, the actual or the total arrival rate that is different from r 1 because the external arrival rate is a Poisson process with the r 1, but there are some customers after finishing the service in the first node. They are again coming back, therefore the total or actual arrival stream into the node 1 that is different from r 1. So, that we are going to calculate later. So, I am going for the routing probability with the notation p of i comma j p of i comma j denotes the probability. That the job leaving queue i goes to the queue j. Whenever this probability is summation of probability, that rho sum is less than 1. Then, the probability of job leaving the system leaves the system after the queue i, that is going to be 1 minus of summation j is equal to 1 to k.

We have k k is the total number of queues in the open queuing network with the feedback. The P i j is the routing probability of jobs going from the node i to node j, either you can say node or queue. The actual or the total arrival rate that lambda i denotes the arrival rate of a job to the queue i, that can be computed using this formula. So ,the lambda i is equal to r i plus, what are all the different rates lambda j. This is the packets are or the jobs are moving from the queue j to queue i. So, if you multiply this routing probability and the arrival rate that summation plus the external arrival that rater I that will give the arrival rate of job to the node queue, where i is running from 1 to k.

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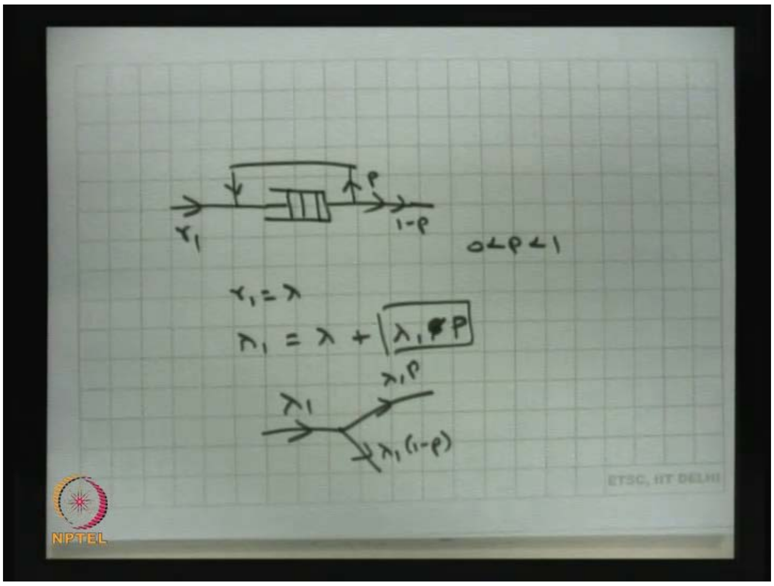

Let me give a one simple situation, how one can calculate the arrival rate for any queue? Suppose, you think of the external arrival rate is r 1 and after the service is over with the probability P, with the probability P the customers, who finishes the service in the first node and come back with the probability 1 minus P with the probability 1 minus P it leaves the first node. Therefore, our interest is to find out, what is the arrival rate for the node 1 with simple only 1 feedback? So, here suppose I make it r 1 is lambda, that is the external arrival rate.

So, the arrival rate to the node 1, that is lambda 1. That is same as the external arrival rate plus, what are all the possibilities in which the node 1 is build up? So, that with the probability P, so this is P times, suppose lambda 1 is the arrival rate and lambda 1 times P that is going to be the proportion. The P is the proportion in which it is coming back and we can use the Burke's theorem. The departure process is the Poisson process. If the arrival rate is lambda 1, then the lambda 1 times p using the Poisson process split one. Poisson process can be splitted into… Suppose, this is a departure process. The departure process is splitted into 2 Poisson process lambda 1 P and lambda 1 times 1 minus p. So, the lambda one times p that is feeded again into the node 1.

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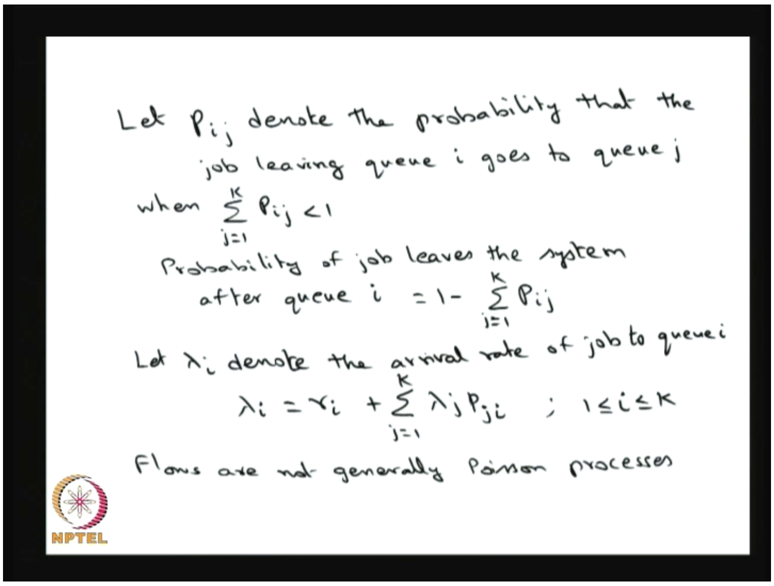

Therefore, that will give lambda 1 times P, so the lambda 1 is equal to lambda plus lambda 1 minus P, that is the way. So, here the arrival rate lambda i is equal to r i plus the summation of lambda j P j i, that is the same thing. I am applying for the i is equal to 1, so the lambda 1 is equal to lambda that is the external arrival rate plus what are all the ways you have input arc to the node 1. So, so this point the departure process is a Poisson process using the Burke's theorem.

Therefore, suppose you make it arrival rate is lambda 1, therefore the departure process is the Poisson process with the parameter lambda 1 and the poisson stream is splitted into 2 P 2 with the 1 Poisson stream with the proportion P. The another Poisson stream with the proportion 1 minus P. Therefore, you have a 2 Poisson stream with the parameters lambda 1 times P and lambda 1 times 1 minus P. These two are the two independent Poisson streams and this Poisson stream is feed it again into the node 1.

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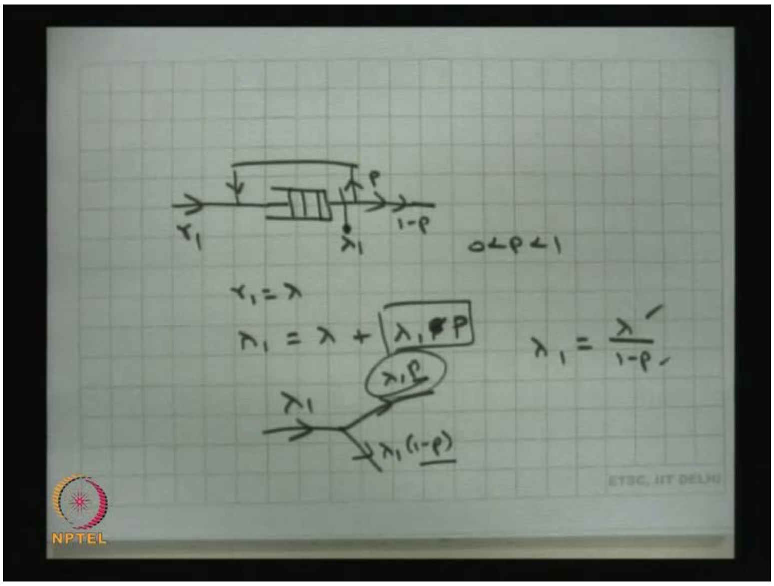

Therefore, therefore this is the only one input for the node 1. Therefore, you will have a lambda 1 is equal to lambda plus lambda 1 times p. Like that if you have a many k nodes you have to write the equation for the all the k nodes. Then you have to solve for lambda i's, then you will get the, what is the arrival rate for the node i. So, here I have only one equation, so I can get lambda 1 is going to be lambda divided by 1 minus P. By solving this equation lambda 1 is equal to lambda plus lambda 1 times P.

Therefore, lambda 1 is going to be lambda divided by 1 minus P, this is known to me, lambda is known to me and p is known to me. Therefore, using these two, I can get lambda 1 that is the arrival rate for the node 1. So, this is the simplest example, but for any open queuing network with the feedback by framing these equations k, equations by solving we get the lambda i's the product form.

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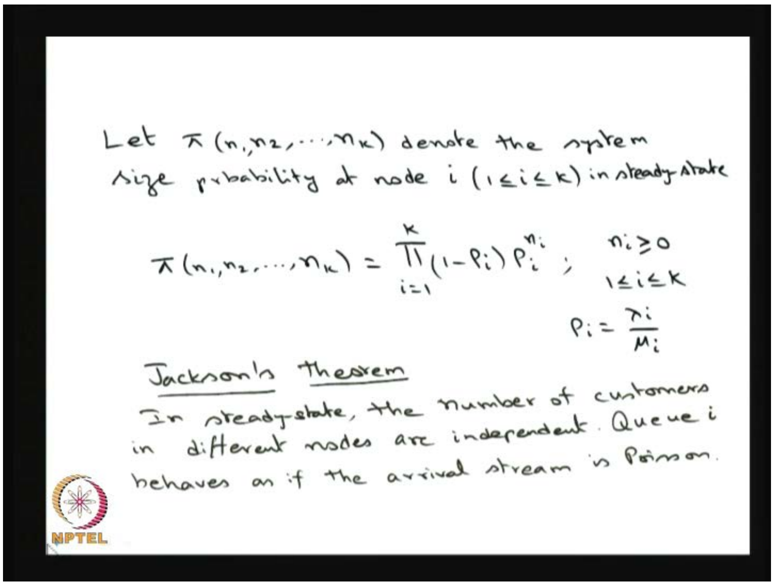

Solution is valid for the stationary distribution that is what given as the Jackson theorem. So, the Jackson theorem says in steady state, the number of customer in different nodes are independent, the number of customers in different nodes are independent. That means the behavior of the queuing system is consisting of behavior of many independent nodes, the queue i behaves as if the arrival stream is Poisson.

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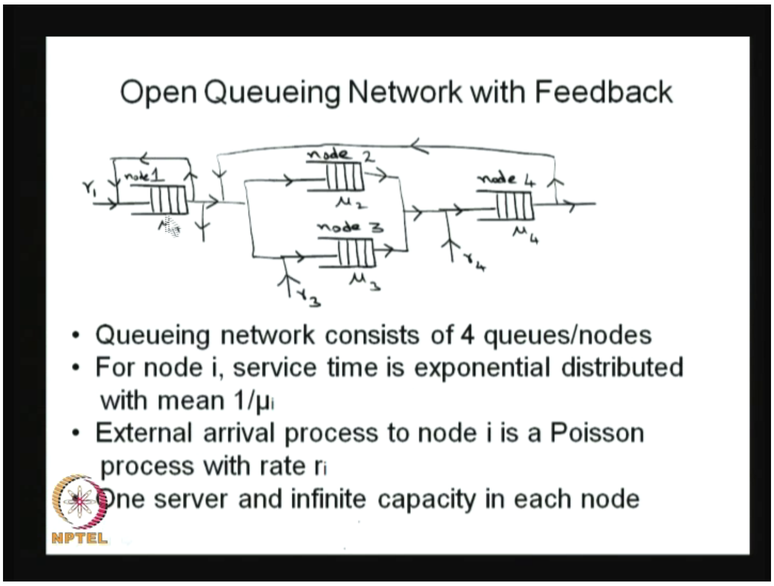

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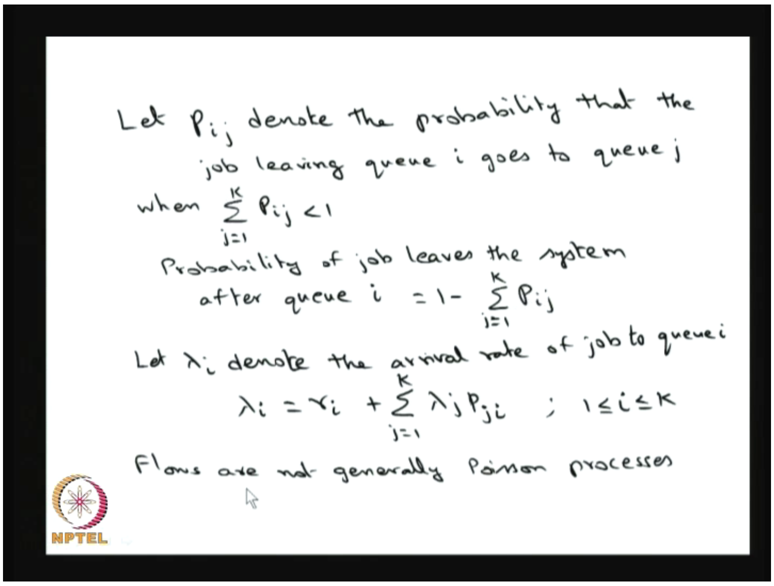

So, not only each node behaves independently, not only each node behaves independently as if it behaves the arrival is going to be a Poisson for each node. Therefore, this point it is a Poisson whereas, this point is not a Poisson arrival process, but still in steady state. The number of customers in this queue number of customers in this queue, this queue and this queue all the queue size are independent as well as in steady state. The arrival process for the each node behaves as if as if Poisson process, but if they are not in general Poisson process.

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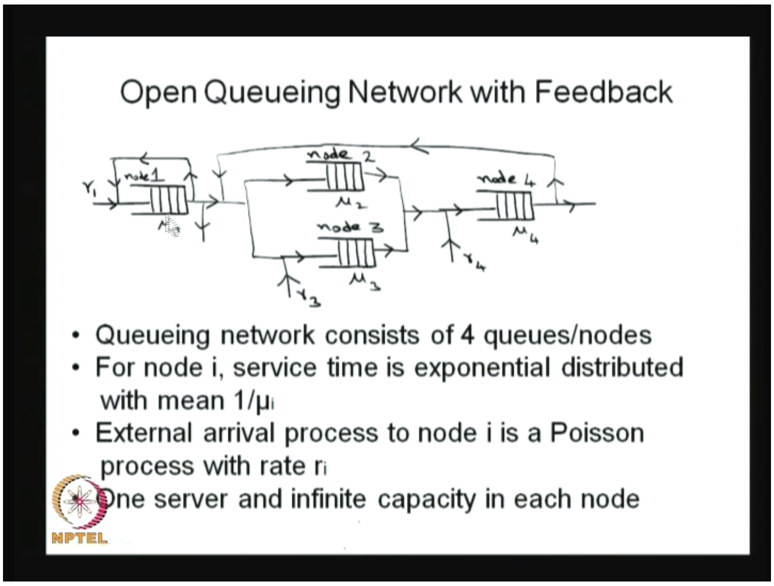

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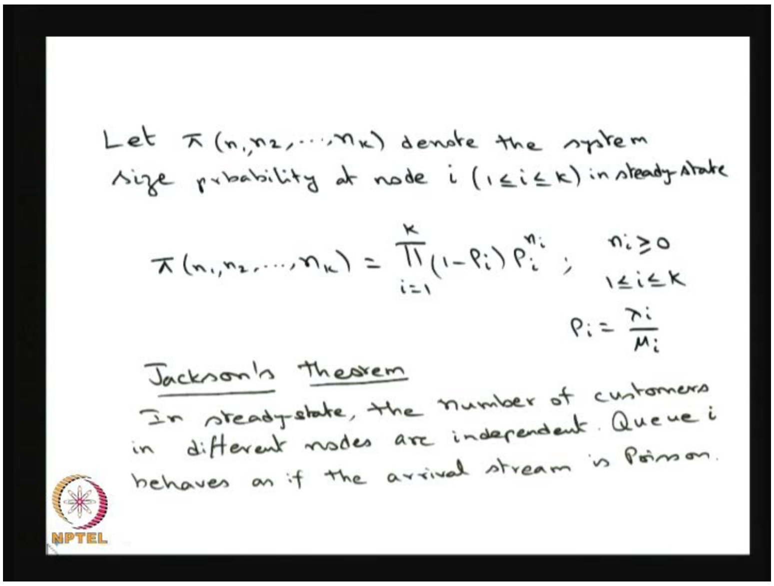

So, in steady state whenever you have a open queuing network with this assumptions in steady state, if this behaves as a independent queuing. Therefore, the joint probability of a n 1 customer in the first queue n 2 customers in the second queue. Similarly, and so on, till the n k customers in the kth node. Since, each nodes behaves independently, so the number of customers in different queues are independent and say arrival are Poisson and already you made assumption. It is infinite capacity in each queue as well as only one server in each queue.

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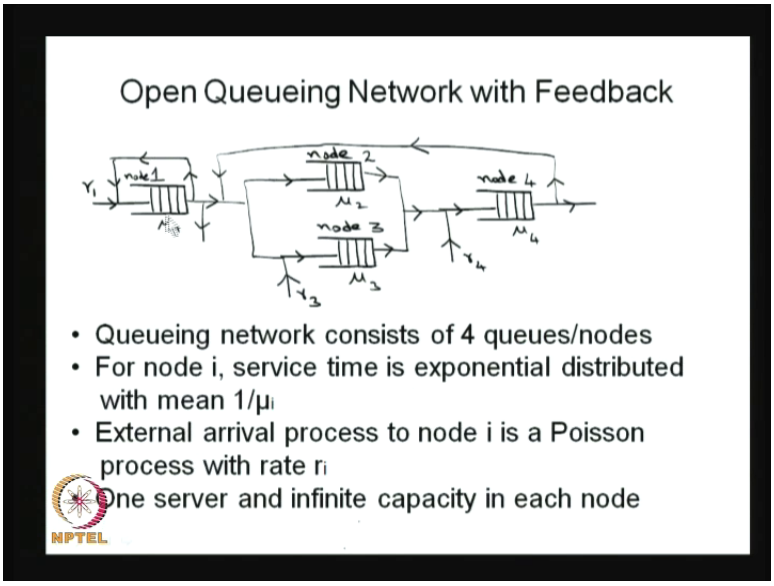

Therefore, the arrival follows Poisson service is exponential only one server infinite capacity. That means even though it is a open queuing network with a feedback each node behaves as if m m 1 queue in steady state. That is important, that means as a time dependent the system may depends on the size of the other number of customers in the other nodes, but in steady state these behaves independently and each one behaves like a m m 1 infinity queue.

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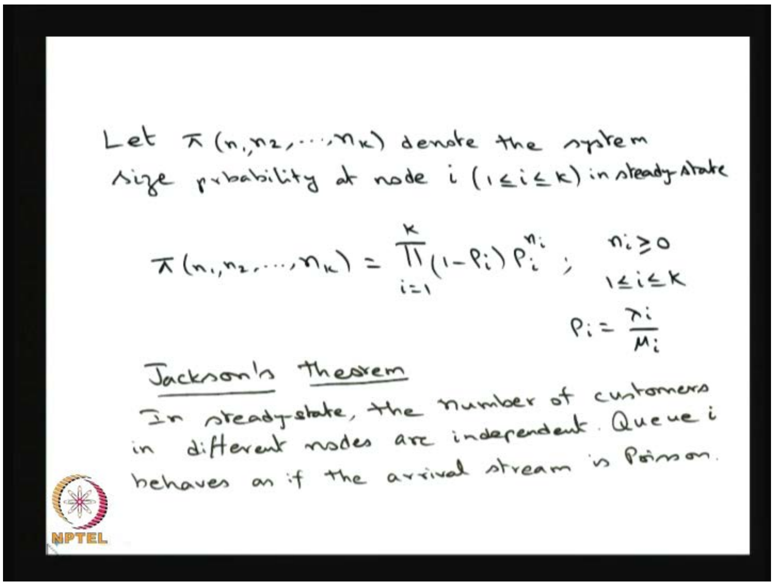

Therefore, you can get the joint distribution of n i customers in ith node, that is the product of n i customers in ith node. So, if you make a product that is going to be the joint distribution because joint distribution is going to be the product of individual probabilities, if each random variables are, if each random variable is independent. Therefore, you can use that logic to use to get the joint probability as the product form solution.

Here rho i's are nothing but lambda i's divided by mu i's and which is has to be less than 1. If about right each rho i has to be less than 1, it has to be stable each queue queuing system has to be stable. So, in steady state you have a product form solution where rho i is lambda i divided by mu i. You have to find out lambda i's by solving the system of k equations.

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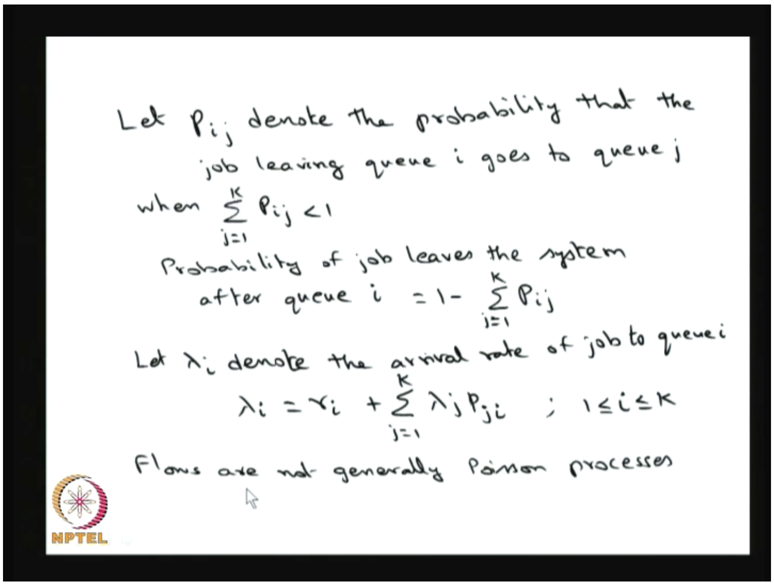

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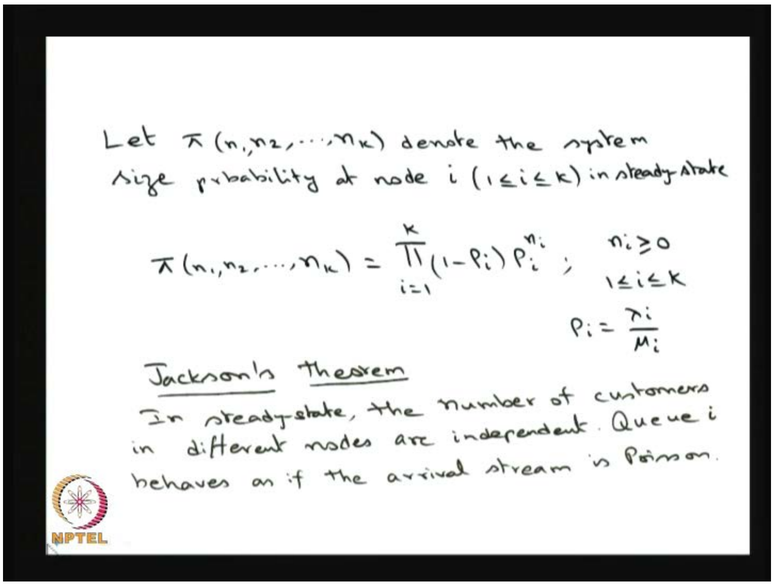

Lambda i's are the unknown r i's are given routing probabilities are given. So, using that solve for solve for lambda i's from these k equations. So, once you know the lambda i's check whether the lambda i's divided by mu i's is less than 1. Then the stationary distribution exist, using the Jackson theorem, the joint distribution is the joint distribution.

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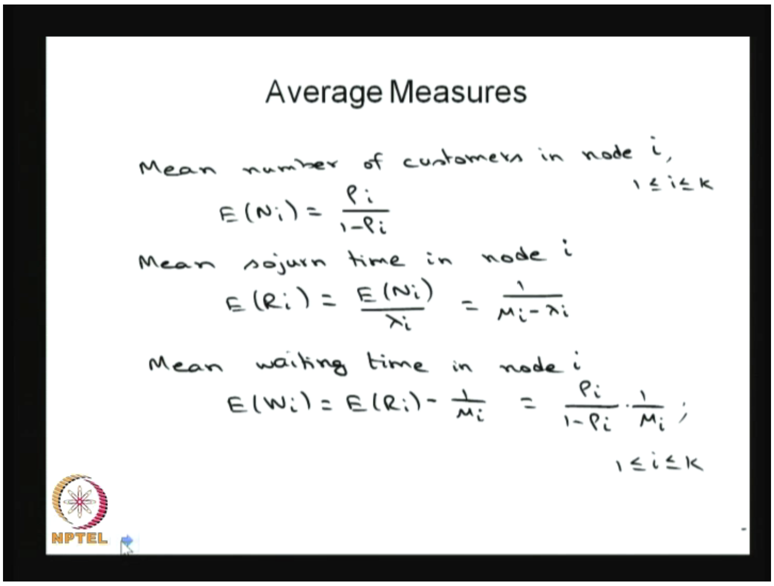

That is the stationary distribution is of the product form solution, assuming that each queue behaves as the m m 1 queue. So, for any general k queues not tandem queue open queuing network with the feedback, you can get the average measures also.

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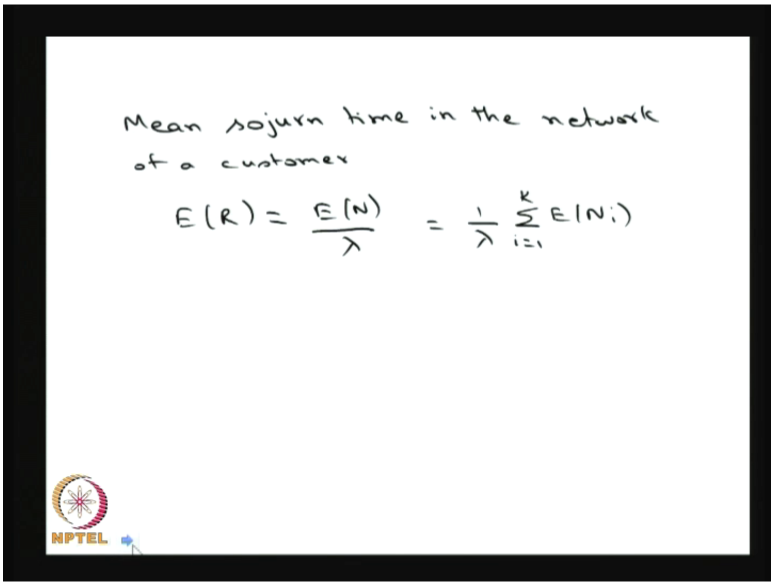

The way we have calculated for the two queues model are the tandem queue model. The same logic can be used for the open queuing networks with a feedback you are getting average number. Then sojourn time in each node, then mean waiting time in node each node by subtracting the average service time for each node. You have k nodes, therefore you are getting the these measures for each node.

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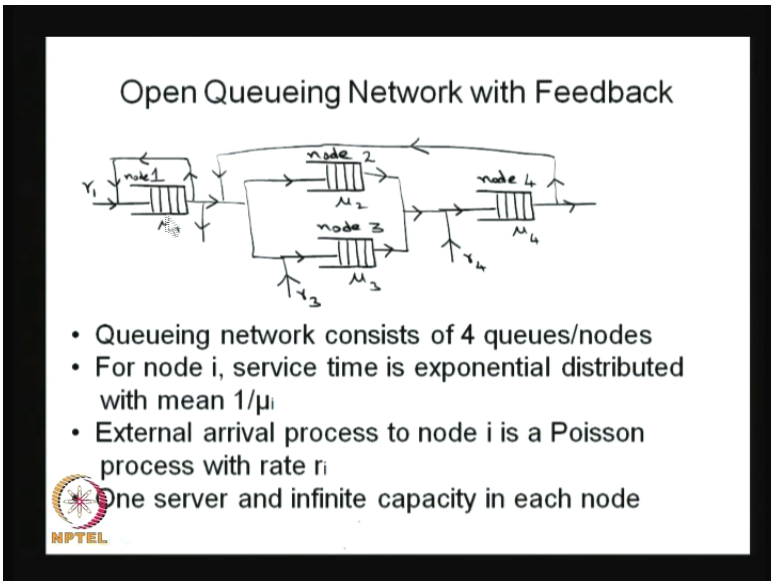

Once you know the result for the each node you can find out the total sojurn time by using the Little's formula because Little's formula is valid. Here the external arrival rate is lambda, so that you have to add… Suppose, here you have to finding out the lambda and for the open queuing network the Lambda you have to compute by adding all the external arrival rates r 1 plus r 3 plus r 4, that is going to be lambda in this example.

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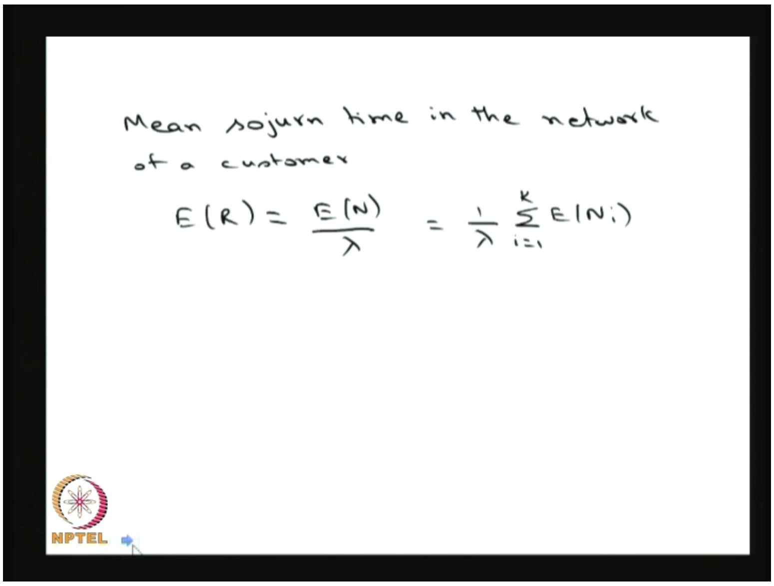

So if you add all the external arrival rates to different queue different nodes that summation is going to be the total arrival rate to the system. Because to apply the Little's formula, you think you consider the whole thing as a one system in which r 1 plus r 2 plus r 3, all are independent. Therefore, the summation is going to be the arrival rate for the system, so the lambda i that example r 1 plus r 2 plus, sorry r 1 plus r 3 plus r 4.

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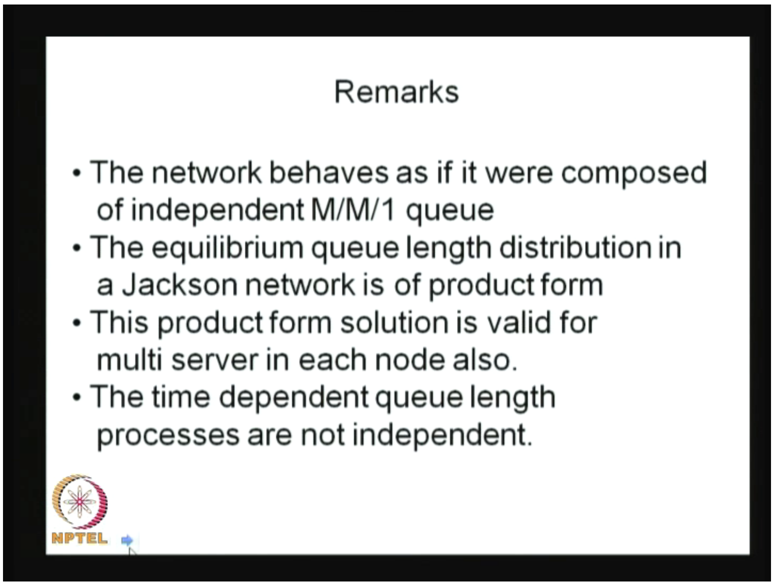

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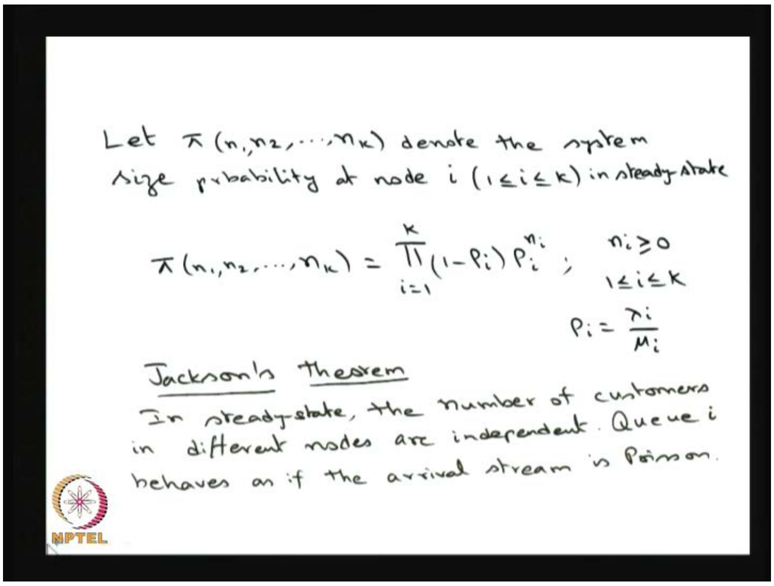

Remarks for the open queuing network the networks behave as if it were composed of independent m m 1, queues in steady state - that is important. I forgot to write that in steady state in time dependent, this is not the case that I am going to have written as a fourth remark.

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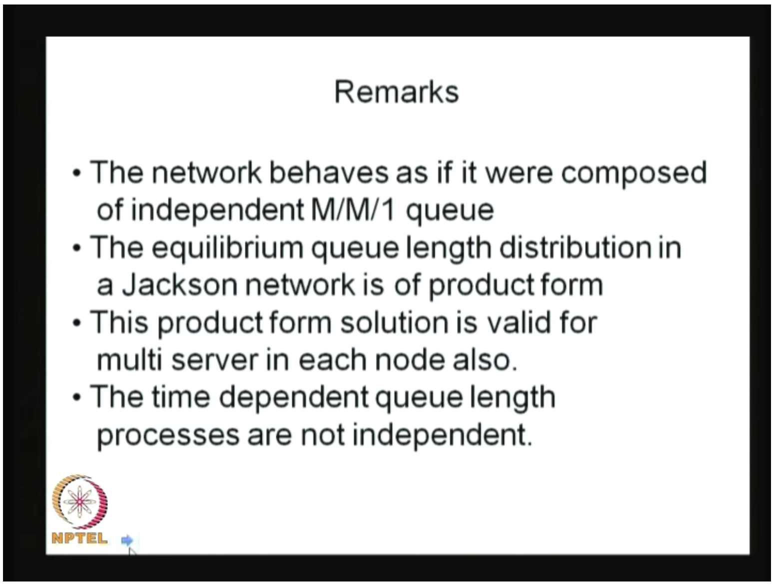

In steady state this network behaves as if it is composed of independent m m 1 queue So, the equilibrium queuing length distribution in a Jackson network is of product form. So, this solution is called a product form, solution for open queuing network with the feedback that equilibrium queuing length distribution is of product form. This the previous example can be extended to the multi server model also.

In that model we have taken it as a single server infinite capacity you can think of each queue is M M c infinity also instead of M M 1 infinity in steady state. This solution is valid with the M M c infinity logic, whereas the time dependent queuing length process are not independent. Therefore, the product, product form solution would not work. The time dependent scenario is completely different and with the steady state or the equilibrium queuing length distribution.

You cannot discuss the behavior of a time dependent. It is completely independent, sorry completely the time dependent, queuing length distributions are not independent for the each queues.

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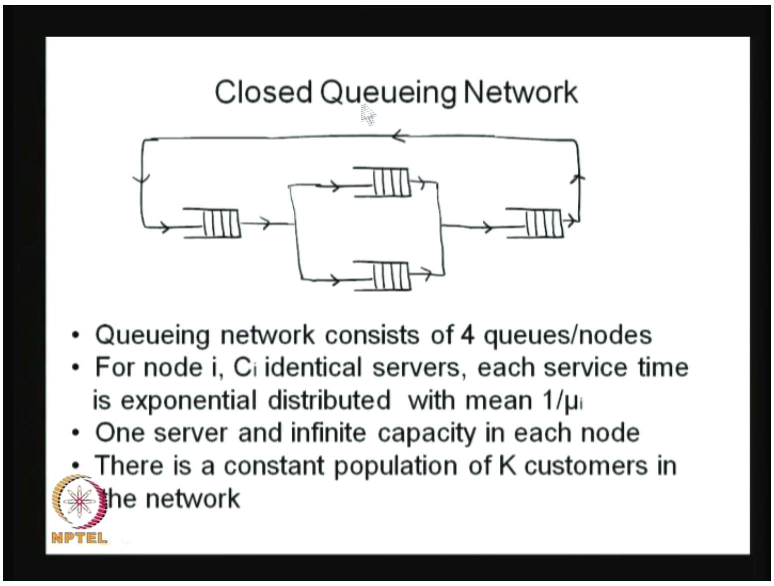

Now, I am moving into the closed queuing network. So, here comparing with the open queuing network, here we have a fixed number of population is moving around the queues. No one leaves and no one enter the system. Therefore, you keep some k customers in the system. This example, I have 4 nodes and instead of either we can have one server or more than one servers also allowed in a infinite queuing capacity system.

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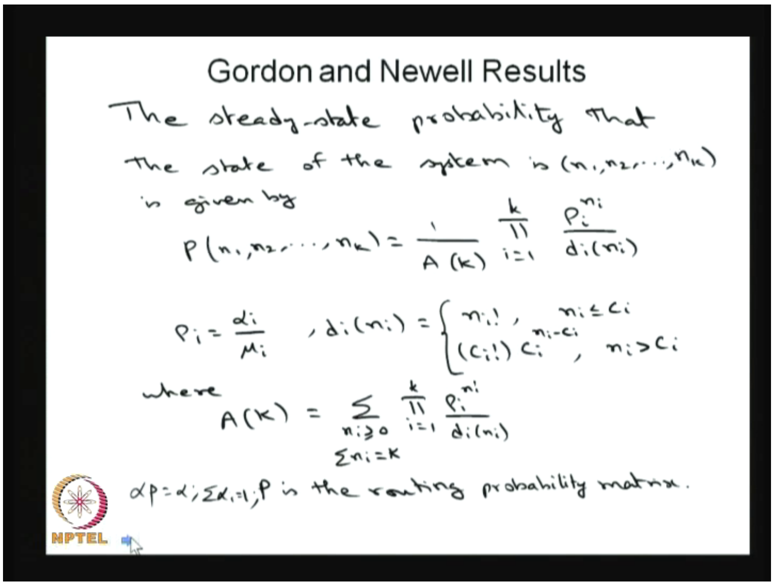

You make the assumption the service time is exponential distribution for each queue and all the servers are identical here. Also you can get the product form solution and the joint distribution of the system size. That is same as the product of rho i's power n i divided by d i's of n i for the k nodes. The small k is the k nodes in the system and capital K is the total number of population. So, the here this k's are nothing but the normalizing constant and the rho i's are in terms of alpha i divided by mu i. Alpha you can calculate by solving this equation where P is the routing probability matrix. You solve alpha times P is equal to alpha and the summation of alpha is equal to 1, using that you get alpha and substitute alpha i's here.

Therefore, you will get rho i's, then you substitute d i's here based on the number of servers in the each node is 1 or more than 1. Accordingly you can use this and once you know the d i n i substitute here, this product form will give a joint distribution of a system size in steady state and this result is given by Gordon and Newell for the closed queuing network.

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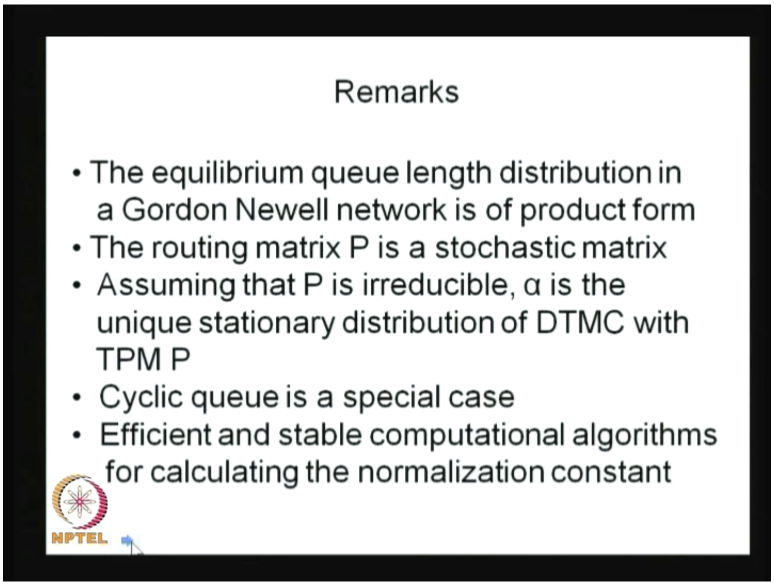

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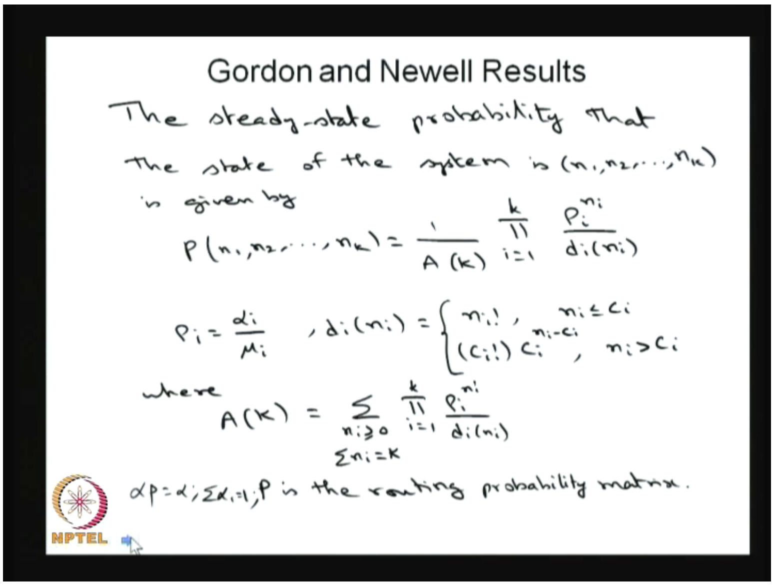

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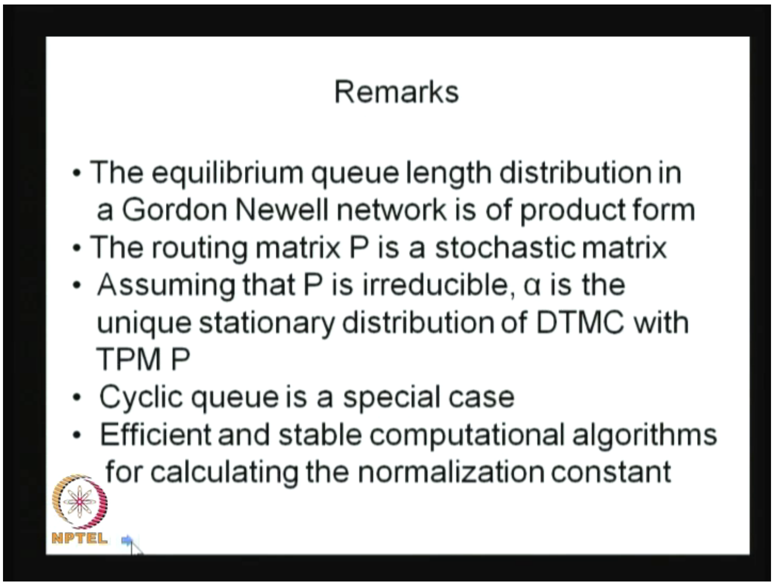

As a remark this equilibrium solution is also product form. The routing probability matrix is a stochastic matrix and suppose you assume P is a irreducible, then the solving alpha P is equal to alpha and the summation of alpha is equal to 1. That is nothing but alphas are nothing but the stationary distribution. Whenever P is a irreducible, not P is irreducible, the underlying DTMC is irreducible with the assuming that the underlying DTMC is irreducible. Then the alpha is nothing but the unique stationary distribution.

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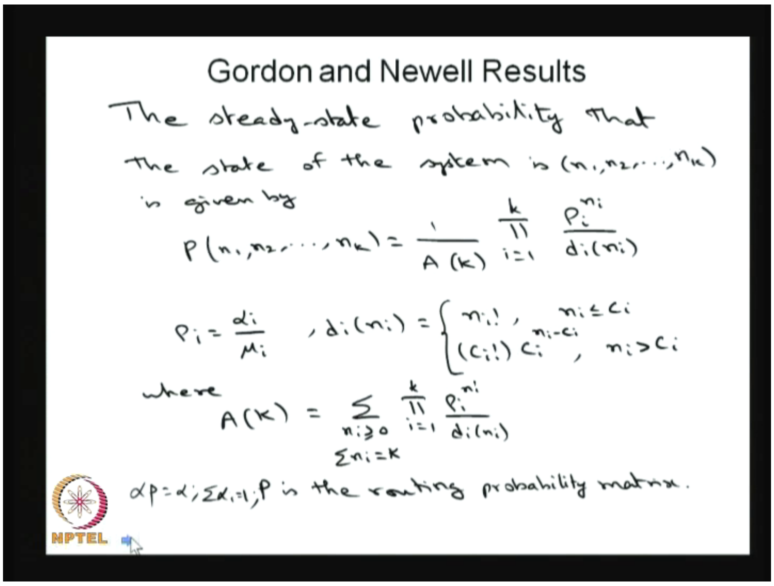

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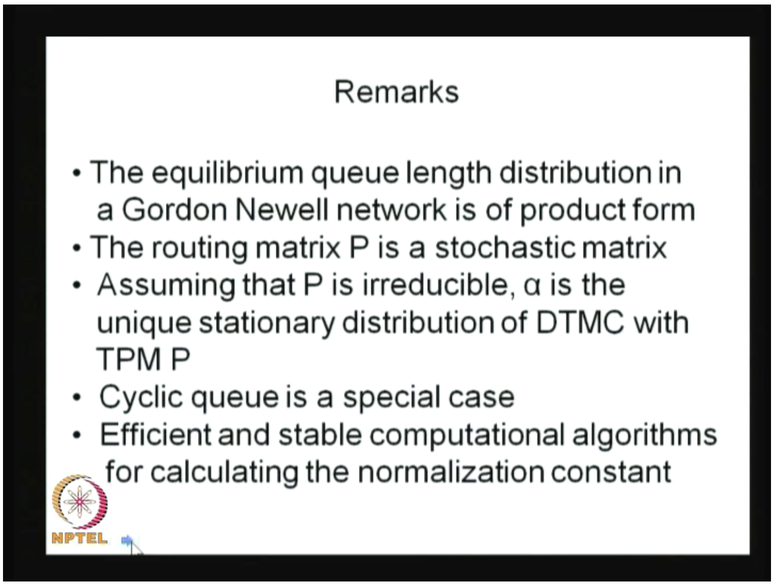

This is valid for the cyclic queue also and the toughness is how to compute a of k where k is the number of, sorry customers in the whole queuing network. So, you need efficient and stable computational algorithm for calculating this normalizing constant a of k.

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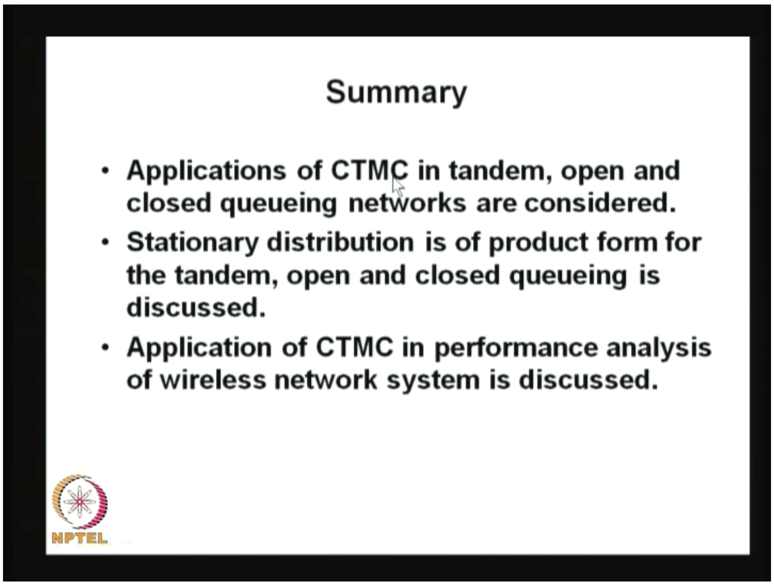

Here some mistake assuming that the DTMC is irreducible the DTMC is irreducible alpha is the unique stationary distribution. Now, we will move into the summary, so in this lecture we have discuss, the applications of CTMC in tandem open and closed queuing networks. We have discuss the stationary distribution and other performance measures for the tandem open and closed queuing networks with the only the product form solutions.

We did not discuss the non-product form solution, we have only discuss the product form solution and application of c t m CTMC in the performance analysis of wireless network system, that I will discuss in the next lecture. Also I am going to discuss the simulation of simple Markovian queuing networks in the next lecture. These are all the reference books.

Thanks.