Stochastic Processes

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Module - 5
Continuous-time Markov Chain
Lecture - 2
Limiting and Stationary Distributions, Birth Death Processes

This is module five Continuous-time Markov Chain. In the first lecture, we have discussed the definition of a Continuous-time Markov Chain, then we have explained how we can derive the Chapman-Kolmogorov equation, then we have defined infinite decimal generator matrix. Then I have given the Kolmogorov differential equations in the first lecture.

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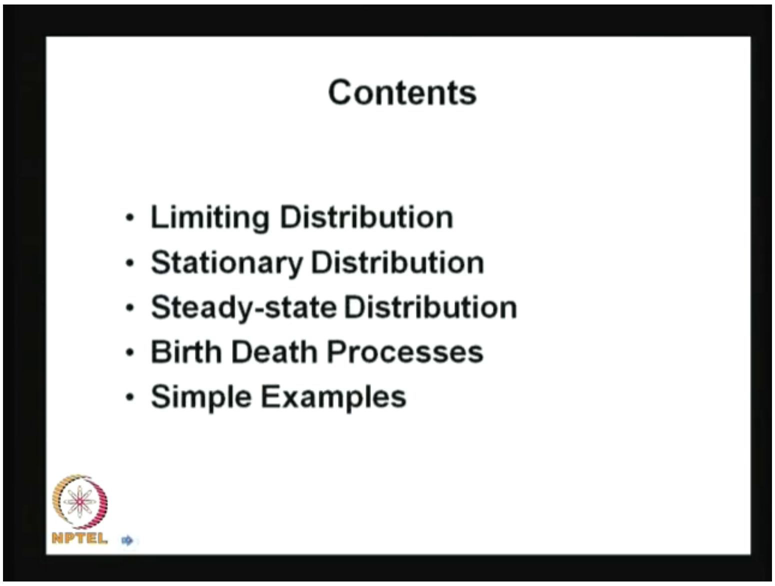

In the lecture 2, I am planning to discuss the limiting distribution, stationary distribution and steady state distribution; followed by that, I am planning to give a description about the birth death processes. And also, some simple examples for the limiting distribution stationary, steady state distributions and birth death processes.

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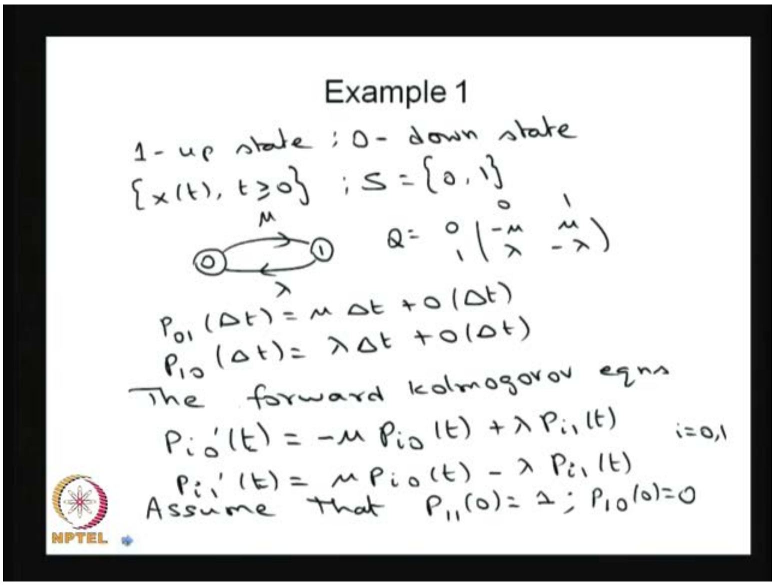

Before I go to the limiting distribution let me explain the, let me give the example for the Continuous-time Markov Chain to get the time dependence solution. This example is the very simplest example that is a two states Continuous-time Markov Chain, the default one is the time homogeneous. The state space are 1 and 0; 1 we can consider as a up state or operational state, and 0 is the down state non operation state.

So, this can be visualized for the any model, in which the whole dynamics can be described with the two state and the Markov properties satisfied. The system going from the state 1 to 0 are the time spending in the state 1 before moving into the state 0 that is exponential distributed with the parameter lambda. Once it is failed that means the system is in the down state, the time spent in the repair time that is exponential distributed with the parameter mu. So, once the repair is over, the system is operation state therefore, it is in the up state. So, the 0 is related to the down state and 1 is related to the up state and the mu is nothing but the mean, 1 by mu is the mean time for the repair and 1 by lambda is the mean time of a failure.

And the failure time is a exponentially distributed with the parameter lambda and the repair time is exponentially distributed with the parameter mu. This is the state transition diagram for the two states C T M C. The corresponding a q matrix, the infinite decimal generator matrix that is consists of it is a two cross two matrix. The system going from the state 0 to 1 that rate is mu. The system is going from the state 1 to 0 that rate is lambda.

And the diagonal values are minus of summation of other values, that row, rows are. So, 0 to 0 is minus mu and 1 to 1 is minus lambda. Therefore, the rates are in the other than diagonal elements and the diagonal elements are minus of sum of the row values, other than that diagonal element. So, this is nothing but in a very small interval of time delta t the system is moving from the state 0 to 1 that probability, the probability of system moving from the state 0 to 1 that is nothing but the down state to the up state in a very small interval of time delta t. Why we are finding, why we are finding the probability of delta t since the model is a time homogeneous, only the interval is matter, not the actual time or you can visualize this as the sometime t to t plus delta t also. So, this is the interval of delta small negligible interval delta t the system is moving from the state 0 to 1 that probability is nothing but the rate in which the rate, the rate is nothing but the repair rate.

So, the mean rate mu times the delta t plus order of delta t. So, it is a small o, order of delta t means, has a delta t tends to 0 order of delta t will be 0. Similarly, we can visualize the probability of system moving from the state 1 to 0 in the interval delta t, in the small interval delta t that is same as the failure rate lambda times the delta t that is the small interval of time plus order of delta t so this order of delta t also tends to 0 as delta t tends to 0. So, using this I can make the forward Kolmogorov equation. I can go for writing a forward Kolmogorov equation or backward Kolmogorov equation but forward Kolmogorov equation is easy to make out.

So, if the system is in the state i at time 0 what is the net rate the system will be in the state 1 at the time t that net rate is nothing but what are all the inflow that probability rate minus what are all the outflows. That is the way you can visualize the right hand side. So, all the positive term terms are related to the incoming rates and the all the negative terms are related to the outgoing rates. So, since it is a two state model if the system is in the state 0 at time t, there is a possibility it it is not moved anywhere from the state 0 or it would have come from the state 1. Therefore, the incoming will be the state 1 therefore, the system will be in the state 1 at time t and starting from given that the starting from the state i that probability multiplied by the rate sort of inflow minus because we are writing the equation for the state 0.

Therefore, it is not moved from the state 0 that is a with the rate mu it can move to the state 0 to 1. Therefore, minus mu times, it does not move from the state 0 therefore, minus mu times the probability of being in the state 0 at time t, given that it was in the state i at time 0, that probability multiplied by minus mu that is outflow and lambda times P i 1 t that is inflow. Therefore, the left hand side it is the derivative of the function t. It is the probability function.

So, P i 0 dash t that is nothing but the net rate being in the system at time 0 sorry at time t in a state 0 given that it was in the state i at time 0, that net rate is same as a inflow minus outflow with the corresponding rates. Similarly, you can write the equation for the state 1 that means you start from the state 1, either you would have move, you would have come from the state 0 to the 1 or you did not move from the state 1. Therefore minus lambda times P i 1 of t plus mu times P i 0 of t, that is the net rate corresponding to the state 1.

Now, we are able to write the forward Kolmogorov equation. So, this is the interpretation of the forward Kolmogorov equation, you can write easily by making a matrix P i j of t dash that is equal to P i j of t times Q where Q is the infinite decimal generator matrix. Then also you will get the same thing so I am just giving the interpretation. Now, my interest is to find out the time dependent or transient solution for the, this two state C T M C. For that this is difference differential equation we need a initial condition to solve these equations. So, I mean the assumption at time 0 the system is in the state 1. Therefore, the transition probability of system the P i P 1 1 of 0 that is equal to 1 since I made the assumption the system was in the state 0 at sorry the state 1 at time 0 therefore, that the being in the state 0 that is going to be 0. So, I need this both the initial conditions to solve the equation.

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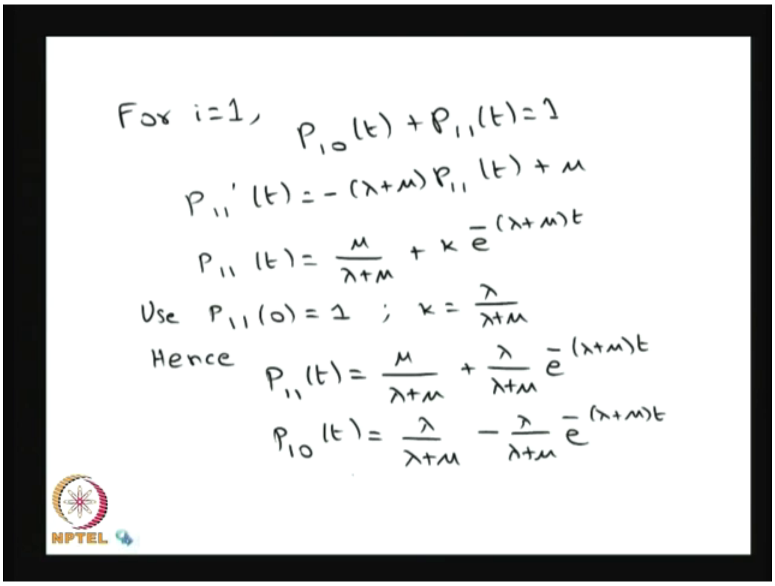

So, let me start since I made the initial condition state is 1 therefore, i is equal to 1 so I have the first equation that is a I always have the summation of the probability at time t, this the transition probabilities are going to be 1, the summation. And also I have a two difference differential equations. So, what I can do? I can take the second equation in these, then instead of P 1 0 of t I can used the summation of probability is equal to 1 therefore, the instead of P 1 0 of t I can use the P 1 0 of t is nothing but 1 minus P 1 1 of t I can substitute in the second equation. Therefore, I will get a P 1 1 dash of t is equal to minus lambda plus mu times P 1 1 of t plus mu substituting P 1 0 of t is equal to 1 minus P 1 1 of t in the second equation, in the previous slide.

Now, i have to solve these differential equation. The unknown is the P 1 1 of t conditional probability at use initial condition P 1 1 of 0 is equal to 1, using that I can get, I will get a P 1 1 of t is equal to mu divided by lambda plus mu plus some constant e power minus lambda plus mu times t. That constant I can find out using this initial condition therefore, k is equal to lambda divided by lambda plus mu. So, the P 1 1 of t is equal to substituting k is equal to lambda divided by lambda plus mu in this equation either the p 1 1 of t.

Once I know the P 1 1 of t, use the first equation. So, I will get P 1 0 of t is equal to 1 minus P 1 1 of t. Therefore, P 1 0 of t that is equal to this expression, you can cross check now if you add both the equations you will get a 1 and if you put the t equal to 0 you will get the initial condition also correctly and if you put t tends to infinity that we are going to discuss in the limiting distribution, if you put t tends to infinity in this expression you will get a mu divided by lambda plus mu lambda divided by lambda plus mu. So, this is for the t tends to infinity.

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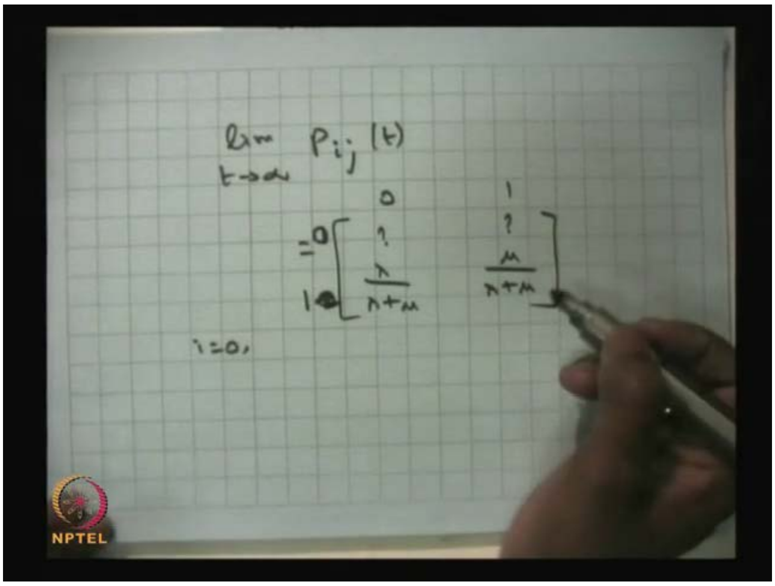

Therefore, if you make a matrix the limit n tends to infinity of limit, if you find out the limiting distribution of limit t tends to infinity of P i j of t. So, you will get the matrix and the this matrix has t tends to infinity for this example it is a two cross two matrix and that consists of, for different values you will have a, for, now we are doing for the second row therefore, that is equal to lambda divided by lambda plus mu sorry and this is equal to mu divided by lambda plus mu.

So, if the system start from the state 1 at t tends to infinity, the system will be in the state 0 with the probability lambda divided by lambda plus mu and the system will be in the state 1 with the probability mu divided lambda plus mu. Similarly, if you go for i is equal to 0 you will get the same derivation and you can fill up what is the element here.

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$$\begin{bmatrix} \dfrac{\lambda}{\lambda+\mu} & \dfrac{\mu}{\lambda+\mu} \\ \dfrac{\lambda}{\lambda+\mu} & \dfrac{\mu}{\lambda+\mu} \end{bmatrix}$$

So, this is the limiting distribution probability matrix and if you see that the rows are going to be identical. So, you will have a, the same identical rows in this row also, that means you will get the limiting distribution.

I will discuss the limiting distribution in the, after giving the one more example I will explain in detail. So, this is the transition probability system starting from the state 1 and being in the state 1 or 0 at a time t.

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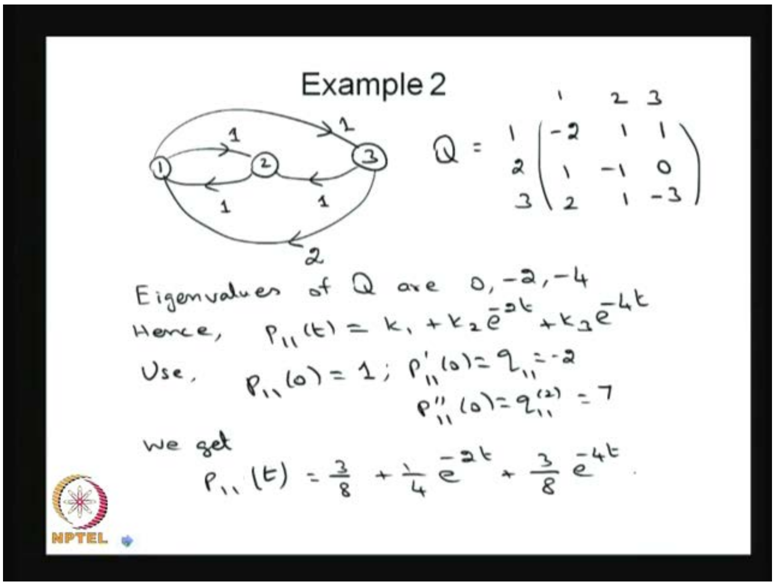

I am going to give one more example. This has a three states and this is the state transition diagram and the values are nothing but the rates in which the system is moving from one state to other states. So, that is the difference between the state transition diagram of a D T M C and the C T M C. So, this is the rate in which the system is moving from one state to another state and some or sorry not the, that means there is no way the system is moving from the state 2 to 3 small interval of time.

Whereas a all the other possibilities are I have given. So, the corresponding Q matrix it is a three cross three matrix and you can make out all the row sums are going to be 0 and the diagonal elements are minus of sum of other values and the same rows and the other than the diagonal elements are the values are greater than or equal to 0. My interest is to find out the time dependence solution for a, this example also. I can make a forward Kolmogorov equation p dash of t is equal to p of t times q.

So, three cross three matrix therefore, I will have three equations and I have a one equation that can have a summation of probability is equal to 1 and I can start with the initial condition the system being in the state 1 at time 0 that probability is 1. I can start with that and I can solve those three equations with the initial condition and i can get the solution that is a one way. Since it is a finite state C T M C there are many ways to get the time dependence solution.

Basically, you have to solve the system of difference differential equations with the initial condition. Here I am using the Eigen value method that means find the Eigen values for the Q matrix therefore, use Eigen value and the Eigenvector concept and get the P 1 1 of t with the unknowns k 1 k 2 k 3 and to find the unknowns k 1 k 2 k 3 use the initial condition. Here I am using the initial condition as well as the Q matrix values, the Q 1 1 that means the element corresponding to the 1 comma 1 that is nothing but the P dash of 1 comma 1 of 0. Similarly, if I go for Q square matrix and Q 1 1 of 2, the element in the 1 comma 1 in the Q square matrix that is nothing but P double dash of 1 comma 1 0. Therefore, now I can use these three initial conditions to get the unknown value k 1 and k 2 k 3, k 1 comma k 2 and k 3.

So, once I know the k 1 k 2 k 3 I can substitute therefore, the P 1 1 of t is equal to this much. Similarly, I can go for finding the P 1 comma 2 of t and P 1 comma 3 of t, I do not want P 1 comma 3 in the same way because once I know the P 1 comma 1 comma t and

P 1 comma 2 comma t. So, 1 comma 3 comma t is nothing but 1 minus of that two state probabilities because summation of probability is equal to 1. So, this is the other way of getting the time dependence solution, the transition probability of system being in the state j given that it was in the state i at time 0.

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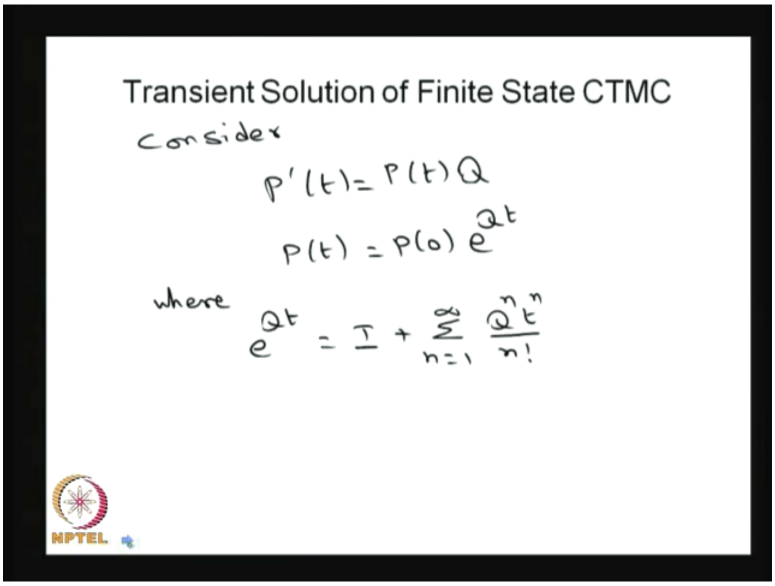

Suppose, the C T M C has the finite state space then I can use the exponential matrix also to get the time dependence solution that is what I have given this way. So, start with the forward equation. Therefore, the solution is going to be P of t is equal to P of 0 e power Q of t, e of t is the matrix, P of 0 is the matrix, e power Q t that is also again going to be a matrix, exponential matrix. Therefore, I am writing e power Q t is nothing but Q is a matrix and the t is the real value.

So, if greater than or equal to 0 therefore, e power Q t is going to be the i matrix, i matrix is nothing but the identical matrix of order whatever the state space number plus the summation i is equal to, n is equal to 1 to infinity of Q power n times t power n divided by n factorial. So, that the whole thing is going to be the exponential matrix and using that you can get the P of t. That is a, I am not going detailed for how to compute this e power Q t and so on. But whenever you have C T M C with the finite space through this method also one can get the time dependence solution. So, with this I have completed the examples for the C T M C to find out the time dependent or transition probabilities.

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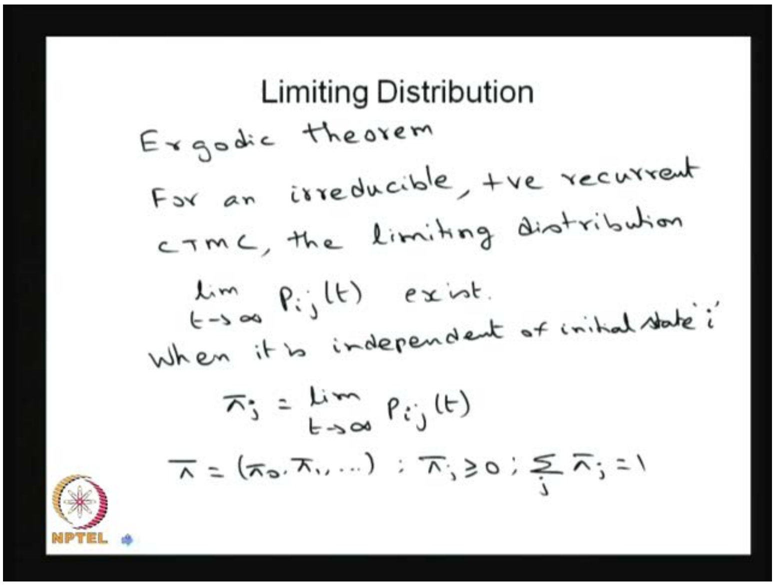

Now I am moving into the limiting distribution. The way we discussed the limiting distribution for the C T M C the same concept can be used for the C T M C also. The change is a instead of the one step transition probability matrix, here we have to use the infinite decimal generator matrix in a different way.

So, I am first giving the Ergodic theorem. Whenever the C T M C is a reducible that means all the states are communicating with all other states. Since, all the states are communicating with all other states if one is of the particular type, it is a positive recurrent then all the other states are going to be a positive recurrent. If one is going to be a null recurrent then all the other states are also going to be a null recurrent.

So, here I am making the assumption the C T M C's irreducible as well as all the states are positive recurrent. Then the limiting distribution always exists. Suppose, it is independent of initial state, it need not be a independent of initial state. Suppose the same thing is independent of initial state then I can write that limiting probabilities P i j of t since it is independent of i I can write it has i j, then I can form a vector and a since it is a limiting distribution, it is a probability distribution therefore the probabilities are, this probabilities are always greater than or equal to 0 and the summation of probability is going to be 1. It would not be defective, it would not be less than 1. That is the Ergodic theorem says whenever you have a irreducible C T M C with all the states are positive recurrent then as t tends to infinity the system has the distribution, limiting distribution.

If it is independent of initial state then you can label with the pi j has a probabilities and this probability distribution satisfies, it is a probability mass function f it satisfies the probability mass function conditions.

That means whenever you have a dynamical system in which it is a irreducible model and all the states are positive recurrent that means the mean recurrence time is going to be a finite value then that system is call it is a Ergodic system or the Ergodic theorem concept can be used therefore, as t tends to infinity you can get the limiting distribution.

If it is independent of initial state means whatever be the see it you are going to do it for the discrete event simulation for the dynamical system, that is Ergodic's, for a Ergodic system then the initial conditions see it does not matter to get the empty distribution. Later we are going to give some few examples how to find out the limiting distribution.

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Stationary Distribution

A vector π is called the stationary distribution of the CTMC if $\pi = (\pi_0, \pi_1, \ldots)$ satisfies:

(i) $\pi_j \geq 0, \forall j$

(ii) $\sum_j \pi_j = 1$

(iii) $\pi Q = 0$

I am explaining the stationary distribution also. The stationary distribution the way I have explained the D T M C sorry in the way I have discussed the D T M C, the C T M C also same. So, I have a vector, if the vector satisfies these three conditions, probabilities therefore, greater than or equal to 0, summation is equal to 1 and you should able to solve this equation and get the pi's. So, homogeneous equation so you need a second condition to have the non 0 probabilities. So, if you solve pi q is equal to 0 along with the summation of pi j is equal to 1 and if this pi j's exist then the C T M C has the stationary distribution.

The similar way I have discussed the stationary distribution for the D T M C model also instead of pi Q is equal to 0 we had a pi P is equal to pi. So, if any vector satisfies that pi P is equal to pi and summation of pi i is equal to 1 and the all the pi's are greater than or equal to 0.

Then that is going to be a stationary distribution for D T M C, the same way if pi Q is equal to 0 and pi summation of pi j is equal to 1, pi j's are greater than or equal to 0. If this is satisfied by any vector, then that is going to be the stationary distribution for a time homogeneous C T M C.

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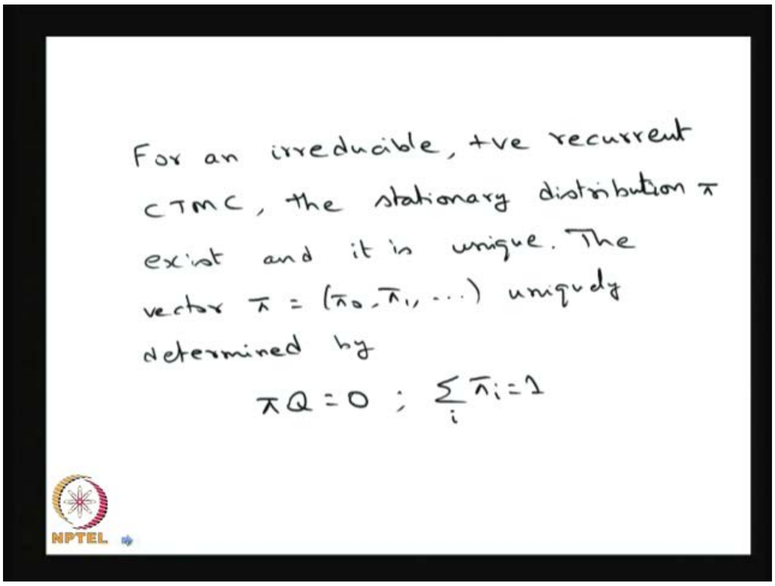

Every time we are discussing the default C T M C that is the time homogeneous C T M C. The main result for the stationary distribution, whenever you have a irreducible positive recurrent C T M C, the stationary distribution exists and that is going to be unique whenever the C T M C is a positive recurrent as well as irreducible.

There is no need of periodicity in the C T M C whereas, the same as stationary distribution the stationary distribution for the D T M C we have included one more condition that is a periodic, but for the C T M C there is no periodicity for the state. Therefore, as long as the system has, system is a irreducible and a positive recurrent 1 then the stationary distribution exists and it is unique and by solving these equations you can get the unique stationary distribution.

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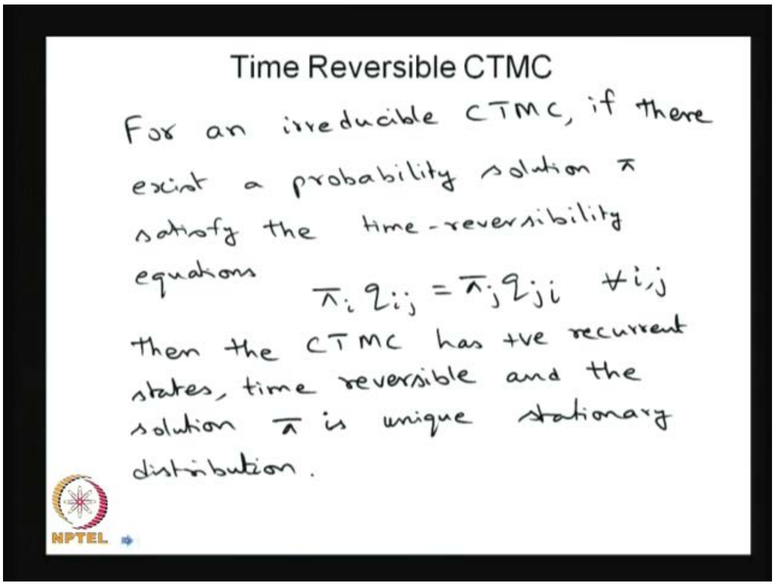

The way I would have explained the time reversible concept in the D T M C, the C T M C also has the time reversible concept. So, the time reversibility equation is a pi i is equal to pi i times q i j is equal to pi j times q j i. The q’s are nothing but the rates and the pi’s are nothing but the probability values. So, if pi is, pi i’s exist, if pi i’s exist this stationary probability or stationary distribution exist then if the stationary distribution exist as well as the time reversibility is satisfied by C T M C, then that C T M C is a positive recurrent and you can say that it is a time reversible and the solution pi can be pi is nothing but the stationary distribution. So, this result says for a irreducible C T M C if there exist a probability solution pi, satisfy the time reversibility equation, this is the time reversibility equation where q’s are rates, pi’s are the probability solution.

If it is satisfied by the irreducible C T M C the time reversible equation then that C T M C has a positive recurrent states and that C T M C is called a time reversible as well as the pi is called the stationary distribution. So, initially we have not taken has a stationary distribution, some probability solution satisfies the time reversibility equations and it is the irreducible C T M C, then that C T M C has a positive recurrent states and pi is nothing but the unique stationary distribution.

So the usage this concept is whenever any C T M C is, first it is a irreducible and satisfies the time reversibility equation of this form then you do not want to solve pi q is equal to 0 and summation of pi i is equal to 1 to get the stationary distribution. Instead of that use this time irreversibility equation instead of solving pi q is equal to 0 and then use the summation of pi i is equal to 1 to get the one unknown. That means use a time reversibility equation repeatedly, recursively and get all these in terms of one unknown either pi naught or pi 1 whatever it is, then use the summation of pi i is equal to 1 to get to find the that unknown instead of solving pi q is equal to 0 .

So, whenever it is model is irreducible and the time reversibility equations are satisfied then you can conclude all the states are positive recurrent and you can find pi the stationary distribution in easy way instead of solving pi q is equal to 0.

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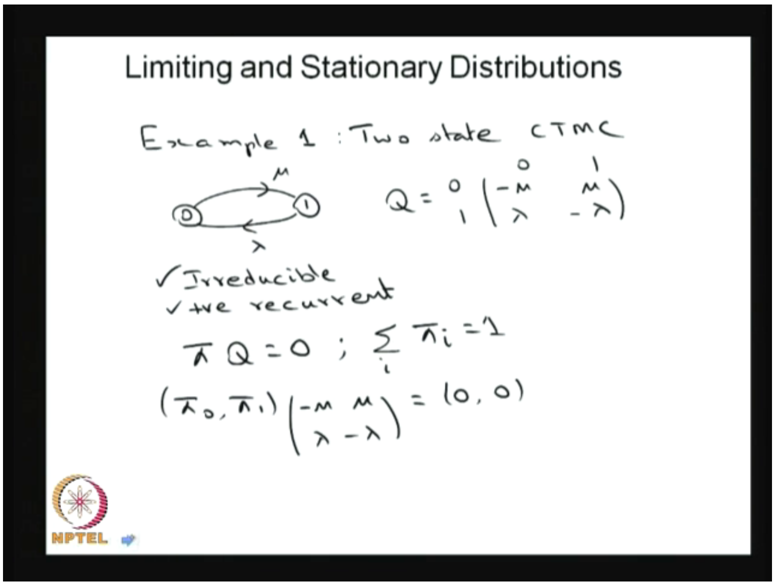

I am going to give a one simple example what is the limiting and stationary distribution. Take the two state C T M C and we know that Q matrix and you can verify whether this is going to be irreducible and the positive recurrent. Since it is a finite state model and both the states are communicating each other. Therefore, it is a irreducible positive recurrent states. So, you can solve pi Q is equal to 0 and the summation of pi i is equal to 1. So, pi times Q pi is the vector, Q is the matrix and again 0.

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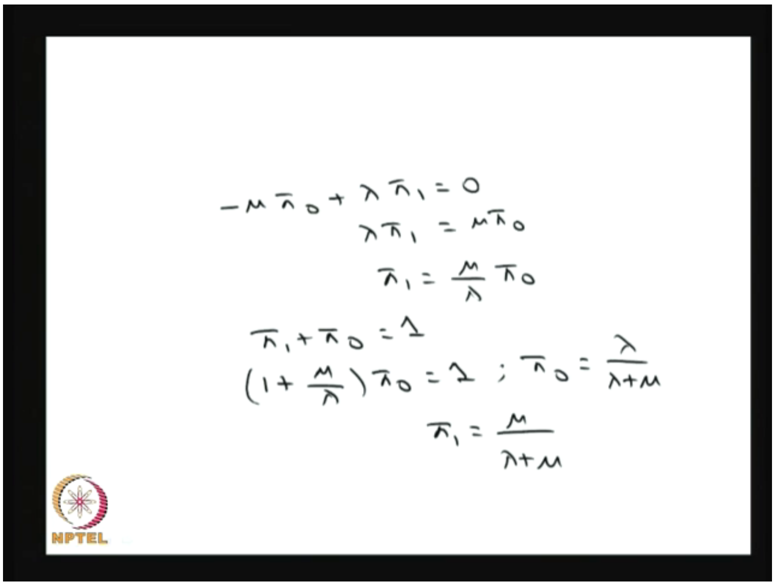

Therefore, if if I take the first equation I will get minus mu times pi naught plus pi lambda times pi 1 is equal to 0 by taking the first equation minus mu pi naught plus lambda times pi 1 that is equal to 0. From this I can get the pi 1 in terms of pi naught since it is a homogeneous equation I have to use a one unknown homogeneous or normalizing condition, summation of pi i is equal to 1. So, using that I will get pi naught is equal to lambda divided by lambda plus mu. Once, I know pi naught then pi 1 is equal to mu divided by lambda plus mu. So, this is the stationary distribution as well as the limiting distribution because it satisfies both the conditions.

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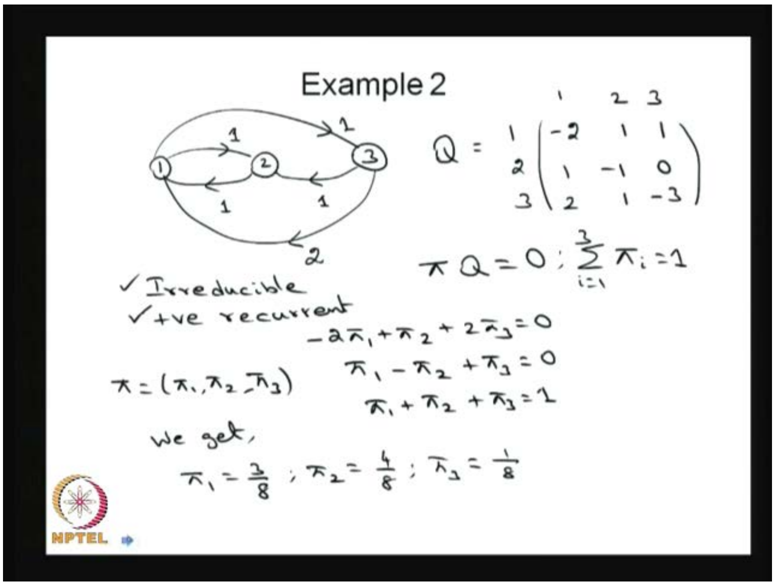

Take the second example. Second example also finite state model, all the states are communicating with all other states. Therefore, it is a irreducible. Since, it is a finite state model you would not have a null recurrent. The, it is a positive recurrent model. So, I can solve pi Q is equal to 0 and the summation of pi i is equal to 1. So, there are three equations. So, I take the first two equation and one normalizing equation and solve these three equation I can get pi 1 pi 2 pi 3.

You can verify that the summation is going to be 1. So, this is the limiting distribution as well as the stationary distribution because the model is the irreducible positive recurrent model. So, this limiting distribution and the stationary distributions are one and the same. Instead of solving pi Q is equal to 0. You can use the time reversibility before that you should verify whether the time reversibility equations are satisfied by this model. If this model satisfies the time reversibility equation for all i comma j then you can conclude, it is a time reversible Markov chain and so on but a example 1 is the time reversible Markov Chain where as the example 2 is not a time reversible Markov chain you can verify it.

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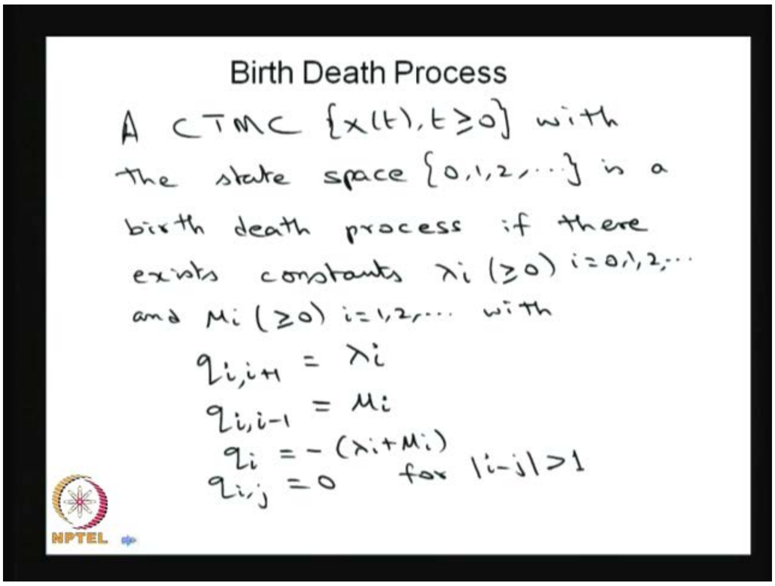

Now, I am moving into the special case of Continuous-time Markov Chain that is a birth-death process. This is the very important time homogeneous Continuous-time Markov Chain because many of the scenario can be mapped with the birth-death process, either with the finite state or infinite state.

Let me first give the definition of birth-death process. I started with the Continuous-time Markov Chain that is the time homogeneous Continuous-time Markov Chain with the state space countably infinite, it can be a finite also and that C T M C is going to be, call it as a birth-death process. If there exists a constants lambda i's and mu i's such that in these are all nothing but the infinite decimal generator matrix elements and this is i to i plus 1 that rate is always lambda i. And the rate in which the system is moving from the state i to i minus 1, that rate is a mu i and the diagonal elements are minus of lambda i plus mu i.

Whereas all the other rate rates the system is moving from the state i to j other than i to i plus 1, i to i minus 1 and i to i and all other rates are is always 0, absolute of i minus j is greater than 1. That means you will have the infinite decimal generator matrix in which you will have only, have a diagonal matrix, tri diagonal matrix and all other elements are going to be 0.

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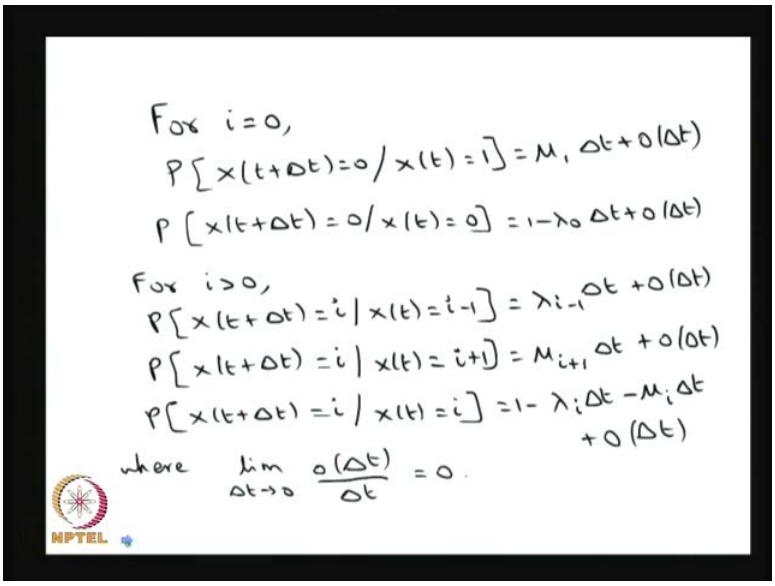

I can write down the condition so that it land up the rates are going to be only lambda i's and mu i's, so 1, not all other rates are going to be 0. So, if I start with i is equal to 0 the system is moving from the state 1 to 0 in the interval of delta t because it is a time homogeneous model. So, this nothing but this probability this is moving from the state 1 to 0 in the interval of delta t. That is nothing but the rate is a mu 1 times delta t plus order of delta t. Similarly, the system is moving from the state 0 to 0 from the time t to t plus delta t or during the interval delta t that is nothing but 1 minus lambda naught times delta t plus order of delta t.

So, this mu i's and the lambda naught and so on, this values are always going to be a greater than or equal to 0, strictly greater than 0 also. For i is greater than 0 the system is moving from the state i to i that is 1 minus lambda i times delta t minus mu i times delta t plus order of delta t. Whereas a system is moving from i plus 1 to I, one step backward that is mu times i plus 1 delta t. The system is moving from the state i minus 1 to i for i is greater than 0 that is forward one step move that is lambda times i minus 1 delta t plus order of delta t.

These order of delta t it may be a function of delta t, it need not be the same. As a t tends to, as a delta t tends to 0 this quantity is a, are going to be 0, order of delta t divided by delta t is going to be 0. Therefore, this is the way the system is moving from the one state to either one step forward or either one step backward or move anywhere. So, these are all the only three possibilities with these probabilities.

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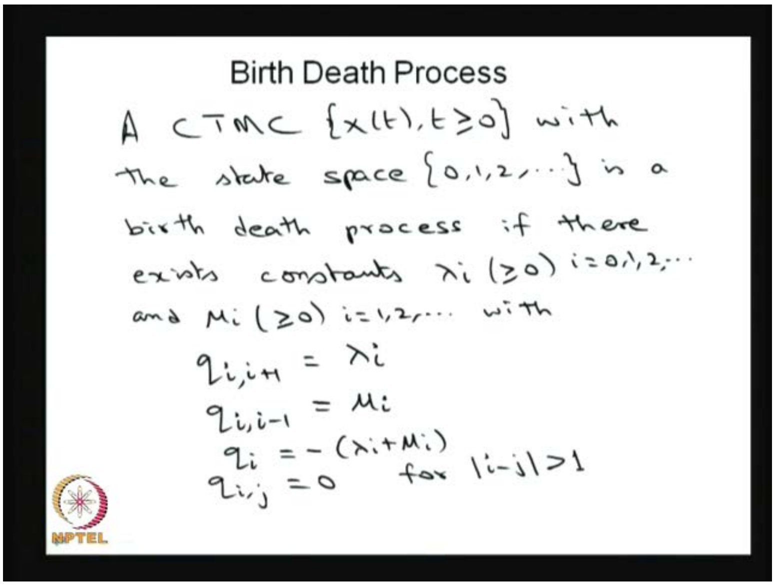

Therefore, we land up the q matrix is going to be the system is moving from the state i to i plus 1 forward one move, that rate is lambda i's and the system is moving from the i to i minus 1 one step backward that is mu i or the system being in the same state, that rate is minus lambda i plus mu i. Therefore, there is no other move from the system from one state to all other states, either one step forward or one step backward.

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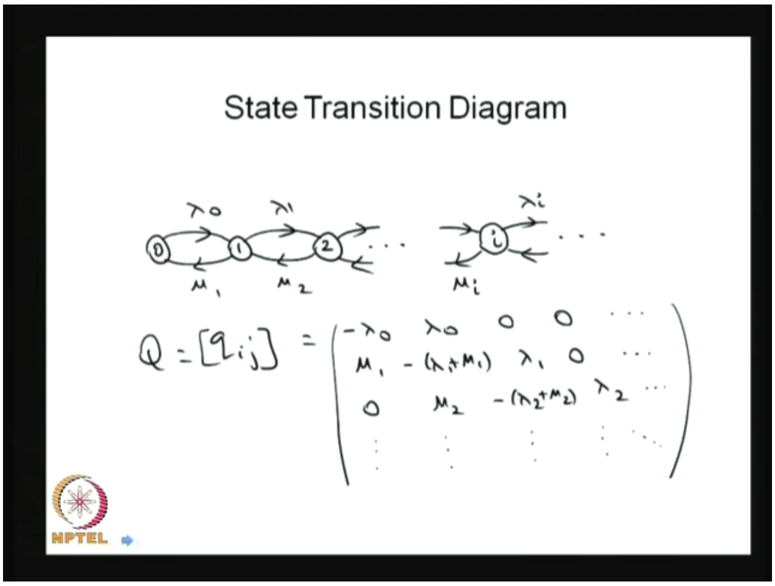

So, this can be visualized in the state transition diagram. Since, I started with the state space 0 to infinity, there is a possibility you can have a label from some negative integers to the positive integers, so you can always transform into something therefore, default scenario or the simplest one I discussed from 0 to infinity. Therefore, you can visualize whatever be the label that can be transfer, you know one to one fashion. So, this the rate in which the system is moving from the state 0 to 1, that rate is lambda naught.

The system is moving from the state 1 to 2, that rate is lambda 1 or the system is moving from the state 1 to 0 that rate is mu 1. Therefore, the time spent in the state 1 before moving into any other states that is a minimum of the time spending in the state 1 before moving into the state 2 or the system time spending in the state 1 before moving into the state 0. So, both are exponentially distributed with the parameters lambda 1 and mu 1 and the minimum of that time is the spending time or the waiting time in the state 1 that is going to be exponential distribution with the parameter lambda 1 plus mu 1 because both are independent, the time spending in the state 1 before moving into the state 2 and similarly, the time spending in the state 1 before moving into the state 0 and both the random variables are independent, that is a assumption. Therefore, it is going to be a exponentially distributed random, the time spending in the state 1 that is exponential distributed with the parameter lambda 1 plus mu 1.

Like that you can discuss for all other states. So, whenever you have the birth-death process the system either move one step forward or one step backward. Then it is called a birth-death process. Therefore, here this is lambda i's are called the system is moving from one state to pass forward one step therefore, this lambda is are called birth rates. The system is moving from one state to the previous one state and the corresponding rates mu i's mu 1 mu 2 mu 3 and so on and these rates are going to be call it as a death rates.

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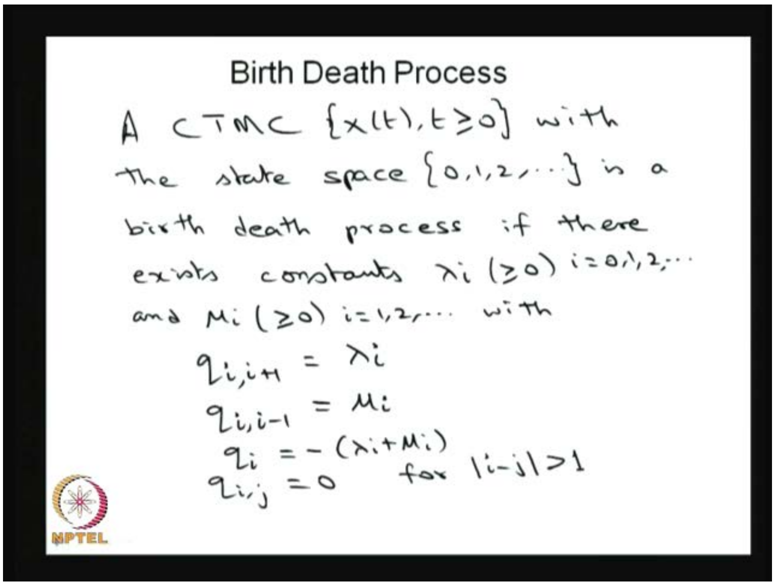

So, lambda i's are nothing but the lambda i's are nothing but the birth rates that means the rate in which the system is moving from the state i to i plus 1 that depends on i therefore, that rate is lambda i. The system is moving from the state i to i minus 1 that is related to the death by 1, that is a function of i therefore, the death rate is mu i. So, the lambda i's are the birth rates and the mu i's are the death rates.

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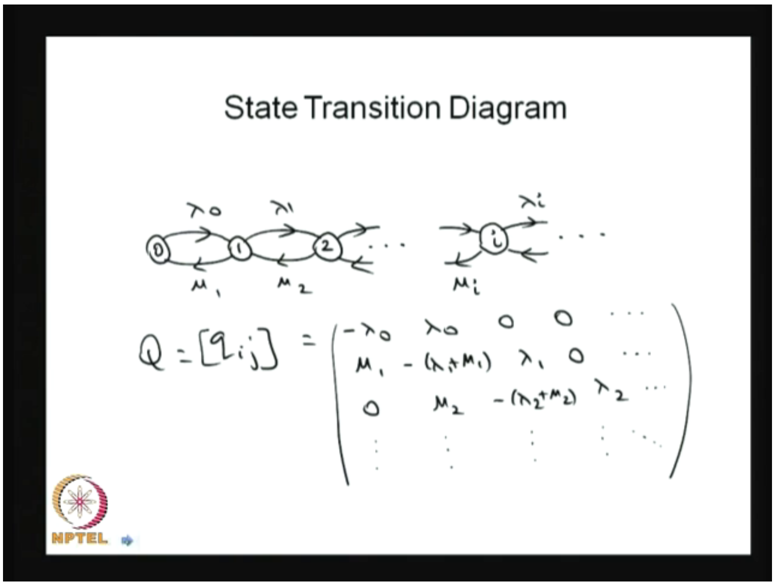

Therefore this is suppose example the system moving from the state 2 to 1 the death rate will be mu suffix 2. So, you can fill up the Q matrix; if you see the Q matrix which a tri diagonal matrix.

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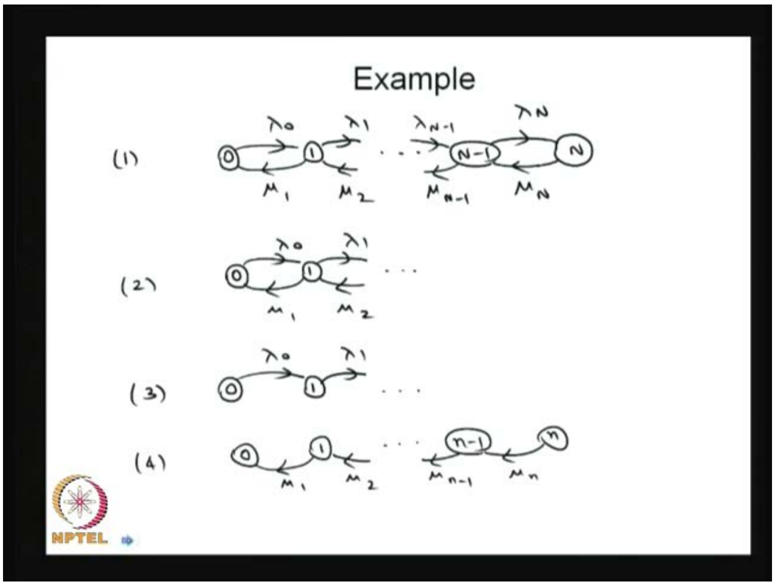

So, here I am giving few examples for the birth-death process. In the first example it consists of, the first example is a finite state model. The birth rates are lambda naught lambda 1, 3 lambda n minus 1.

The death rates are mu 1 mu 2 and mu n. It is a finite state birth-death process. The second example is the infinite state birth-death process, the third example the all the death rates are 0, that is also possible. The fourth example, all the birth rates are 0 that is also possible, but you can discuss the, one can discuss the straight classification also. The first one, all the, it is a finite state model, all the states are communicating with all other states. Therefore it is a irreducible positive recurrent birth-death process. The second one is the infinite state, all the states are communicating with all other states. It is irreducible, but one cannot conclude without knowing the values about the lambda naughts and lambda i’s and mu’s i’s, one cannot conclude it is a positive recurrent or null recurrent, if the mean recurrence time that is going to be a finite one then we can conclude it is a positive recurrent otherwise it is null recurrent.

So, as you choose one cannot discuss now the positive recurrent or null recurrent, but you can conclude it is a recurrent state. The third example, the system is keep moving forward. Therefore, all the states are transient states. It is not a irreducible, it is a reducible model, all the states are transient states. That means as the t tends to infinity, the system will be in the, some infinite states. So, one cannot define a infinite state therefore, the limiting distribution would not exist in this situation.

The fourth example it is a finite model, but all the states are not communicating with the all other states therefore, it is a not a irreducible, it is a reducible model. Whenever the system starts from some state other than 0 over the time the system is keep moving backward and once it reaches the state 0 it will be forever. Therefore, the state 0 is a observing barrier, state 0 is the observing state and all other states 1 to n, those states are the transient states. The limiting distribution exist. The system will be in the state 0 at t tends to infinity with the probability 1 and the all other states are transient state therefore, the probabilities are 0.

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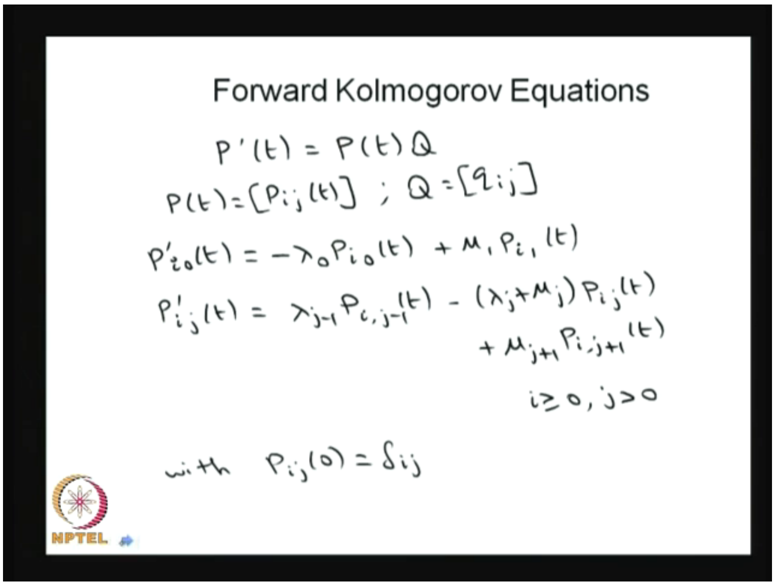

We are discussing the forward Kolmogorov equation for a special case of Continuous-time Markov Chain that is a birth-death process or a birth-death process the Q matrix is a tri diagonal matrix. Therefore, you will have a, in the equations from the forward Kolmogorov equation you will have a only two terms in the right hand side for the first equation and you will have only three terms the diagonal element and to of diagonal elements and therefore, the second equation one can, the first equation one can discuss first the P dash of i comma 0 that is nothing but the system is not moved from the state 0, moving from the state 0 that rate is a lambda naught.

Therefore not moving minus lambda naught times the probability and (( )) or the system can come from the state 1 with the rate mu 1. Therefore, mu 1 times P i comma 1 of t. For all other equations either the system comes from the previous state with the rate lambda j minus 1 or it comes from the forward 1 state with the rate mu j plus 1 or not moving anywhere. So, these are all the, all possibilities therefore, with these three possibilities you have a three terms in the right hand side and that is the net rate for any state to j. So, if you solve this equation with these initial condition Kolmogorov delta i comma j you will have the solution of P i comma j.

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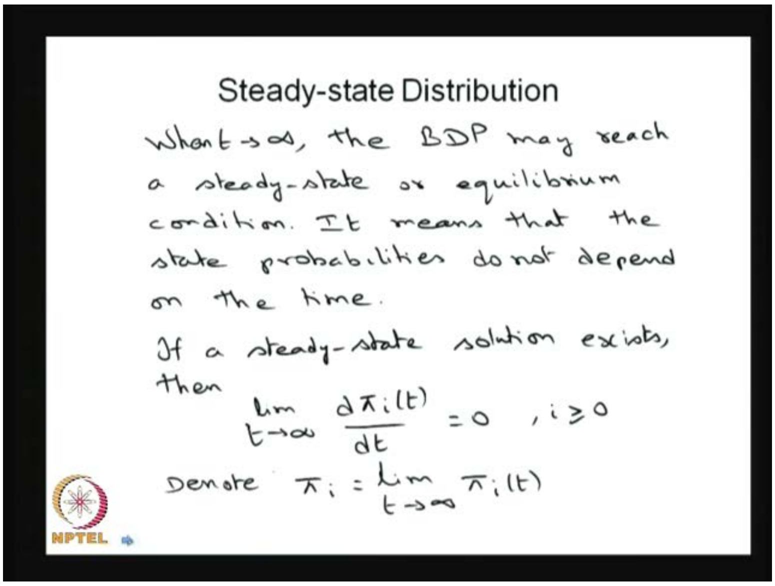

Here I am discussing the steady state distribution. The way I have discussed the limiting distribution that is the limit t tends to infinity, probability of i comma j of t insist then it is called a limiting distribution and the stationary distribution is nothing but for the D T M C it is a pi p is equal to p, summation of pi i is equal to 1, for the C T M C it is pi q is equal to 0 and the summation of pi i is equal to 1. That is going to be a steady state distribution, stationary distribution.

Now, I am discussing the steady state distribution that is nothing but when t tends to infinity the birth-death process may reach steady state or equilibrium condition. That means the state probabilities does not depend on time. That is a minimum of steady state distribution as a t tends to infinity, whenever we say the birth-death process reaches a steady state or equilibrium that state probability does not depend on time. That means if a steady state solution exist since the time depend, since the state probabilities does not depend on time t the derivative of the time dependence state probability at time t, that derivative at t tends to infinity becomes 0, if the steady state solution exist. Since the state probabilities does not depend on time t as a t tends to infinity I can write as a pi i's a limit t tends to infinity of pi i of t.

So, this is different from the way we discuss earlier that conditional probability p i j of t, but using p i j of t one can find out what is pi i j, pi i of t that is nothing but the pi i of t that I have given in the first lecture for the C T M C.

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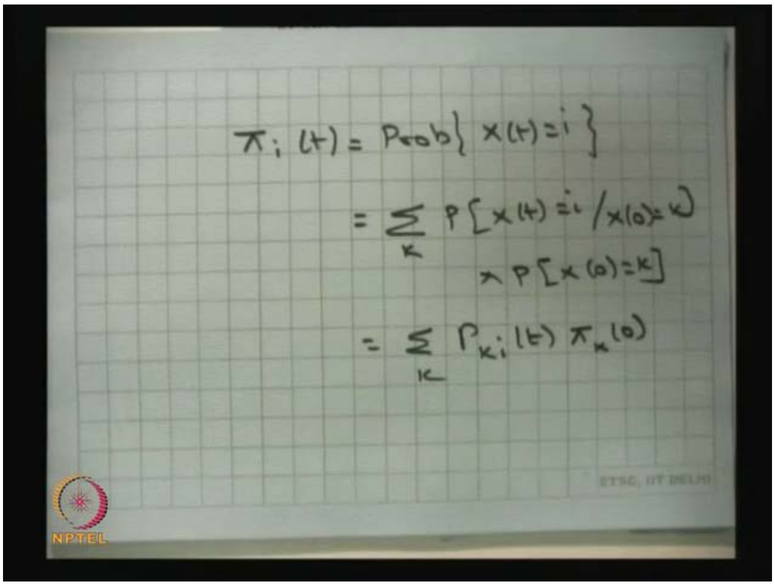

The pi i of t that is nothing but what is the probability that the system will be in the state i at time t. That is same as what is the probability that the system will be in the state i given that it was in the state some k at time 0 multiplied by what is the probability that it was in the state k at times.

That is nothing but summation of k and this is nothing but the transition probability and this is nothing but the initial probability at the limit. So, using P k i of t or P i j of t that is the conditional probability one can get the unconditional probability, this is the nothing but the distribution of x of t. So, this is the probability mass function, probability mass at state i. Now, what I am defining whenever the steady state distribution exist that means it is independent of time t. Therefore, as t tends to infinity the pi i of t can be written as the pi i. And whenever the steady state solution exist I can use the limit t tends to infinity, the derivative of pi i of t that is going to be 0.

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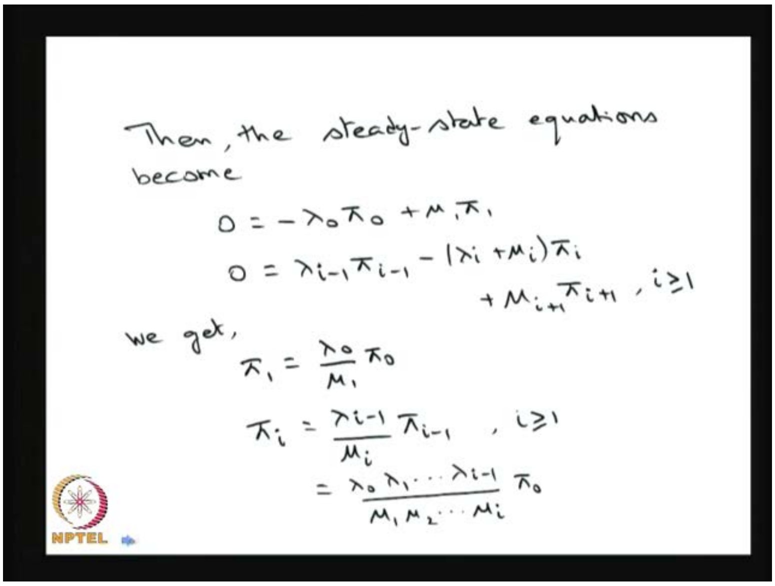

Therefore, I am going to use these two to get the steady state probabilities for the birth-death process. Since, as t tends to infinity the derivative of pi i j of t is equal to 0 therefore, the all the left hand side in the forward Kolmogorov equation that is going to be 0, the right hand side you will have a as a t tends to infinity the pi i of t that can be written as the pi 0 and pi 1. So, the way we write the conditional probability for p i j in the Kolmogorov forward equation you can write the similar equation for the unconditional probability pi i's also.

Now, I am putting the left hand side zero's because of these condition limit t tends to infinity the derivative is equal to 0 and the right hand side I am using as a t tends to infinity this probabilities is nothing but the pi i's therefore, it is going to be minus lambda naught times pi naught plus mu 1 times pi 1 and all other equation has a three terms in this homogeneous equation and you need a one normalizing condition.

So, from this homogenous equation I can get recursively pi i's in terms of pi naught. So, from the first equation I can get a pi 1 in terms of pi naught and the second equation I can get pi 2 in terms of first pi 1, then I can get a pi 1 in terms of pi naught. Therefore, recursively I can get pi i's in terms of pi naught, for all i greater than or equal to 1.

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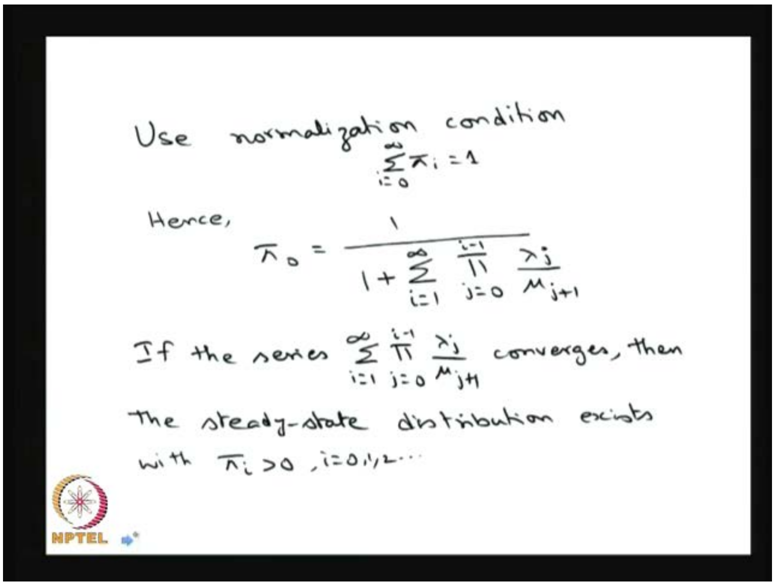

Now, I can use the normalizing condition summation of pi is equal to 1. Therefore, I will get a pi naught is equal to 1 divided by summation of this many terms in the product form. Since, we need a steady state probabilities and all the pi i's are in terms of pi naught as long as the denominator is converges you will have a pi 0 is greater than 0.

So, once pi 0 is greater than 0 then you will get all the pi i's with the summation of pi i is equal to 1. So, whenever these series converges then I will have a steady state distribution with the positive probability and summation of probability is going to be 1.

So, this is the condition for a steady state distribution for a birth-death process because we started with the birth- death process forward Kolmogorov equation using these two conditions we have simplified into this form and use the normalizing condition and get the pi naught as long as the summation is or the series is converges then you will have the steady state, if the series diverges that means by substituting the values for the lambda i's and mu i's and if the series denominator series diverges then the pi naught is going to be 0, in turn all the pi i's are is equal to 0 therefore, the steady state distribution will not exist if the denominator series diverges.

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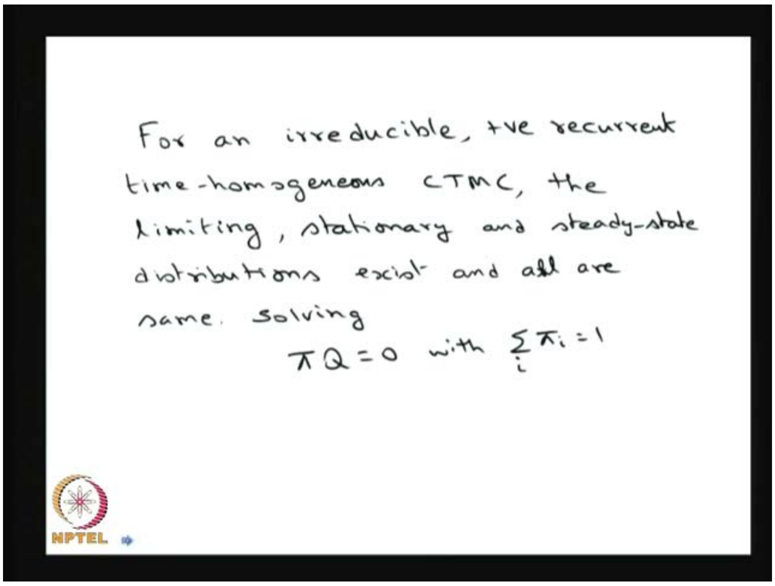

I am going to give a one simple result for a irreducible positive recurrent time homogeneous C T M C. We know that the limiting distribution exist, the stationary distribution exist. Now, I am including the steady state distribution also exist. I have given for the steady state distribution for the birth-death process, not for the C T M C, but here I am giving the result for the C T M C. All the three distributions exist and all are going to be same.

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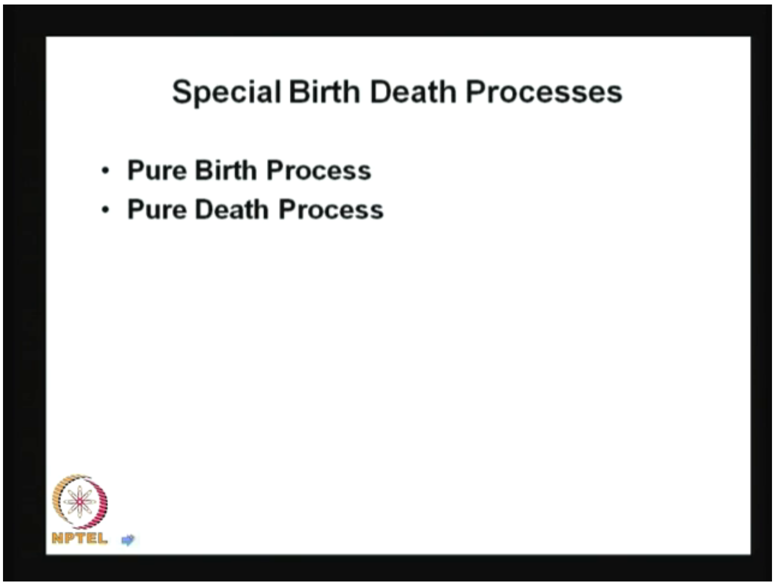

Whenever the C T M C is a time homogeneous irreducible positive recurrent all these three distribution are same and one can evaluate, one can solve this two equations pi Q is equal to 0 and if the summation of pi i is equal to 1 you can get the limiting distribution, stationary distribution or steady state or equilibrium distribution. As, a special cases of birth-death process I am going to discuss these two process in this lecture.

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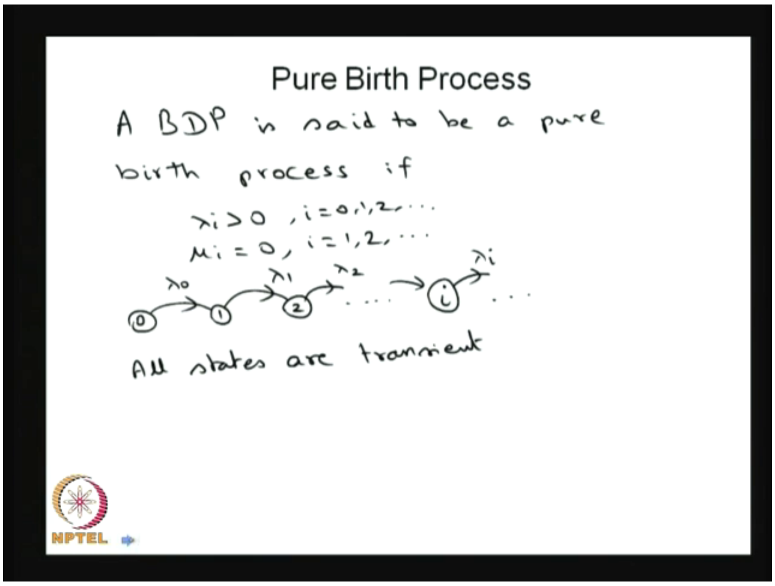

Whenever we say the birth-death process is a pure birth process that means all the death rates are going to be 0. We started with the birth-death process with the only lambda i’s are greater than 0 and the mu i’s are going to be 0, then it is going to be called as a pure birth process. There is a one special case of pure birth process with lambda i’s are going to be constant that is lambda, that is a Poisson process. That I am going to discuss in the next lecture and in this pure birth process this lambda i’s are the function of i. Here all the states are transient states.

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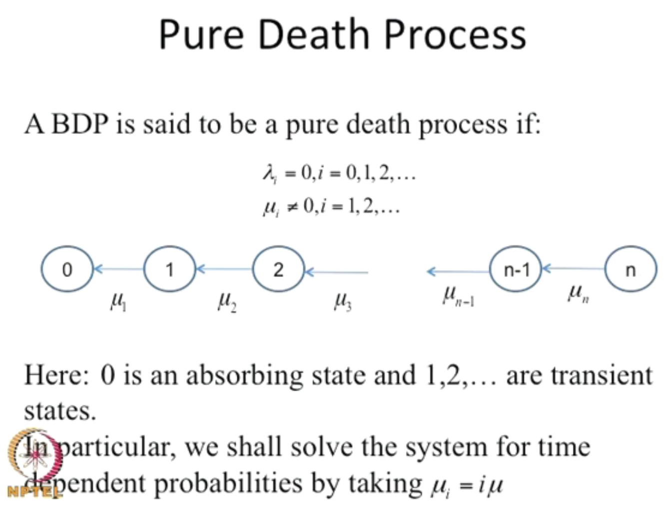

Here I am discussing the pure death process. A birth-death process is said to be a pure death process if the birth rates are 0 and the death rates are non 0. In particular we shall obtain the time dependent probabilities of a pure death process in which the death rates mu i’s are equal to i times mu. As a given the example as a fourth example in the birth death process, this state 0 is the observing barrier. Therefore, the state 0 is a observing state and all other states are going to be transient state and here the limiting distribution exist and one can also find the time dependent probabilities for this model.

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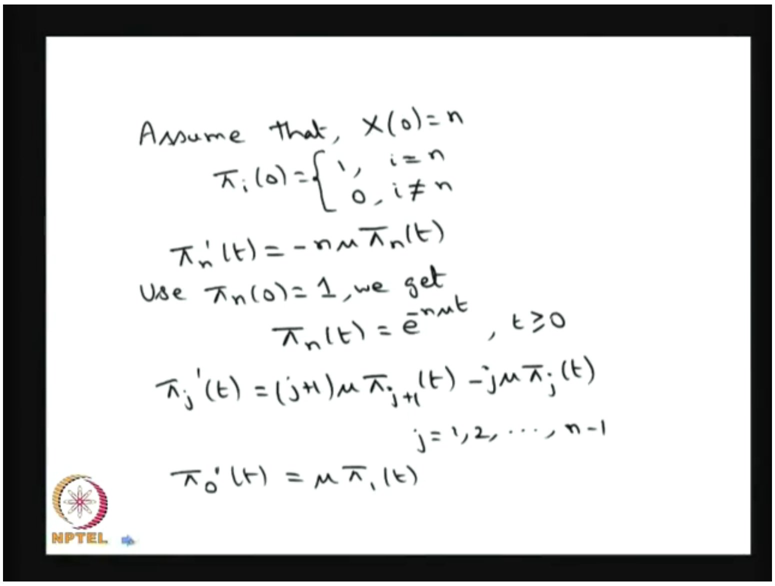

Suppose you start with the assumption the system at time 0 in the system is in the state n at time 0.

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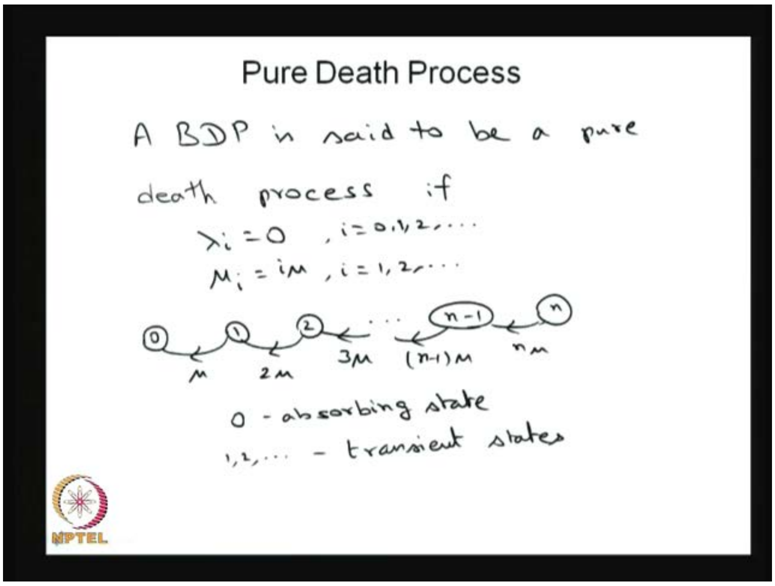

The systems in the state n at time 0 with that assumption I can frame the equation that is pi n dash of t is equal to minus n times mu of pi n of t. That means the rate in which the system is moving in the state n that is nothing but not moving to the state n minus 1 with the rate n minus n times mu. Therefore, the equation for the state n that is pi n dash of t that is equal to not moving from the state n therefore minus that outgoing rate that is n times mu being the state is n therefore, pi n of t. I can use the initial condition pi n of 0 is equal to 1.

So, I will get pi n of t. For the second equation I have to go for what is the equation for the state n minus 1. So, the pi n minus 1 dash t that is nothing but either the system come from the state n or not moving from the state n minus 1. Therefore, system coming from the state n that is a n mu times the system being in the state n minus n minus 1 times mu i n minus 1 of t. So, we will have a two terms in the right hand side coming from the one forward state or not moving from the same state. So, you will have two terms for j is equal to 1 to n minus 1. For the last state that is the state 0 the systems come from the state 1. Since, the state 0 is observing states there is no second term. So, it is going to be mu times pi 1 of t.

So, you know pi n of t, use the pi n of t in the equation for n minus 1 and get the pi n minus 1. Like that you will find out till pi 1, use the pi 1 to get the pi 0 of t.

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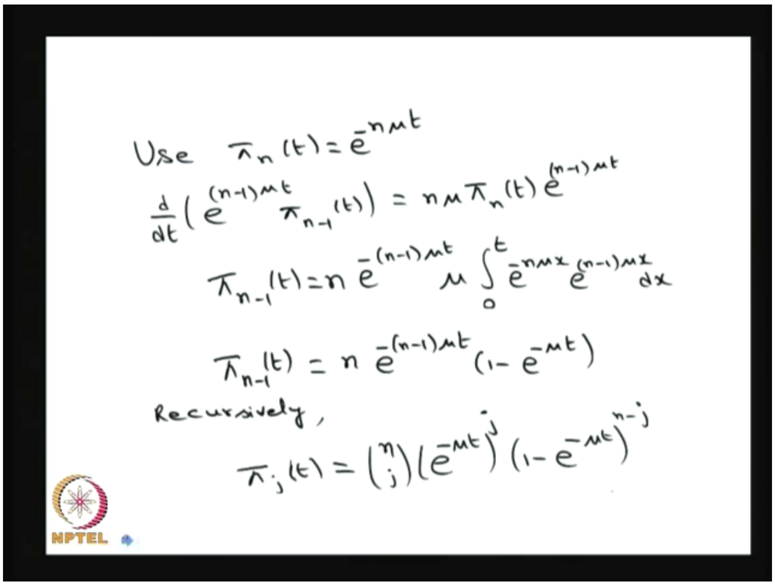

Use the recursive way. So, using the recursive way you will get the pi j of t is equal to n c j combination n c j and e power minus mu times t power j. This is survival probability of system being in the state and 1 minus e power minus mu of t n minus j. Suppose the system being in the state j that means from the state n this many combination would have come and the survival probability is e power minus mu times t, that power.

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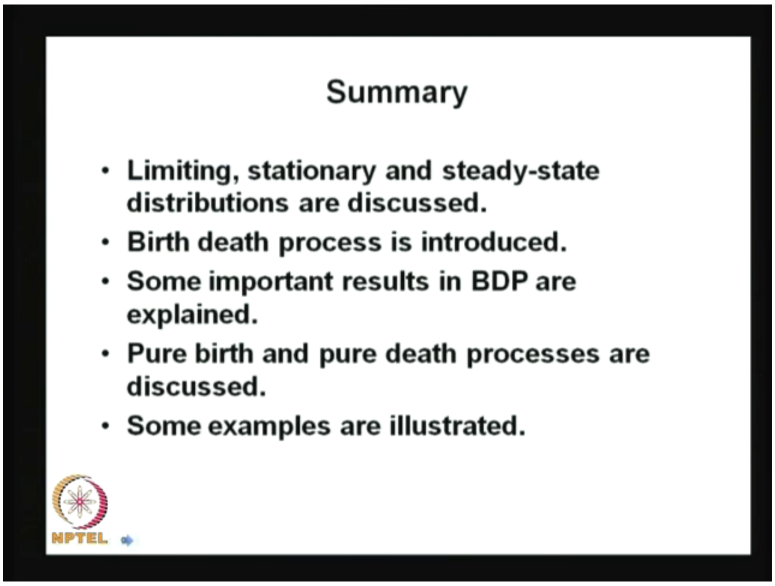

So, this is nothing but the probability p power j and 1 minus p power n minus j. Therefore, this pi j follows a binomial distribution with the survival probability e power minus mu t being in the state j.

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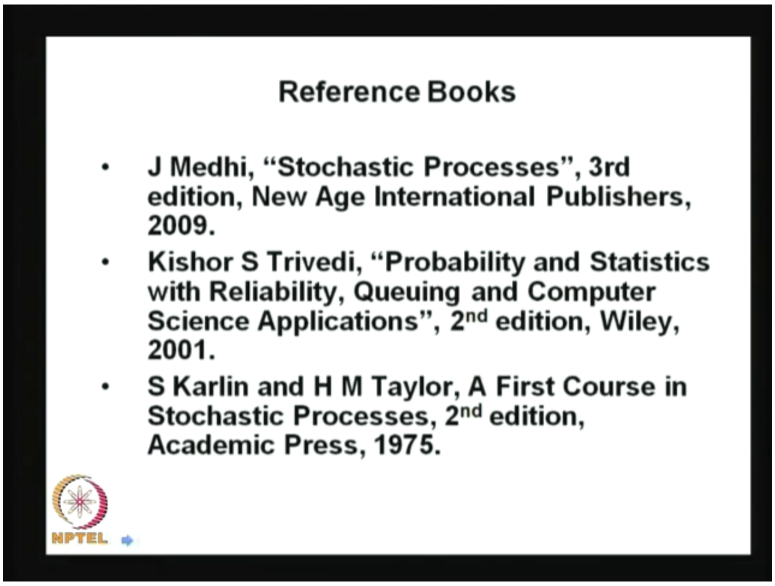

So for the pure death process, I have explained the time dependent probabilities of being in the state j that is unconditional probability. So, with this the summary of this lecture is I have discussed the limiting stationary and steady state distributions. I have introduced birth-death process, some important results also discussed; and at the end, I have discussed the pure birth and pure death processes also. In the next lecture, I am going to explain the important pure birth process that is the Poisson process, and these are all the reference books. Thanks.