Stochastic Processes
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Module - 4
Discrete-time Markov Chain
Lecture - 7
Reducible Markov Chains

Module 4 Discrete-time Markov Chain, lecture 7 Reducible Markov Chain. The last three lectures we have discussed irreducible Markov chain, and this lecture we are going to discuss reducible Markov chain.

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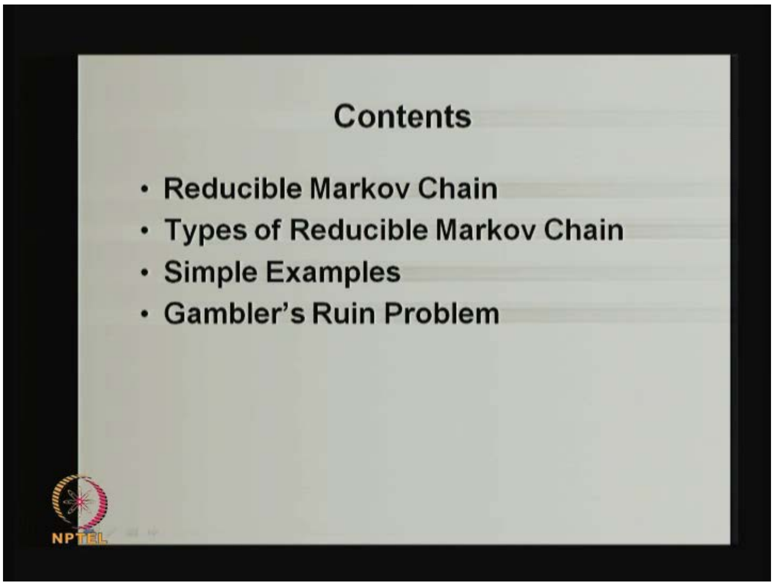

So, in this lecture I am going to start with the concept of reducible Markov chain, then I am going to give the different types of reducible Markov chain. And also I am going to present some simple examples, then finally one important application of reducible Markov chain; that is Gambler's ruin chain problem, Gambler's ruin problem going to be discussed.

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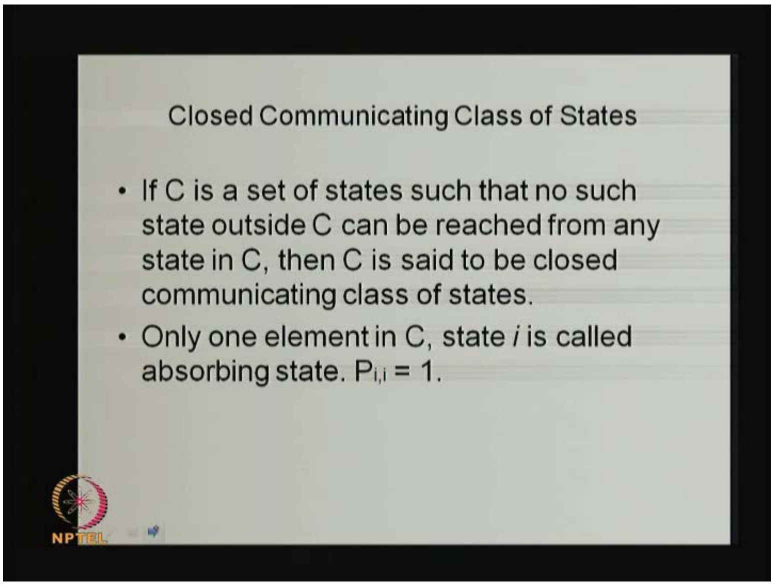

Before we discuss the reducible Markov chain, let me explain the closed communicating class of states, this is this definition we have already given in the few lectures earlier also, again I am giving. Using this, we are going to conclude the Markov chain is a reducible Markov chain or irreducible Markov chain. The closed communicating class, suppose you collect set of states that you name it label with the C.

If that collection of or set of states is going to be call it as a closed communicating class of states, if it satisfies no such state outside C can be reached from any state in C, then C is said to be closed communicating class of states. If in a set of states forming a closed communicating class, and it has only one element only one state, you cannot include one more state, so that it is going to be a closed communicating class of states, then that class is in that class the state is going to be call it as a absorbing state.

And you know the definition of absorbing state, that means the one step transition probability i to i that is equal to 1. So, there are two ways you can have a absorbing state, either p i i is equal to 1 or the closed communicating class has only one element, then that state is going to be a absorbing state. So, using a closed communicating class of states we are going to distinguish or we are going to make the reducible Markov chain and irreducible Markov chain.

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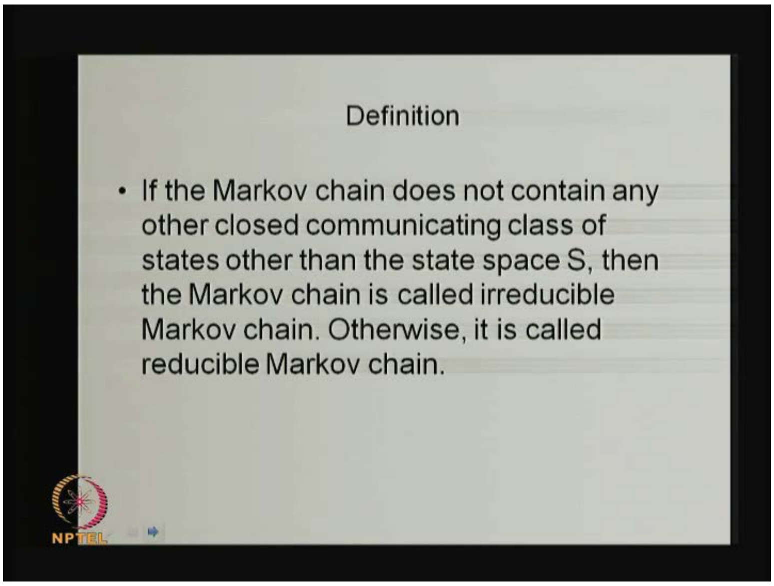

How, let me see the definition of a irreducible Markov chain, if the Markov chain does not contain any other closed communicating class of states other than the state space S. Then the Markov chain is called a irreducible Markov chain, otherwise it is a reducible Markov chain. That means, you have a Markov chain with the state space capital S, you are trying to create the closed communicating class.

If that class and the state space S both are one and the same, that means all the states are going to form a one closed communicating class. That means each state is communicating with each other state and that is same as the state space then, that Markov chain is a irreducible Markov chain. Otherwise, that Markov chain is going to be call it as a reducible Markov chain; before we go to the various reducible Markov chain, I am going to give few examples.

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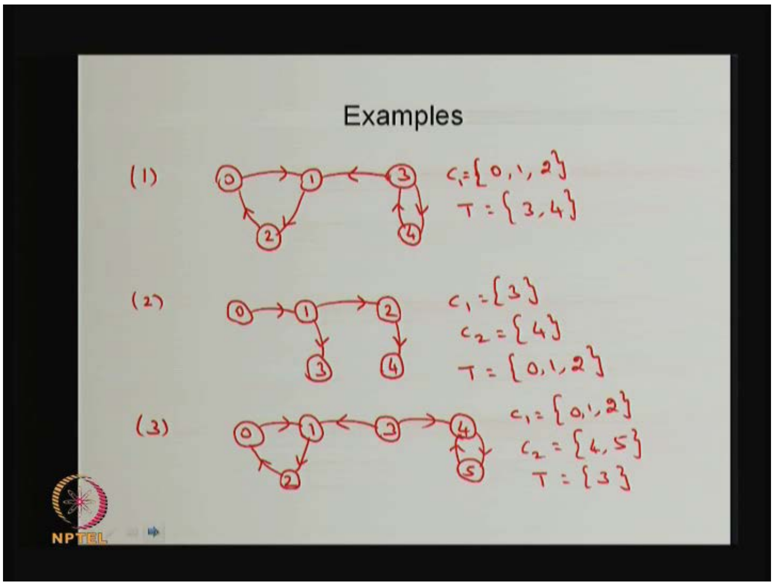

So, through this example we can make the classification, over the reducible Markov chain. You see the first example it has 5 states, instead of the one step transition probability matrix, I have drawn the state transition diagram. So, using this you can easily able to conclude, whether it is going to be a reducible Markov chain or irreducible Markov chain.

If you see the arc from 3 to 1 and the states 0, 1 and 2, all 3 are connected, therefore you can conclude 0 1 and 2 is going to form a closed communicating class. Because all the states inside that class are communicating each other, there is no state going away from this collection to outside; it satisfies the closed communicating class definition. Whereas the 3 and 4, even though there is a communication between 3 and 4 states, once the system goes from 3 to 1 it would not be back. Therefore, the state's 3 and 4 are going to be transient states.

The first visit if you find out F i i capital F i i for a state 3 and 4, it is going to be less than 1. Whatever be the probability, here I have not assign the probability, you can assign the probability positive 0 to 1 and you will get the conclusion, the state's 3 and 4 are going to be the transient states. So, since it satisfies the definition of reducible Markov chain that means, you have a closed communicating class, which is other than the state space.

That means, you have a closed communicating class with the fewer elements than the state space 0 1 and 2, and a few transient states. Therefore, this Markov chain has the reducible Markov chain of some type, I am going to discuss later. See, the second example, this also has the 5 states, if you observe you will conclude the states 0, 1 and 2 are going to be the transient states.

Whereas, the state 3 as well as 4 are going to form a two different closed communicating class, but it consist of only one element in it, only one state in it. You cannot include the state 1 along with 3 or you cannot include the state 2 along with 4 to create a closed communicating class. Therefore, the state’s 3 and 4 will form a closed communicating class with one single state in each in it respectively, and these 3 these 2 states are absorbing states also. So, this is also going to form a reducible Markov chain.

See the third example this has 3 plus 3 6 states, out of 6 states there is no backward arc to the states to the state 3. Therefore, state 3 will be a transient state, whereas state states 0 1 and 2 form a close communicating class. Similarly the state’s 4 and 5 will form a closed another closed communicating class of states. So, in the third example we have a 2 closed communicating classes of states; whereas the first example you have 1 closed communicating class and transient states.

You see that all these three examples, you have a collection of transient states and close communicating class, either one or many or the close communicating class consist of only one element, but all the states, all the model has the fewer transient states. Therefore, you can easily find out the reducible Markov chain whenever, it is not going to form a only one closed communicating class with all the states. If that is not there, then all other things are going to be the or on and all other things are going to be a reducible Markov chain. So, based on these three examples, there are some more examples I can create with the infinite state but, here I have not made it.

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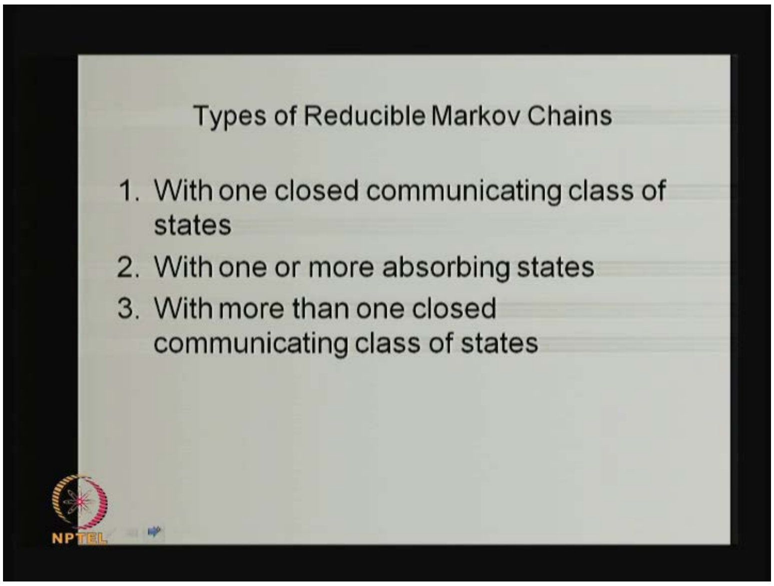

But, based on these three examples you can have some idea, how one can have a various types of reducible Markov chain. I am listing here, in this the default is all the types has the few transient states along with that, it has 1 closed communicating class of states that is a one type, one or more absorbing states that is similar to the example 2. The first one is similar to the first example, the third one is with more than one closed communicating class of states, that is related to the third example. But here, I have not specified whether it is a finite state or infinite state Markov chain.

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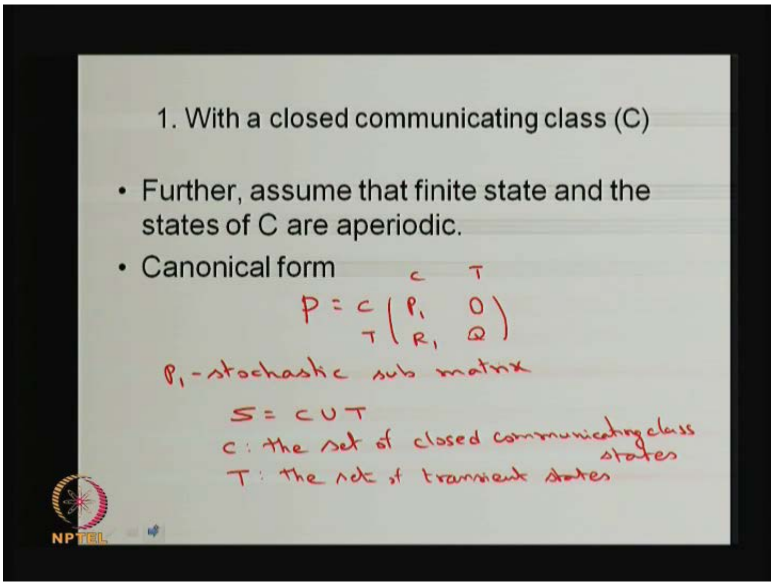

So in material of that the reducible Markov chain can be classified into these 3 in general. So, we are going to discuss out of these 3, the first 2 we are going to discuss in detail and the third I am not going to discuss. So, the way we are discussing the first model, the similar logic can be used to study the third type also. The first type, it is a reducible Markov chain that means, it has the fewer transient states and one closed communicating class. My interest is to study the stationary distribution, therefore I am making further assumption so that, I can go for studying the stationary distribution. For that, I am making the first assumption, it is a finite state model, state space is finite.

And also this model state space has the one closed communicating class and the set of transient states. So, whatever the states in the closed communicating class, that state I am making at a aperiodic. Aperiodic is important to study the stationary distribution therefore, I am making the aperiodic state. So, this state space is the collection of the transient states as well as one closed communicating class.

Therefore, I am making a 2 notation C and T, C for the set of closed communicating class only one, the T is set of all transient states. Therefore, the state space S is going to be the C union capital T. Since it has one closed communicating class and set of transient states, I am reordering the one step transient probability matrix such a way that, the first few rows are corresponding to the states of the closed communicating class.

Therefore, I make it a C but inside, suppose the state space then number of states in this reducible Markov chain is capital N, there is a possibility some fewer elements fewer states may be in the capital C. Therefore, fewer rows that will make a sub matrix that is p 1, that means C to C that sub matrix is one step transition probability sub matrix is p 1. Whereas, the one step transition probability going from closed communicating class that states to the transient states, that probability 0.

Therefore, all the entities are 0, therefore this 0 is nothing but a matrix sub matrix with the number of rows is the number of states in the closed communicating class and the number of column that is same as the number of transient states. This is the way we reorder the one step transient probability matrix, therefore C to capital T that is the sub matrix of 0's. The remaining elements are capital T, that you reorder it in the other remaining rows.

Therefore, T to C will be a some non-zero fewer elements, that is a R 1 matrix R 1 sub matrix. And similarly, T to T there is possibility of possibilities, therefore the probabilities may be greater than or equal to 0 therefore, that matrix is the Q matrix. Therefore, the whole P matrix is a divided partition into 4 sub matrices P 1, 0 matrix, R 1 and Q matrix. Since it is a 0 matrix, entries of 0's therefore, this P 1 is also going to be a stochastic matrix.

The row sum is going to be 1 and the entities are greater than or equal to 0 lies between 0 to 1. So, this values are, this sub matrix will form a stochastic this is called a stochastic sub matrix that means, I am just reordering this P matrix that labeling the state. Such a way that, first I am collecting all the states corresponding to the closed communicating class of states then, set of transient states and this form is called the canonical form. For reducible Markov chain, this canonical form is very important, because once you are able to make a canonical form then you can study the stationary distribution in a easy way.

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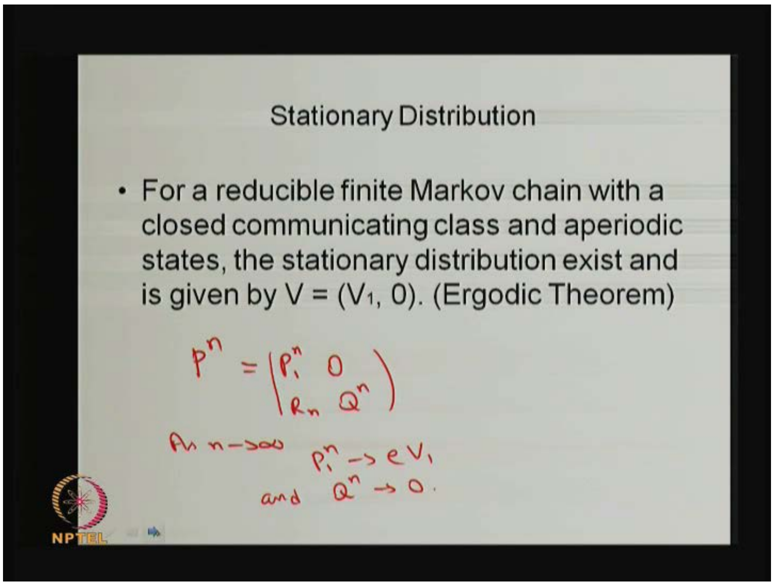

Now, we are moving into the stationary distribution, how to study the stationary distribution for a reducible Markov chain along with the assumptions, one is a periodic and the finite state. Here I am making one more here I am giving the stationary distribution, so I am giving the result for a reducible finite Markov chain, Markov chain is as in the finite states space and it is reducible one.

With the closed communicating class has a aperiodic states this is a mistake a closed communicating class of states has a aperiodic states aperiodic the closed communicating class of states has aperiodic. Then, the stationary distribution exist, that is going to be unique also and that is given by the vector v it consist of a two sub vectors v 1 comma 0 vector, that you can find out.

And this is nothing but, the ergodic theorem for the reducible Markov chain with the assumption finite state space and the states of a closed communicating class as aperiodic states. In that case, you will get the unique stationary distribution and that unique stationary distribution has a two sub one that vectors are v 1 and vectors of 0 units. Before we get the stationary distribution, we can find out what is the n step transition probability for the same reducible Markov chain model.

So, the P n is going to be, you have a sub matrix, stochastic sub matrix P 1, therefore that is going to be P 1 power n. Whereas, for every n this is going to be a function of n, R is the sub matrix, which is the one step going from the transient state to the closed communicating class. Now, the R n is nothing but, a function of n, that elements that sub matrix is corresponding to the transient state to the closed communicating class. Whereas, the transient to transient, that is going to be a power n.

That is a Q matrix, Q matrix is the sub matrix for one step T to T whereas, Q power n is the element corresponding to the n step transition probability matrix. So, as n tends to infinity, the stochastic sub matrix that power n, that will tends to e is the vector of V 1. V 1 is the sub few elements, that is corresponding to the stationary state probabilities for the states corresponding to the closed communicating class of states. So, e is the vector of entities 1 1 1 and so on, multiplied by the V 1.

And the transient to transient n step transition probability P power n, as n tends to infinity, this will tends to 0. This is obvious because, since the states are transient state, for a finite n you have a probability Q power n whereas, as n tends to infinity the system would not be in the transient state. Therefore, the long run proportion of the time the system being in the transient states, that is 0 as n tends to infinity. Therefore, Q n will tends to 0 whereas, this will tends to the stationary state probabilities.

Therefore, this stationary distribution vector v consist of few elements of zeros, that is corresponding to a transient states transient state probabilities in a longer run. And the v

1 is the steady state probabilities in a longer run, it is not a steady state, a stationary distribution, stationary state probabilities in a longer run for the closed communicating class of states. So, this one can solve by using the equation pi P is equal to pi, you can get this pi i’s, this pi i’s is in the notation here it is p i’s p 1.

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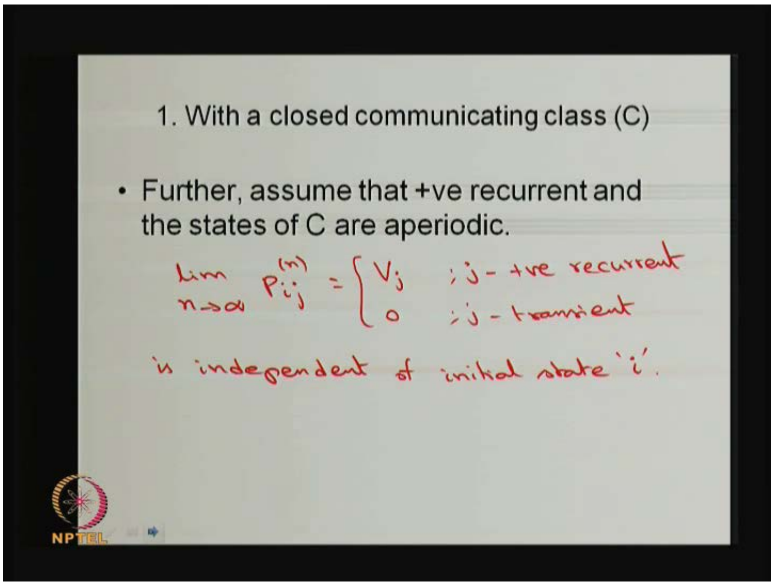

So, now I am making a further assumption, the states are going to be a positive recurrent. So, already I made a aperiodic states, now I am making the one more assumption it is the positive recurrent. Once it is the positive recurrent, then the limiting probability is limit n tends to infinity, that probability is going to be v j’s for the positive recurrent states and for all the transient states, the probabilities are going to be 0.

And, since we have a reducible Markov chain with one close communicating class and all other states are transient states, this stationary distribution are stationary state probabilities, these probabilities are independent of the initial state i. That means, either the system can start at time 0 in the one of the states in the closed communicating class or transient states.

In a longer run, ultimately the system will be in one of the states in the closed communicating class, whether it is started initially from the closed communicating class or transient states. Therefore, this stationary distribution is independent of initial state I and for transient state you can conclude immediately this probabilities are zeros, and for a positive recurrent states you can make it v j’s and you can compute this v j’s.

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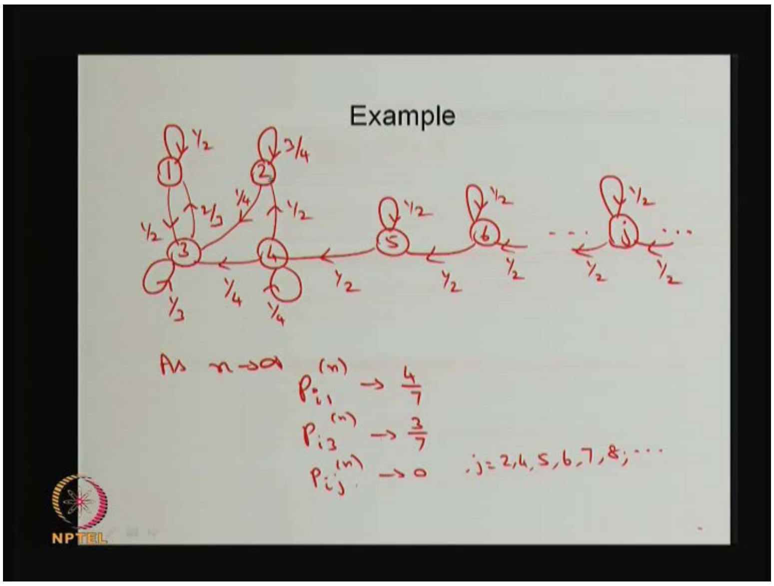

Now, I am going to give one simple example in which, we have a infinite state, this is going to be a reducible Markov chain. Because, the states till 5 not till 5 including 4 and 2, the system come to the state 3 there is no arc from 3 to 4 or 3 to 2. Therefore, the states 2 4 5 6 and so on, all those states are transient states whereas, the state's 1 and 3 are going to form a one closed communicating class. Therefore, this is the reducible Markov chain with one closed communicating class 1 and 3 and all other states are going to be the transient states.

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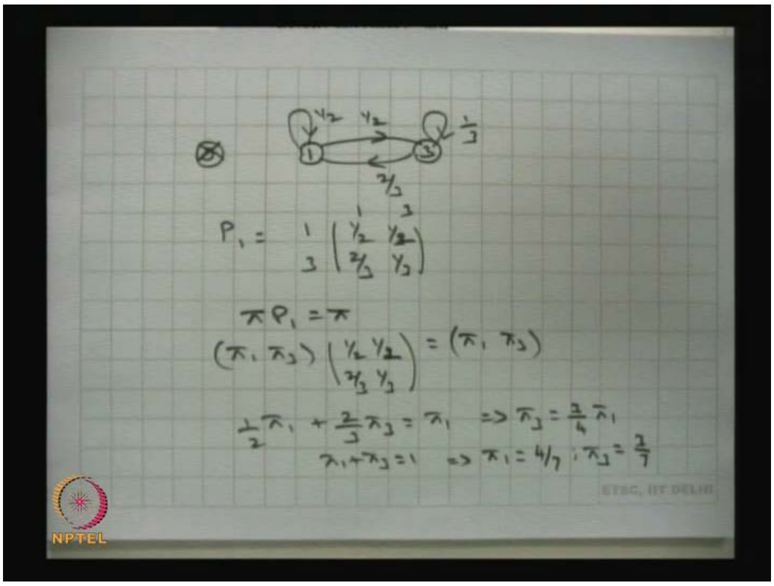

Therefore, as n tends to infinity, this probabilities is are going to be 0 for these states 2 4 5 and so on and this probabilities are independent of the initial state i. So, wherever the i, whether the i is belonging to the one of the element one of the states in the close communicating class of states or transient states, immaterial of that this is stationary distributions are zeros for the transient state. For the for the closed communicating class of states, you can find out this probability by separately making the Markov chain.

The state's 1 and 3 you can make it separately and there is a arc from 1 to 3 with the probability of, there is a self-loop with the probability 1 by 2. And there is a self-loop in the state 3 with the probability 1 by 3 and the arc from 3 to 1 is 2 by 3. So, what you want to find out, the stationary distribution for these 2 states. Therefore, you make a stochastic sub matrix with the state's 1 and 3, that is 1 by 2 1 by 3 2 by 3 and 1 by 3, this is also stochastic matrix you can verify.

Now, if you want to find out the stationary distribution for these two states, you solve pi P 1 is equal to pi. That means, pi 1 pi 3 times p 1 that is 1 by 2 1 by 3 oh sorry 1 by 2 this is 1 by 2 made a mistake 1 by 2 1 by 2 and this is 2 by 3 1 by 3, that is equal to pi 1 pi 3. You take the first equation, that is pi 1 half times pi 1 plus 2 third pi 3 that is equal to pi 1, so from here you will get pi 3 is equal to 3 by 4 pi 1. Now we use pi 1 plus pi 3 is equal to 1, so using this you will get pi 1 is equal to 4 by 7. Once you know the pi 1 the pi 3 is going to be 3 by 7.

So, you do not want to find out the stationary distribution for the whole model instead of that, you can find out what is the close communicating class and you can solve only the close communicating class that sub matrix pi p 1 is equal to pi. And you will get pi 1 and pi 3, and that is going to be in a longer run, that is equal to 4 by 7 and 3 by 7 and all other states are going to be 0. So, this is the way one can find out the stationary distribution for a reducible one close communicating class and transient states.

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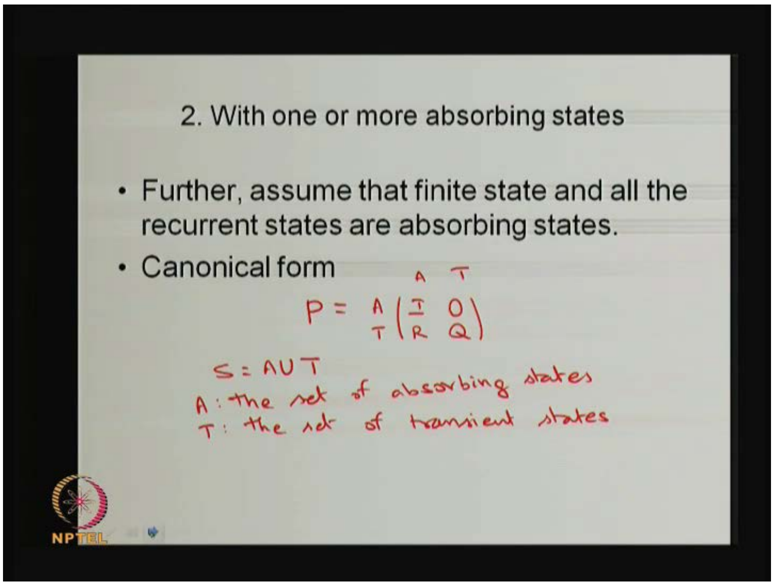

Now, we are moving into the second type, in the second type this is our reducible Markov chain. But, here each closed communicating class consists of only one element that is nothing but, the absorbing states. But, more than one closed communicating classes are possible therefore this type is called a with one or more absorbing states. Here also my interest is to find out the stationary distribution, the stationary distribution here the interest of the different way one is the probability of absorption, the other one is what is the mean time before absorption.

So, for that I making a further assumption, the state space is going to be finite so, with that I am making a canonical form. The canonical form consist of all the absorbing states that I will label it as a capital A and all the transient states as a capital T therefore, the state space S is the A union capital T. Therefore, the canonical form I collect all the absorbing states in the first few rows and then, remaining will be the all the transient states.

Since the absorbing states p i i is equal to 1, therefore you will have a identity matrix for the sub matrix of the matrix p, corresponding to A to A. Whereas, A to T absorbing state to transient states, that elements are going to be 0. So, that is the sub matrix with the entities 0 whereas, T to A will be sum matrix capital R and T to T will be sub matrix Q.

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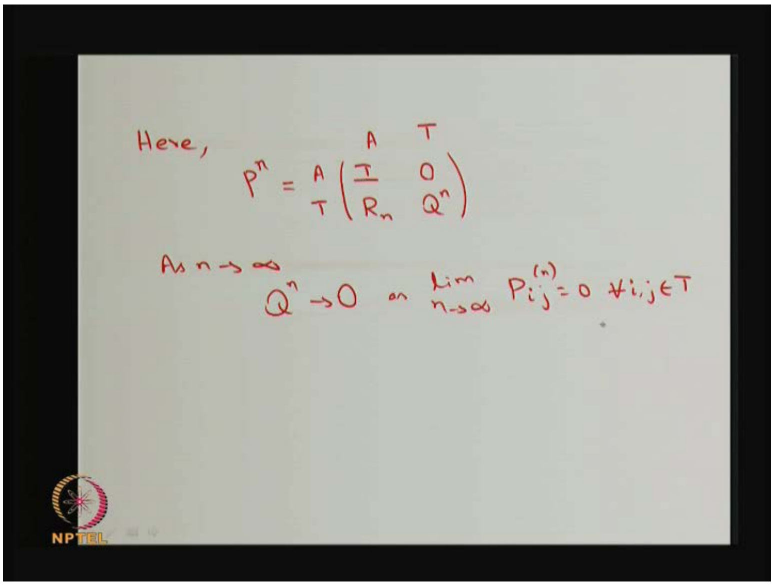

So, if you go for what is the n step transition probability, since it is identity matrix, again also you will have identity matrix whereas, T to A that is going to be a function of n. Whereas, T to T will be a power n, that is Q rise to power n, as n tends to infinity the system would not be in the transient state. Therefore, Q n will tends to 0 sub matrix as n tends to infinity and this probabilities are going to 0 for all i comma j belonging to T, T is nothing but the set of transient states.

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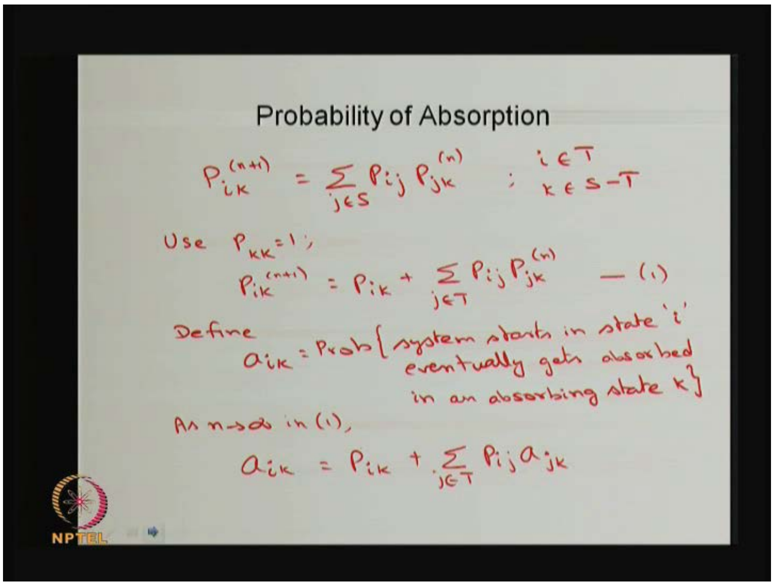

Our interest is, here what is the probability of absorption because, we have a few one or more absorbing states. So, if the system start from some transient state, what is the probability that the system will be absorbed into this absorption state. So, for that I am going to start with the Chapman Kolmogorov equation, that is Chapman Kolmogorov equation for the n plus 1 th step the system going from the state i to k, that probability same as what are all the possible the system can go make a one step from i to j and then j to k in n steps, all the possibilities j belonging to S, where S is the state space.

I know either I have a one either I have a transient states or all other states are absorbing states. Therefore, if k is going to be the absorbing state then, p k k is equal to 1 that means, one step transition probability of system moving from state k to k that is one. Therefore, i to k in n plus 1 steps that probability I can split, I can make i to k in one step then forever I will be in the state k, plus i would have move to the state i to j, where j is the another transient state. It could be same also it could be same also then, j to k in n steps.

Now, I am defining what is a meaning of a probability of absorption, that I am denoting with the letter a suffix i comma k that is nothing but, the probability that the system starts in state i it starts in state i, eventually get absorbed in absorbing state k. So, the first letter is the starting state and the k is the absorbing state. So, this is the probability of absorption starting from the state i to the absorption state k.

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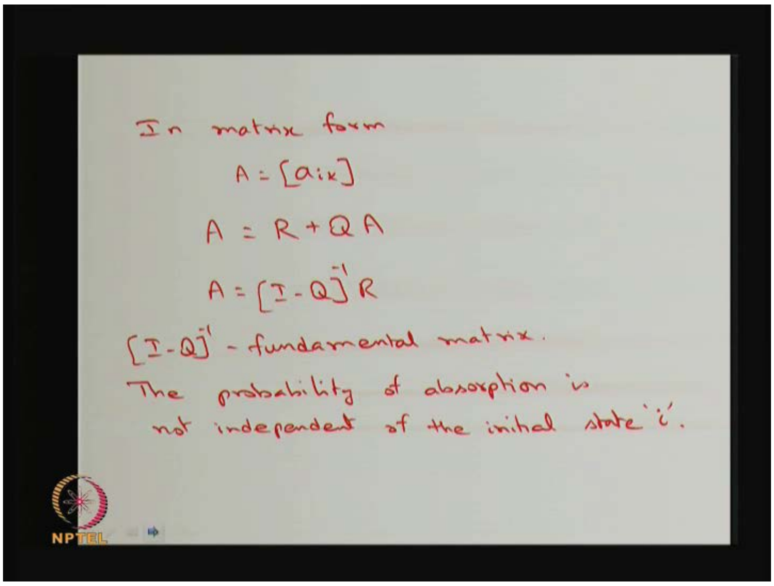

Now I am taking the equation 1, as I make n tends to infinity in both side, the left hand side will be a of i comma k. Because, as n tends to infinity so, this will be a i k similarly, p j k of n that is also a j k. Therefore, I will have a a i k this side and a j so, this is sort of regressive equation. So, this is in the element form, I can go for in the matrix form. So, I can write a i k as a matrix capital A therefore, in the matrix form the previous this equation for all values of i. This equation as in the matrix form capital A is equal to R matrix because, this is p i to k where, i is the transient state and k is the absorbing state. So, transient state to the absorption state transient state to the absorption state that sub matrix is capital R.

Therefore in the matrix form capital A is equal to R matrix plus Q matrix that is, the one step transition of system is moving from transient to transient multiplied by A matrix. So, I can do the simplification so I get A matrix is equal to i minus Q inverse R matrix and here i minus Q inverse that is nothing but, the fundamental matrix. So, once you are able to find out the fundamental matrix multiply by the R matrix, that will give the probability of absorption starting from the transient state and reaching absorption state.

And this probability is not independent of initial state that is very important whereas, the previous type of reducible Markov chain, that is independent of initial state. Whereas, here the probability of absorption is not independent of the initial state i. So, this we can visualize through one example that I am going to present later.

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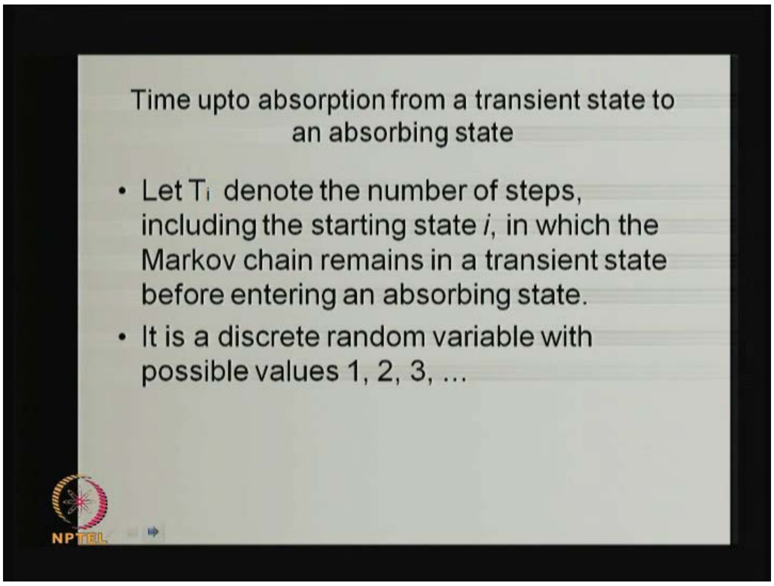

The next result interested in the reducible Markov chain with one or more absorbing states, that is what is the time to absorption. Basically, our interest is to get the mean time to absorption starting from the transient state to a absorbing state. That means, how much time on average, the system is spending in the transient states before absorption that is very important. Because, many application has reducible Markov chain in which, more than one absorption states are there with the transient states.

Therefore, what is the mean time up to absorption that means, how much time spending in the transient states before the absorption. So, for that I am going to define the random variable capital T i. The T i denotes the number of steps including the starting state i in which, the Markov chain remains in a transient state before entering a absorption state absorbing state. So, there is a possibility the system would have been spending at least one step before absorption or two steps or three steps and so on.

Therefore, that is going to be a random variable, it is a discrete random variable with the possible values are 1 2 3 and so on. Our interest is not only finding out the distribution of T i, our interest is to find out what is the mean time up to absorption from the transient state to a absorbing state.

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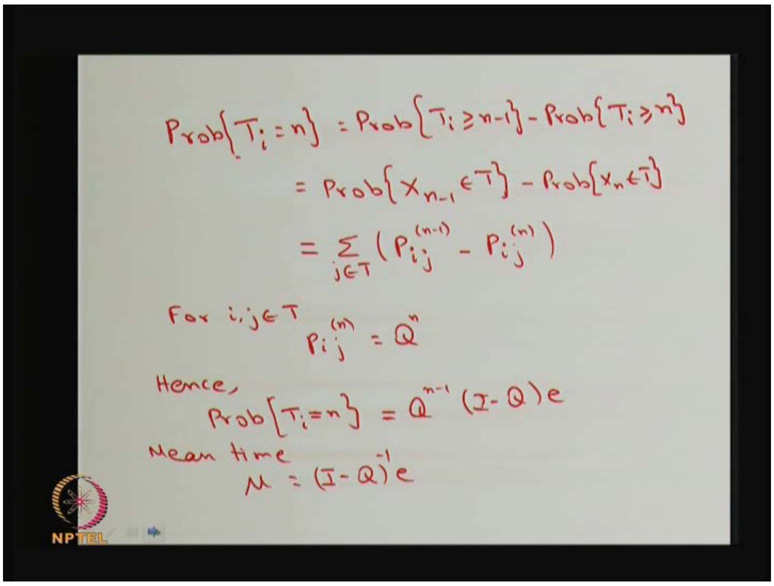

So, this probability can be computed by, find out what is the probability of T n is equal to n for some n, n can take the value 1 2 3 and so on. So, that discrete random variable probability probability mass function can be computed in this way, you find out what is the probability of T i is greater than or equal to n minus one minus, what is the probability that T i is greater than or equal to n. If you find the difference, that is same as the probability mass at n.

But this is same as the T i greater than or equal to n minus 1, that is same as the n minus 1 th step the system is in the transient state. If T i is going to be greater than or equal to n minus 1 that means, the system spends at least n minus 1 steps in the transient states. Once it goes to the absorption state, then it cannot come back to the transient states. Therefore, the meaning of T i greater than or equal to n minus 1, that is same as the n minus 1 th step the system in the transient states. So, both the events are equivalent therefore, the probabilities are equal.

Similarly, you can argue T i greater than or equal to n means, at least n steps the system in the transient states before absorption, that is same as in the n th step the system is in the transient state. The probability of n minus 1 th step the system is in the transient state, that is same as what is the what are all the possibilities the system would have move from the state i to j in n minus 1 steps. You add all the possibilities j belonging to T, you add all the possibilities of the transient states, that summation will give this probability.

Similarly, for the X n belonging to capital T this is in the for fixed i where, i is belonging to the transient state. Now, I will go for, I know that for i comma j belonging to t the n step transition probability is nothing but, the sub matrix, that is Q power n. If you recall, the way we made a canonical form of a p matrix, the T to T that is a Q matrix therefore, as n tends to for any n th step that is going to be Q power n. So, this is what I am using for i comma j belonging to capital T, the sub matrix of p power n, that is Q power n.

Therefore, for i comma j belonging to t the n step transition of system moving from i to j that is Q power n. Therefore, I can substitute here the above equation so, the probability mass at n, that is same as Q power n minus 1 into i minus Q into e vector. Once I know the probability mass function for the discrete random variable T I, then I can find out the mean. Mean is nothing but, summation n times the probability mass at n, T i is equal to n.

If I add summation over n, that is going to be the mean time up to absorption that is going to be, do the simple calculation we will get I minus Q inverse into e vector. This I minus Q inverse is nothing but, the fundamental matrix that means, if you find out the fundamental matrix multiplied by the R sub matrix you will get the probability of the absorption. If you multiply by the e vector, you will get the mean time up to the absorption.

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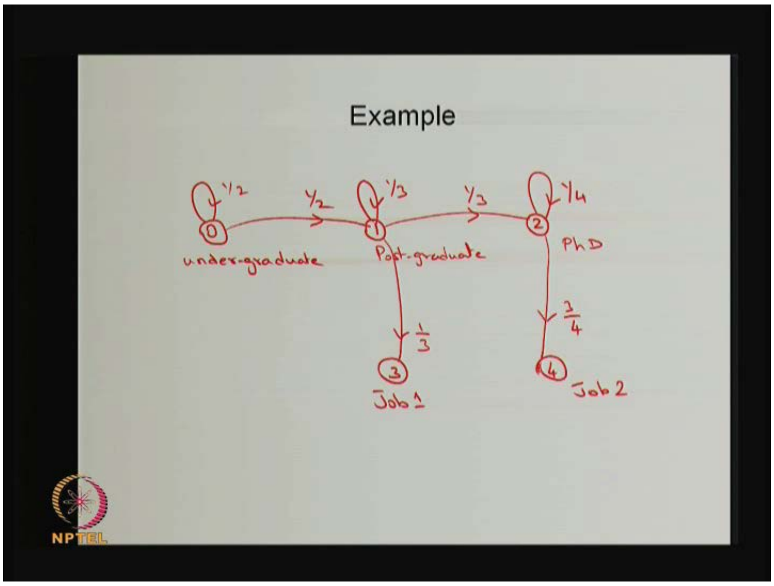

I am going to give one simple example for this type of reducible Markov chain with the transient states and one or more absorption state, I am making the assumption that is a positive recurrent. So, instead of positive recurrent, I have a finite Markov chain so the finite Markov chain at least one state is the positive recurrent. Therefore, this is both are going to be a absorbing state, therefore we do not want those conditions also. So, here in this model the states 0 1 2 are the transient states, 3 and 4 are absorbing states.

This is the easy example in which, you can visualize someone is doing the undergraduate with the probability of, he is not able complete the undergraduate in the next step with the probability of, he is moving into the postgraduate in the next step. So, I making a DTMC, with the assumption the memory less property is satisfied and so on. From the postgraduate either someone gets the job one with the probability one third or not able to complete the postgraduate, that probability is one third or he completes and go to the PhD program one third. From the PhD, one fourth is not able to complete the PhD in the next step or with the probability three fourth, he is getting the job 2.

Now you can visualize the questions, what is the probability that I absorbed into the state job 1 or job 2, that is the probability of absorption. The next question, how much time on average I will be spending in the transient states in the study before I get the job. So, this is the way you can visualize the reducible Markov chain with this type. So, these two questions are going to be answered, by finding the probability of absorption and mean time up to absorption.

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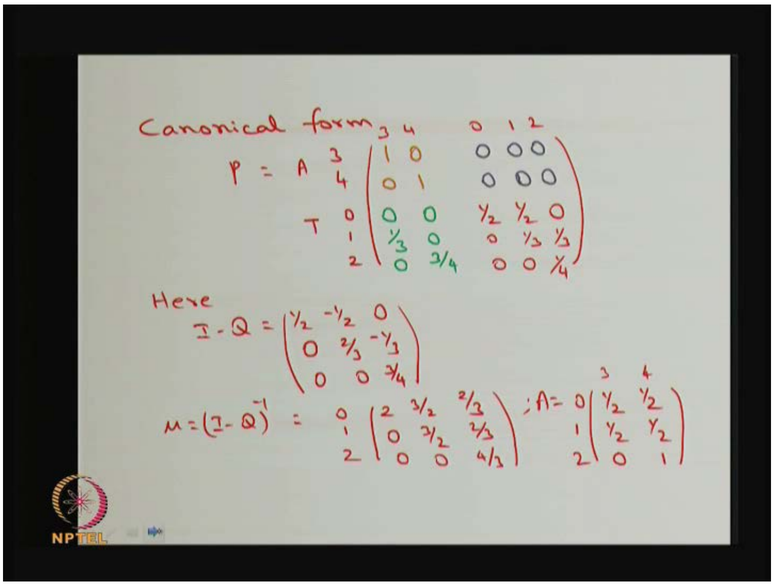

First, let me write the p matrix in the canonical form and all the sub matrix, I made it in the different colors. So, 3 and 4 are going to form a, each one is going to be absorbing states. So, therefore A to A that is identity matrix, A to capital T that is a zero matrix sub matrix then, T to A that is again A matrix that is R then, T to T that is a Q matrix. So, what we need the Q matrix and the R matrix, both are sub matrix of capital P, that is a one step transition probability matrix.

So, you find out what is I minus Q, I is the identity matrix of the same model, I 3 here minus Q matrix. So, you know the Q matrix is this so I minus Q matrix, find out the inverse, that inverse is this much. So, from these if you multiplied the vector e, that is 1 1 comma 1 you will get the mean time to absorption and also you can find out the probability of absorption. After finding the I minus Q inverse, that is the fundamental matrix multiplied by the R, this matrix will get the probability of absorption. I am not giving here the numerical calculation, see the result.

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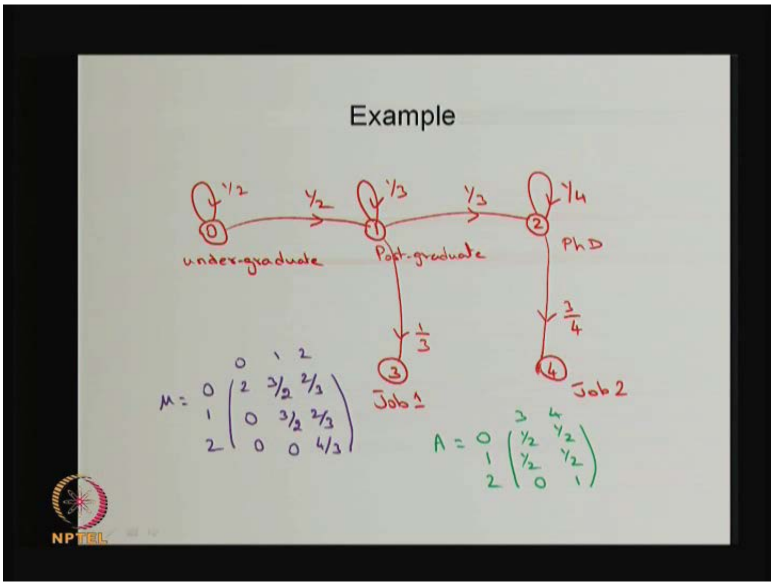

So, this is the mean time up to the absorption and this is the probability of absorption, first let us discuss the probability of absorption. If the system starts from the state 0, state 3 is nothing but, the job 1 so with the probability half, you would have been absorbed into the job 1 with the probability half, if the system start from the state 0 from the undergraduate. Similarly, with the job 2 that probability is half, it is a it is a probability mass function, either you will be in the job 1 or job 2 that is a probability of absorption.

If you would have started starting with the postgraduate, then with the probability half and half you may be in the job 1 and 2, whereas if you beginning with the PhD program not these two programs, that is not possible but, still this is the just example. So, if you start with the PhD program, then definitely you will land up with the job 2 with the probability 1 because, there is no arc from 2 to 1 and land up the job 1. Therefore, the probability of absorption into the job 1 that probability 0, this for illustration.

Therefore, you can make out how the calculation goes so here, the probability of absorption starting from the state 2 that probability is 0 to the job 1. Whereas, job 2 that probability is one so, this is the probability distribution of probability of absorption starting from this transient states. Similarly, you can visualize the mean time up to the absorption, this zeros can be discussed first.

So, if system starts from the state two, what is the average number of steps the system goes from the state 2 to 0, then it goes to the absorption state. That is not possible, the system is going from 2 to 0 therefore, the mean time is going to be 0. Because, the minimum time is 1 or minimum number of steps system spending in the transient states are one and so on, therefore mean is 0 here.

Similarly, the system is starting from the state 2 and land up 1 and from there it goes to the absorption state, that is also not possible therefore, that mean is also 0. Whereas, all other values the greater than 0, that gives what is the average number of steps the system is starting from these transient states, and reaching this transient states before absorbed into any one of the absorption states, accordingly will have this values.

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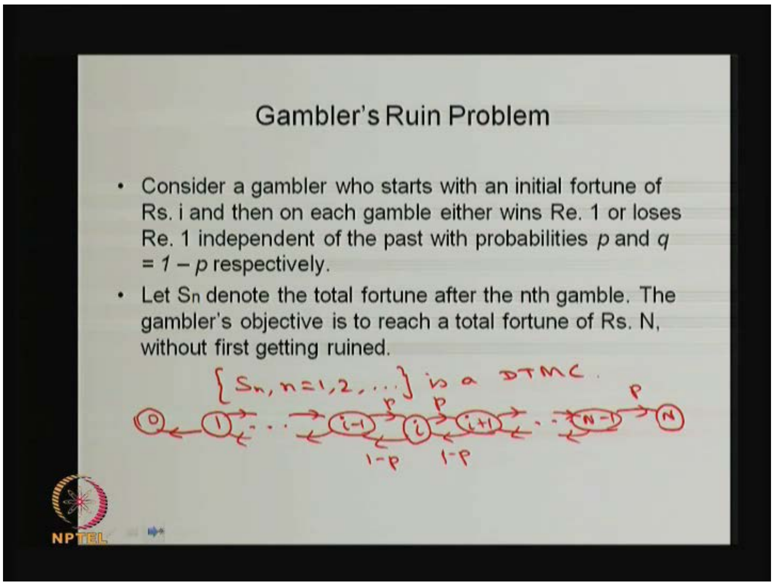

With this example I go to the next example, that is a reducible Markov chain and this is a special case of a random walk also. Let me discuss what is the example, this is called a gambler's ruin problem, let me define what is the gambler's ruin problem. Consider a gambler who starts with the initial fortune of rupees i, i amount he has at the time 0 and then, on each gamble either wins rupees 1 or loses rupee one, independent of the past with the probabilities p and 1 minus p respectively.

So, in this game there is no draw, there is no die, either he wins or he loses, wins with 1 rupee loses 1 rupee and the corresponding probabilities are p and 1 minus p. And he started with the initial amount small pi and S n denote the total fortune after the n th gamble. That means, S naught is small i and S 1 becomes a, if he wins his total fortune after the n th first gamble, that will be i plus 1. If he loses then his money would have been i minus 1, that is a way s 1 s 2 s 3 sample arcs goes.

The gambler's objective is to reach the total fortune of rupees capital N, where N is a some number, some positive integer, without first getting ruined. That means, you can make a state transition diagram for this Markov chain, the S n is going to form a time homogeneous discrete time Markov chain. Because of, each games are independent and with the probability p and with the probability 1 minus p, he wins or he loses therefore, the Markov property is going to be satisfied therefore, this stochastic process will form a discrete time Markov chain.

If you notice if he is a land up 0 amount at the n th game, then he is ruin. If he is getting a first time n rupees, then the game is over, that is objective. Therefore, this is a special case of random walk one dimensional random walk in which, the state's 0 and n are going to form a absorbing barrier. Once the system goes to the state 0 the system is absorbed in the state 0, once the system reached the state capital N, then system is absorbed in the state N.

Therefore, the state's 0 and N are absorbing states and all other states are states from 1 to N minus 1 are going to be the transient states. Therefore, this DTMC is a reducible DTMC with transient states and two absorbing states so this will fall under second type, the one we have discussed. Our interest in this model is, what is the probability of absorption, what is the probability that he loses all the money at the end of some give or what is the probability he reaches capital N that is a subject so, that is the probability of absorption. The other one is, how much time he is in the transient states on average, what is a mean time of absorption till he reaches the absorbing states either 0 or N.

So, for that I making the notation first p suffix i, that denotes the probability that the gambler wins when s naught is equal to i, that is i; i means initially i amount he ask that is s naught. So, what is a probability that the gambler wins, clearly p naught is equal 0, similarly, p n is equal to 1. Because no way, if he is having initially 0 amount he cannot win, therefore that probability is 0. If he has if he is having initially, the gambler has the amount n amount in at the time 0 itself, then he need not play at all, therefore that probability is going to be 1.

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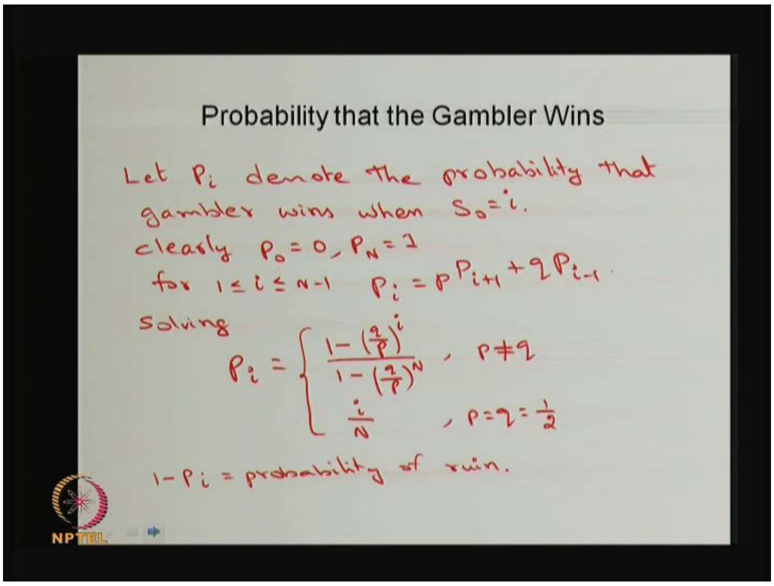

Therefore, the probability the gambler wins that probability is going to be 1, if he is having N amount initially. For all i in between 1 to N minus 1, you can make a recursive relation using the Chapman Kolmogorov equation, that means the probability that the gambler win with the I amount initially, that is same as either he has initially N plus 1 sorry i plus 1 amount initially, and with the probability p he wins or with the probability i minus 1, the gambler wins multiplied by the probability q, q is the he loses. So, these two combinations will give the probability of gambler's win.

You can do the simple calculation the way you have p i in terms of p i plus 1 and p i minus 1, you can write p i minus 1 also, then you find out the difference, then you will get the recursive way and you will get in terms of p 1 and everything you will get it. So, you can use p capital N is equal to 1, using that you will get all the p i's. You can use this relation p naught is equal 0 and capital P N capital P capital N is equal to 1, using these two values you find out the difference and you make regressive relation, he will get a p i's.

So, whenever the p is less than q and p is greater than q, you will get and the p i's is 1 minus q minus p power i divided by 1 minus q power p power n. For p and q is equal to same, that means it is half because, q is 1 minus p therefore, you will get the probability of gambler's win that will be i divided by n that you can get. And here the interest is, what is the probability? That he is going to ruin; that means this is the probability that he is going to win. And the 1 minus of that is going to be the probability that he is going to ruin in this gamble.

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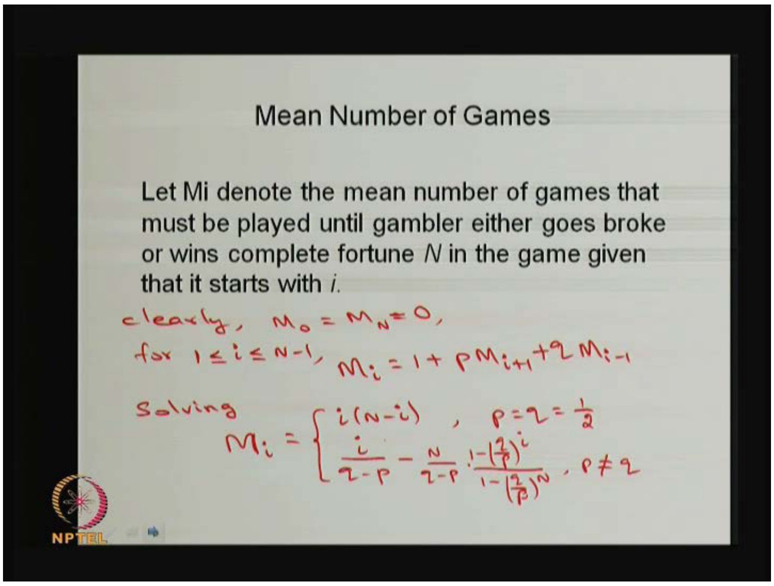

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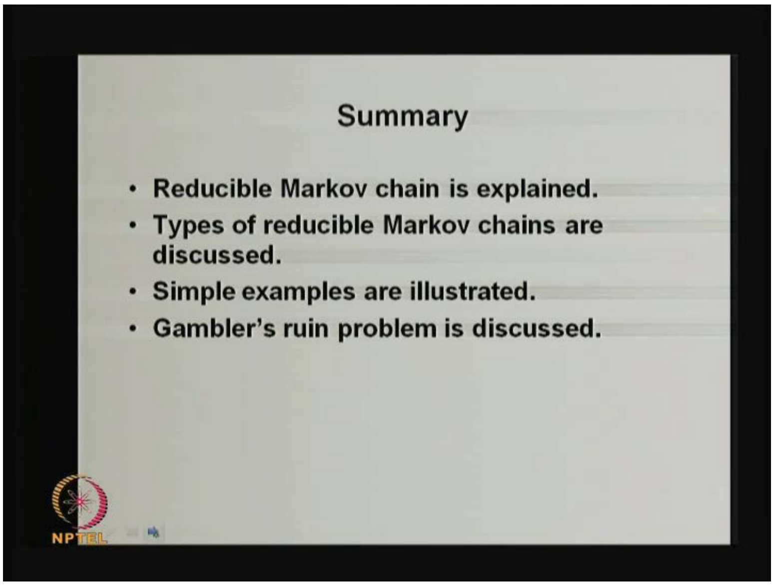

The next one is, our interest is mean number of games because the objective is he has to reach the capital n amount. So, the game is going to be over either he completely ruin or he is going to get the N amount therefore, I am making here the random variable M suffix i this is suffix i. So, M suffix i is denote the number of sorry mean number of games, I am directly making a random variable for mean suffix i and I know the relation for this and here also, I am making the similar relation by solving that, I will get the M i's. So, this is the mean number of games in the mean number of games played by the gambler, until he goes to broke or wins completely fortune N.

So, in this lecture I have discuss the reducible Markov chain and types of reducible Markov chain and some examples also. And finally, I have given gambler's ruin problem, references are these.

Thanks.