Stochastic Processes
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Module - 4
Discrete-time Markov Chain
Lecture - 6
Time Reversible Markov Chain, Application of Irreducible Markov Chain in Queueing Models

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Good morning, this is module four discrete time Markov chain, lecture 6, time reversible Markov chain. Then application of irreducible Markov chain in queueing models. So, in this lecture I am planning to give the time reversible Markov chain, and how to compute the stationary distribution in an easy way. Then I am going to give applications of irreducible Markov chain in queueing models. Here irreducible Markov chain means it is a DTMC model, because later we are going to give the applications of irreducible continuous time Markov chain in queueing models also in the later lectures. Also I am going to give few simple examples.

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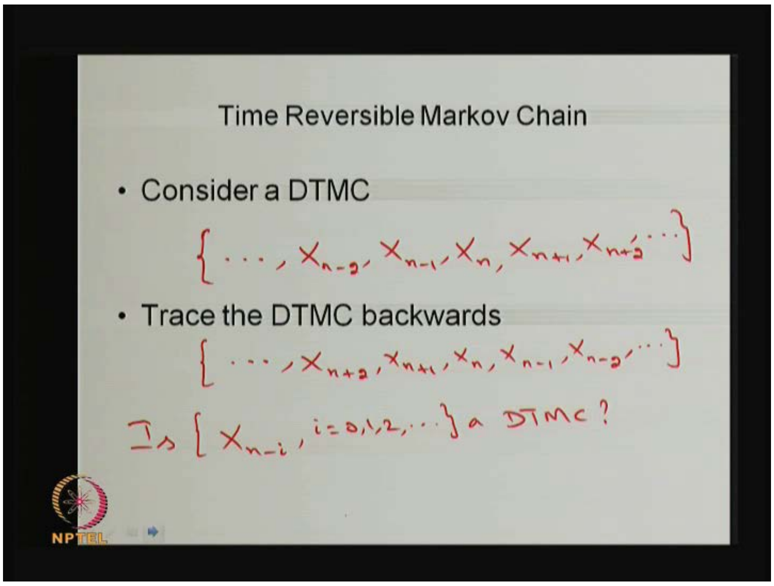

What is the meaning of time reversible Markov chain? First let me explain how to construct the time reversible Markov chain. Consider the DTMC, it is of course time homogenous discrete time Markov chain. You see the collection, it is in the usual way X n minus 2 X n minus 1 X n X n plus 1 x n plus 2, and so on. Now, you trace the DTMC backwards, that means you know the first all X n plus 2, then X n plus 1, then you know, then you know what is X n, then you collect X n minus 1, X n minus 2, and so on. Now, the question is whether if you make a DTMC backwards, and that sequence, that sequence of random variable that is a of course, it is a stochastic process whether this is going to be a DTMC.

Any stochastic process is going to be a DTMC, if it satisfies the Markov property, and state space is a discrete, and time space is also discrete. But, if you see this, the DTMC backward, that is also going to satisfies the Markov property, the way the given situation, the future depends only on the present not the past history that is the Markov property. The same thing is going to be satisfied in the reverse also. The Markov property satisfied by the reverse the backward DTMC, therefore this is also going to be a time homogenous discrete time Markov chain.

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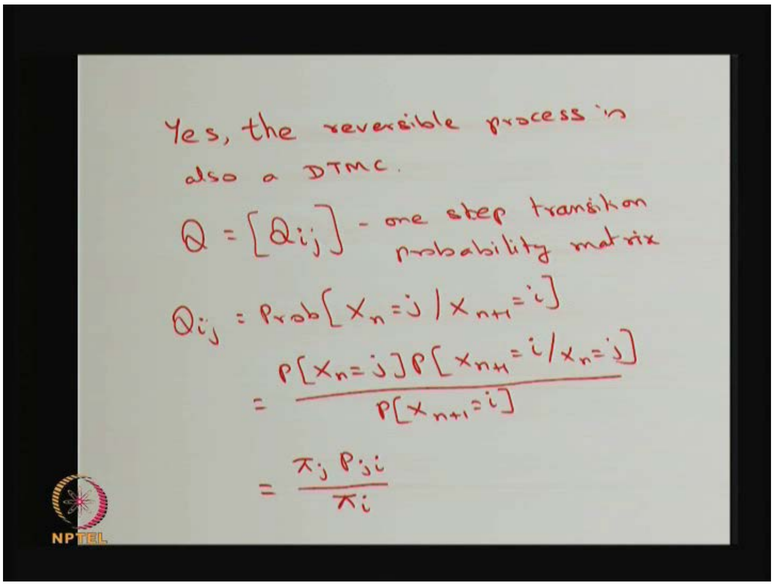

Now, I am going to define, I am going to give, how to find out the one step transition probability for the reversible process, that you take it as the matrix Q that consists of elements Q i j. You can find out the entries the Q i j that is nothing but, what is the probability that the system will be in the state j at the n th step given that it was in the state i at the n plus 1 th step, that is different from the original DTMC. The originalistic DTMC it is what is the probability that the system will be in the X n plus 1 in the state j given that X n was i, whereas here it is X n is equal to j given that x n plus 1 is equal to i.

This conditional probability you can compute in this way the product of probability of X n is equal to j, multiplied by the probability of X n plus 1 is equal to I, given that X n is equal to j, divided by what is the probability that X n plus 1 is equal to i. That is same as the probability of X n is equal to j is nothing but in a steady state, in a, what is the probability that in a at the n th stage in the system being in the state that is equal to pi j multiplied by, what is the probability that this is the one step transition probability of system is moving from j to i divided by pi i. That means the q i j is going to be pi j, p j i divided by pi i. Assuming that the stationary distribution exist, otherwise pi j is equal to limit n tends to infinity of p i j n.

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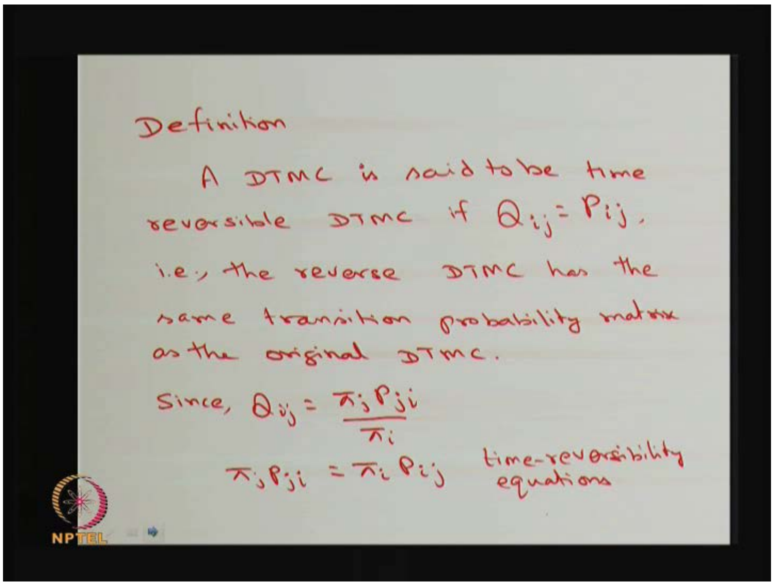

Now, I am going to give the definition of time reversible. A DTMC is said to be a time reversible DTMC if both the transition probabilities are one and the same. That means the one step transition probability of the new, or the time reversible process Q i j is same as what is the one step transition probability of the original DTMC, that is p i j. That is the reverse DTMC has the same transition probability matrix as the original DTMC.

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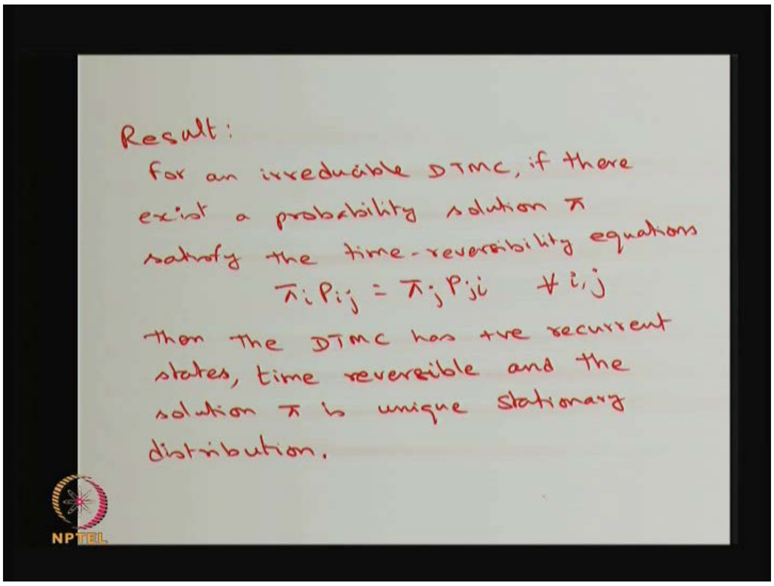

Now, I am going to equate this the Q i j is equal to this much, therefore that is same as pi j of p j i is equal to pi i of p i j. If this equation is going to be satisfied, then that DTMC is going to be call it as a time reversible Markov chain. This collection of equation for all p i j that equation is called a time reversibility equations.

Now, am going to give the few results on time reversible Markov chain. Let me take irreducible DTMC. If there exist a probability solution that is a pi is a vector, satisfies the time reversibility equations, that is a, the vector pi consists of pi 1 pi 2 and so on. So, if that entries satisfies the time reversibility equations, that is pi i is equal to pi i times p i j is equal to pi j, p j i for all pairs of i comma j, then the DTMC has a positive recurrent states, also it is a time reversible, the solution pi is a unique stationary distribution.

That means a whenever you have a irreducible DTMC, and if you have their exist probability solution vector pi satisfies the time reversibility equation, then you can conclude the DTMC has the positive recurrent states as well as the DTMC is a time reversible Markov chain. Also the vector pi that satisfies the time reversibility equations that vector pi is a unique stationary distribution.

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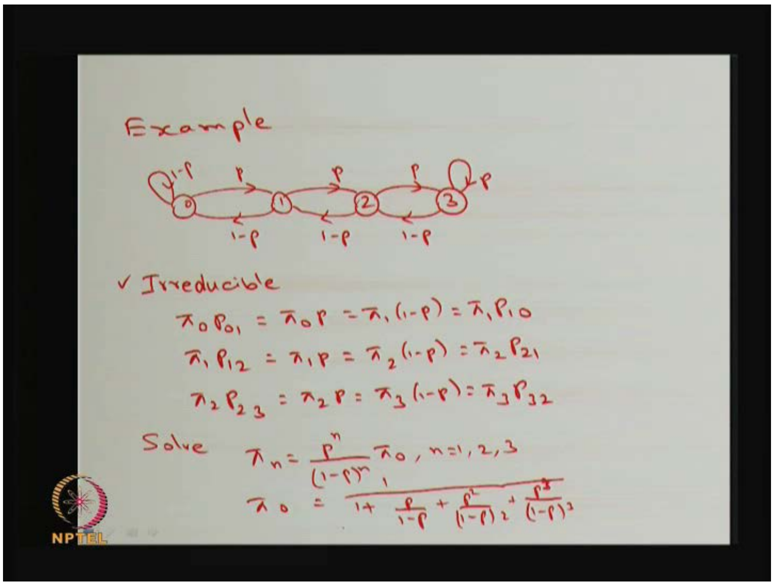

So, the, how you can one can use the time reversible concept in finding the stationary distribution that I am going to explain in the next example. Let me take a simple example which consists of four states, it is a finite model, finite state space, also it is irreducible, because each state is communicating with each other states. I assume that the probability p that is lies between 0 to 1, therefore this is going to be a aperiodic states. So, this Markov chain is a finite irreducible. You have the result for a finite irreducible Markov chain at least one state is going to be a positive recurrent. So, since it is irreducible all the states are of the same type, therefore all the states are going to be a positive recurrent states, also it is aperiodic.

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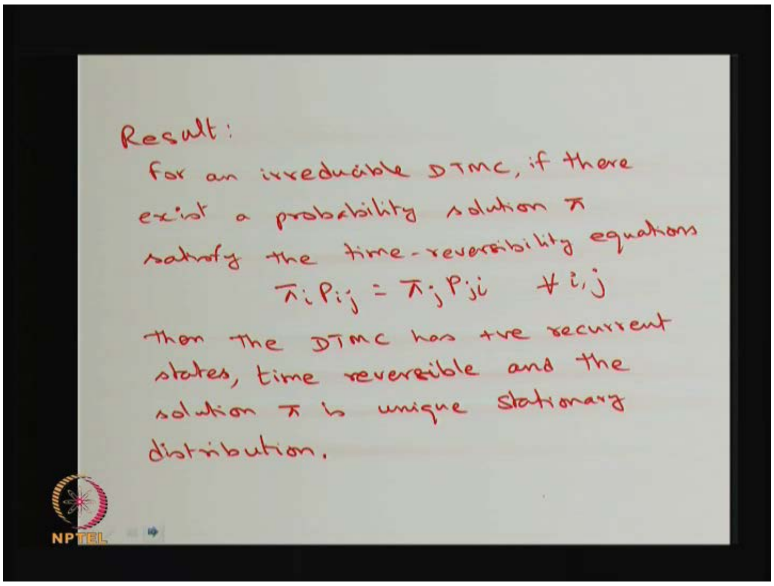

So, you can use a result of irreducible, aperiodic positive recurrent, and also the finite states going to give the unique stationary distribution. That can be computed by solving pi p is equal to pi where pi is a stationary probability vector. Here you can use the time reversibility concept, therefore you do not want to solve actually pi p is equal to pi, but you can start from the time reversible equation from that you can get the solution. That is what I have done it in this example.

First I have check it is irreducible, then I check whether the time reversible equation is going to be satisfied by this irreducible Markov chain. So, since it has the four states, I am just checking all the states whether, whether the time reversible, reversibility equations are going to be satisfied, that is satisfies. Since it is you see the previous result for a irreducible DTMC, if there exist the probability solution that means I started with their exist a solution but, since I know the result it is the irreducible aperiodic positive recurrent the stationary distribution exist. Therefore, I started with the probability solution pi and I have checked with the time reversibility equations for the example also. Then I am concluding it is going to have a unique solution.

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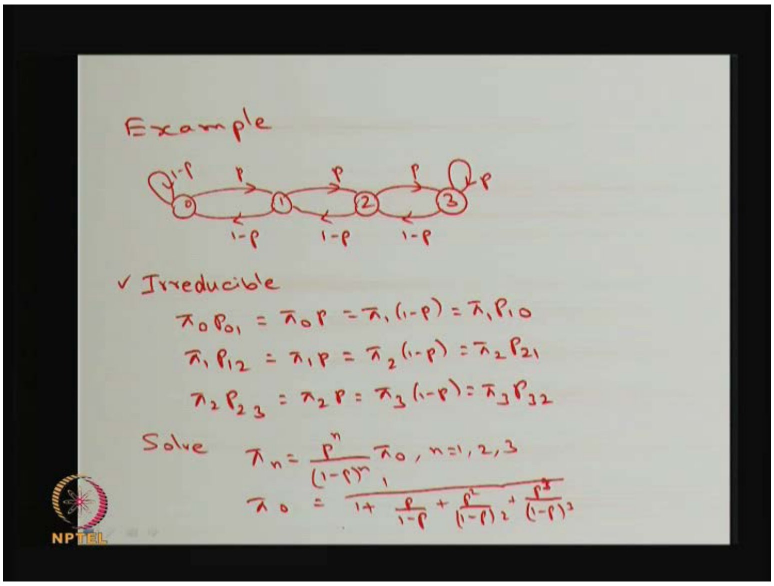

So, I have checked, I have verified whether, I have verified the time reversibility equations after the time reversibility equations from that I am getting the pi n in terms of pi naught, because the way the recursive relation goes you can make out pi 1 from the first equation you can get pi 1 in terms of pi 0. Then the second equation pi 2 you can get it in terms of pi 1 then in turn you can get pi 2 in terms of pi 0. Similarly, you can get pi 3 in terms of pi 0.

Now, you have to find out what is pi 0. pi 0 you can use the normalization equation that is summation of pi i is equal to 1. That is pi 0 plus pi 1 plus pi 2 plus pi 3 that is equal to 1. From that you can get pi 0 is equal to 1 divided by 1 plus p divided by 1 minus p plus p square by 1 minus p whole square plus p cube by 1 minus p whole power 3. So, this is going to be the pi 0 substitute pi 0 in this pi n, therefore you got the, you get the pi n also. So, you are getting the unique stationary distribution, because this DTMC is a time reversible. Therefore, without solving the pi p is equal to pi, you are using the time reversibility equation itself and summation of pi i is equal to 1, you are getting the pi I's. So, that is a easy way whenever the DTMC is going to be a time reversible Markov chain.

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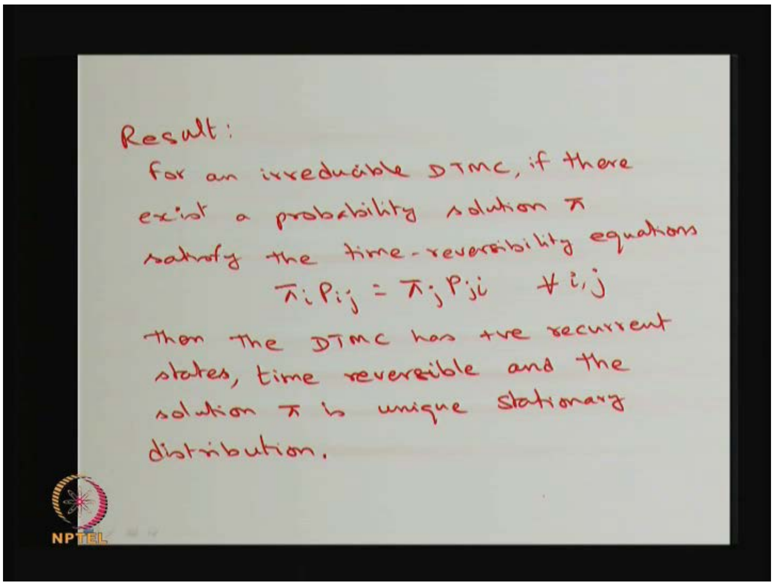

Now, I am moving into the applications of irreducible Markov chain in queueing models. Yes, so in this example we have used time reversibility property to get the unique stationary distributions. The result we said there is a, and the solution pi is a unique stationary distribution, so in the result needs only the proof whether the pi is going to satisfy the equation pi p is equal to pi.

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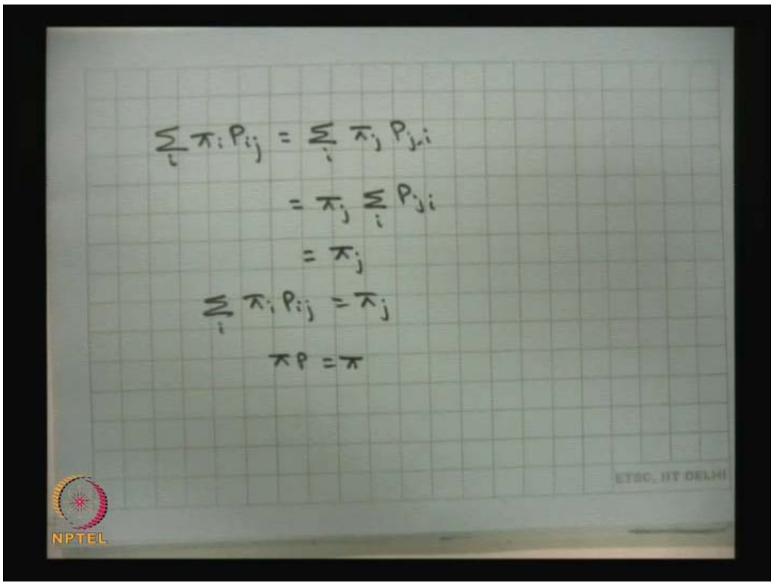

So, that can be easily proved by taking summation of pi i p i j that is same as summation over i pi j of p j I, because it satisfies the time reversibility equation. We can write summation over i pi i times p i j is same as the summation pi j p j comma i. That is same as you can take out the pi j outside, that is same as the summation over i the p j i. You know that the summation over i the p j i that is going to be one therefore, this is going to be a pi j. So, hence we get summation over i the pi i of p i j that is equal to pi j. So, this is nothing but in the matrix form pi p is equal to pi.

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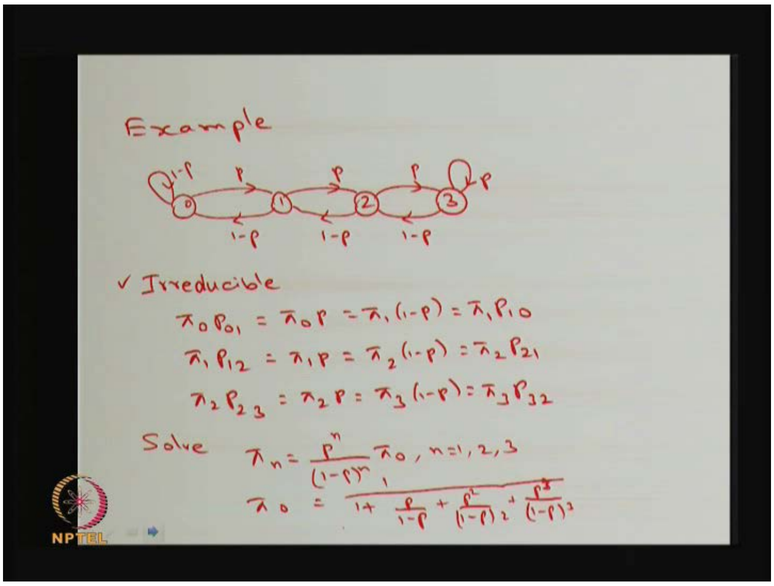

So, whenever you have a irreducible DTMC, and satisfies the time reversibility equations then you have a unique solution pi, that pi unique solution is a stationary distribution. So, and also you can prove easily it has the positive recurrent state and time reversible Markov chain also. So, with this with this proof we have got the result the pi is going to be a unique stationary distribution and also I have given the example how to use a time reversibility equations to get the unique stationary distribution.

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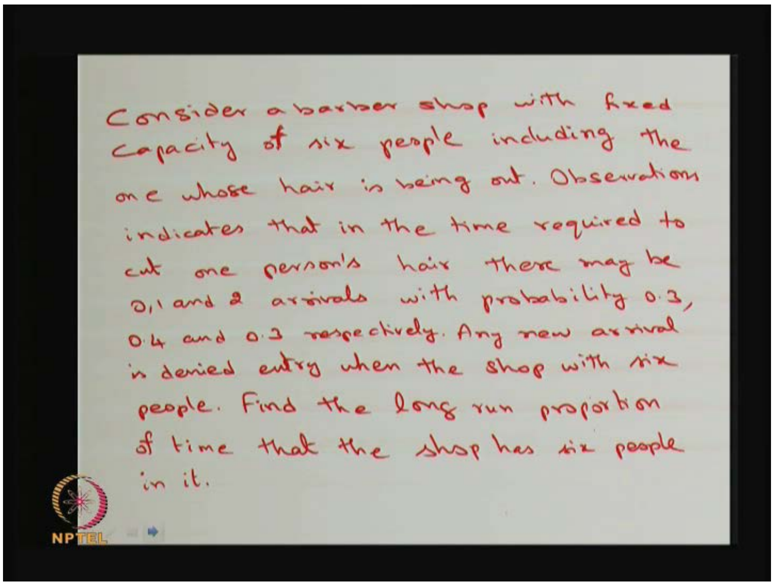

Now, we are moving into the application of irreducible Markov chain in queueing models. For that I am going to start doing the one example through that I am going to explain the queueing models. Later also we are going to give the application of irreducible continuous time Markov chain in queueing models. This is for the discrete time Markov chain.

You see the example, consider a barber shop with a fixed capacity of 6 people including the one whose hair is being cut, hair is being out cut. Observation indicates that in the time required to cut 1 person’s hair there may be 0 comma 1 and 2 arrivals with the probability 0.3 comma 0.2 and 0.3 respectively. Any new arrival is denied entry when the shop with the 6 people.

Find the long run proportion of time that the shop has 6 people in it. So, here we have given only few information that is there is a barber shop, which has the maximum capacity 6 people including 1 is getting the hair cut. That means at any time the number of people getting the hair cut goes from 0 to 6. During the haircut time there is a possibility no one can arrive with the probability .3 or there is a possibility 1 person can arrive during the 1 person’s any 1 person’s hair cut that probability is .4 and also there is a 2 people can arrive during 1 person’s haircut that probability is .3. Whenever the system is with the 6 people already there, that means 1 is under haircut and remaining 5 people are in the waiting, then the whoever enters it cannot be enter into the system, it will be always denied.

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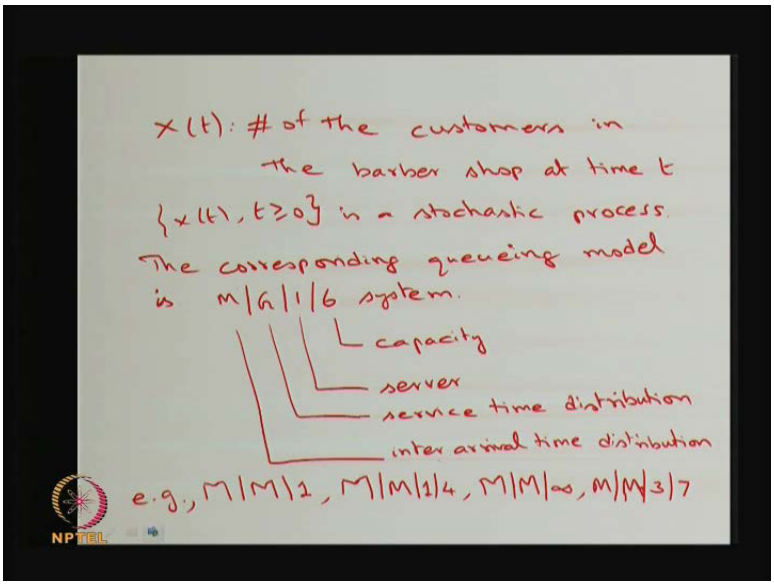

The question is, what is the probability that in a longer run the system will be in the state six? So, this can be analysed with the queueing model, for that we need first stochastic process. So, the stochastic process we need a random variable, so the random variable is here x t denotes the number of customers in the barber shop at time t. For fixed time how many people in the system that is a random variable.

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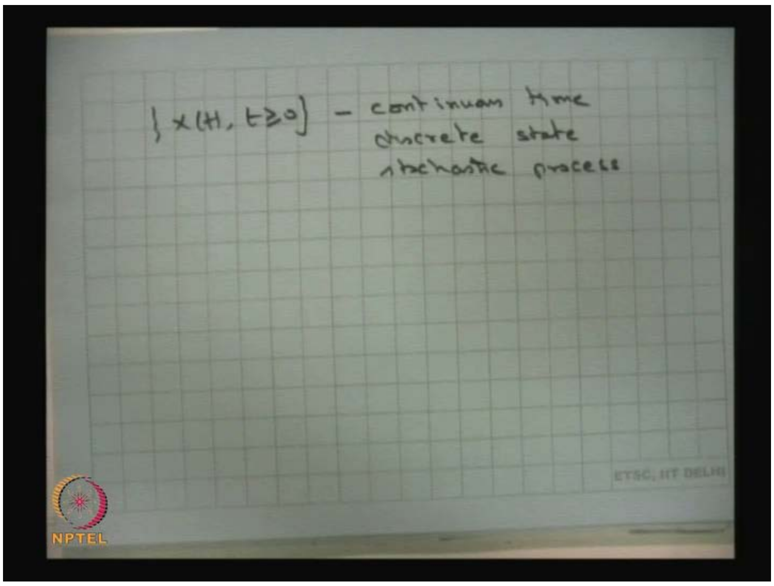

Therefore, over the time if you collect then that is going to be a stochastic process. This stochastic process is the continuous time discrete state stochastic process. This x t is a, x t is a continuous time discrete state, because the possible number of customers in the system that will goes from 0 to 6. So, this is a continuous time discrete state stochastic process.

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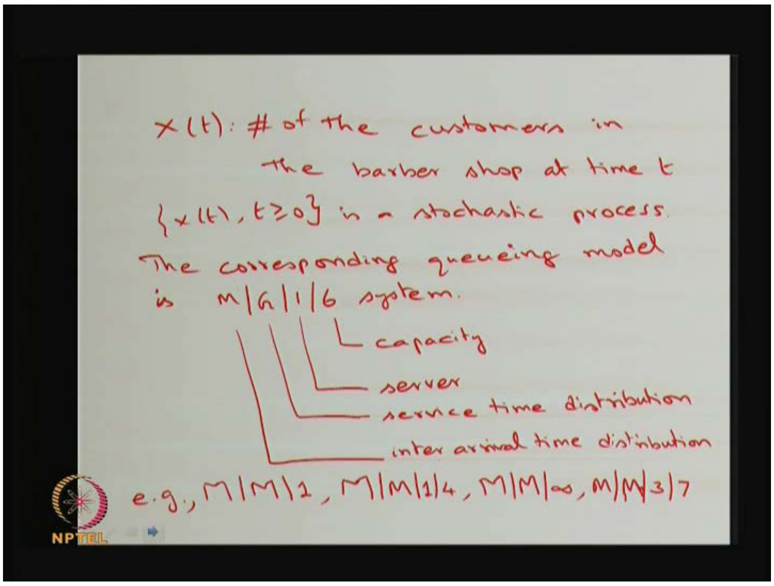

The corresponding queueing model, it is a m g 1 6 system this is a one way of visualizing this is the m g 1 6 system like that you can visualize in a different way, but the queueing system how one can define ? Now, let me explain that the queueing system is the system in which the customers are getting served and leave the system. Here the people who is entering into the barber shop that is going to be the customers who are entering into the system, and the system is nothing but the barber shop, the haircut is nothing but the service time and whenever the haircut is over, then it leaves the system.

So this can written in the Kendall notation with the first letter is denotes for what is the inter arrival time distribution, slash second letter that is denotes for what is the service time distribution, slash the third is denote how many servers in the system. The fourth one what is the capacity of the system that is what is the maximum number of customers can be stay in the system, stays as well as getting the service. The 5th one is nothing but what is the population, the population of the input source that is usually make it as the infinite. Therefore, we would not write it, here I did not write it here.

The next one is what is the service discipline? Here, the default, default is a first come first serve. So, that also we would not write if it is default, so the default input source capacity that is infinite, and the default service discipline is first come first serve, that also we would not write whenever it is changes then we write down in the notation. So, here this is the m g 1 6 system, because the inter arrival time that satisfies the Markov property that is the assumption we have taken here therefore, it is a m denotes for Markov property satisfied or the inter arrival time distribution is a exponential distribution.

The g is for service time distribution which is general here. If the service time distribution is exponential then we write m there, because the Markov property is satisfied. In this model we have only one server, therefore the third letter denotes 1. The fourth is the capacity, the system capacity here that is 6, the capacity of the system means maximum number of people can wait in the system, and the person who is getting service. So, here the capacity of the system is 6, like that we can give many more examples.

Here I have given m slash m slash 1 that means inter arrival time distribution is exponential, then service time is exponential which is independent of the arrival. Only one customer only one server in the system, since the fourth letter is not there that means it is a default the default capacity of the system is infinite. The second example is m m 1 4, that means similar to m m 1 system, but here the capacity of the system is 4. The third example m slash m slash infinity, that means it is inter arrival is exponential, and service is exponential, whereas the number of servers in the system that is also infinite, that means it is sort of a self service discipline, therefore the customer himself act as a server.

The fourth example that is m slash m slash 3 slash 7, that means it is 3 servers in the system, the capacity of the system is 7 that means at most 3 people can get service, at most 4 people can wait in the system. Therefore, the maximum capacity for the capacity of the system that is 7. So, I have given a very simple way of explaining the queueing system, but in this example we have I have explained how the queueing system is attached to with this problem, I have given the interpretation, but later we are going to give the different type of queueing models which is going to be the application of irreducible continuous time Markov chain that will be the later.

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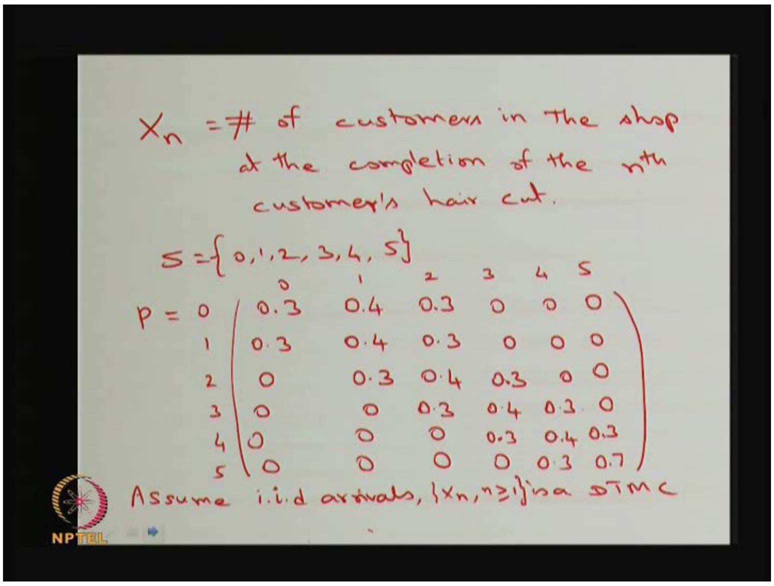

With this the information given in this problem I can go for writing another random variable that is x n x suffix n that is nothing but number of customers in the shop at the completion of the n th customer’s haircut. This is also going to be a stochastic process, because over the n I can collect the random variables and that will form a stochastic process, but this stochastic process is going to be a discrete time, discrete state stochastic process.

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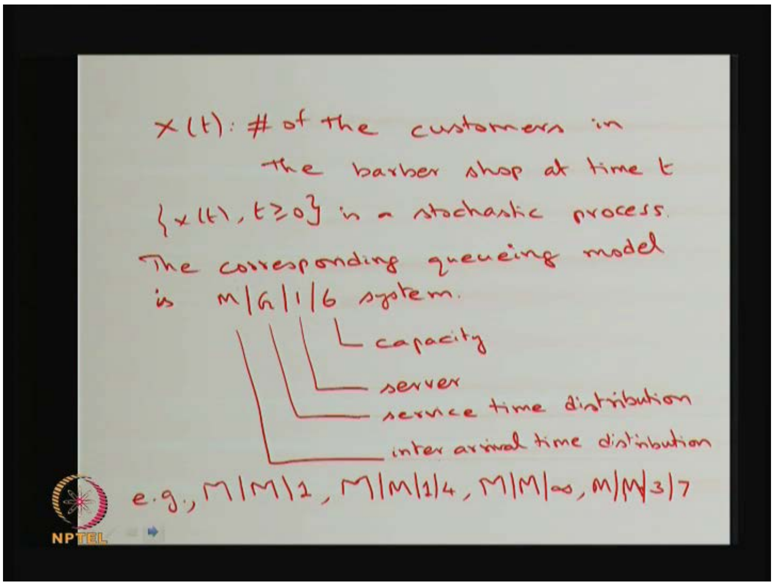

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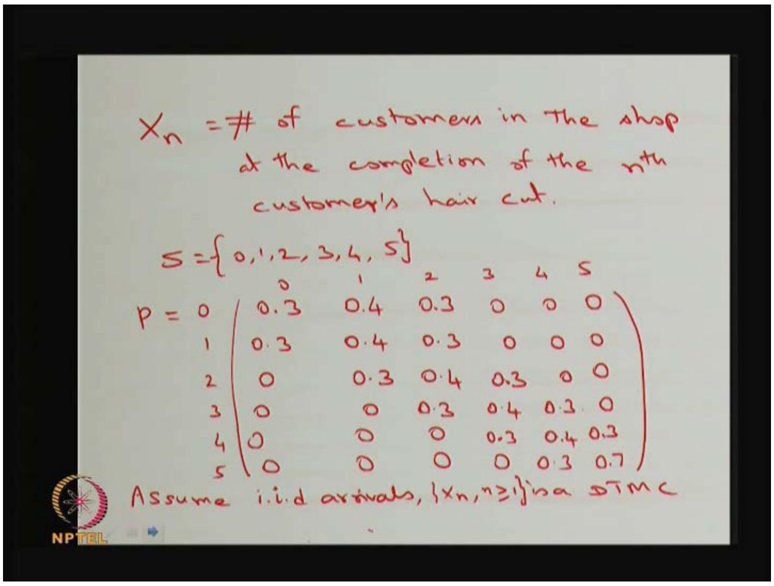

Because, at the completion of every customers we will find maximum 5 customers in the system behind him. Therefore, the possible state space, the state space is going to be 0,1,2,3,4 and 5. This stochastic process that x n is embedded in the stochastic process x of t. The x of t is the stochastic process that is the continuous time discrete state stochastic process, whereas x n you are observing the system at the completion of n th customer haircut.

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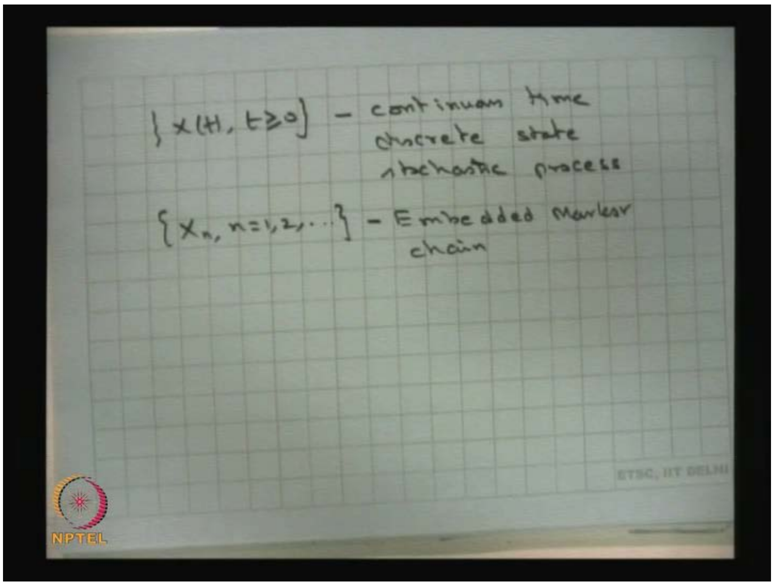

Even if you can say it is a after the after the n th customer's leaves excluding him, how many customers in the system that is going to be the, that is going to be the state space. So, you are observing the system at the completion of a first customer, then second customer, then third customer, and so on. Therefore, this is embedded in x of t. Therefore, this stochastic process is called embedded Markov chain. Why it is Markov chain? You can verify the x n the stochastic process x n x suffix n n is running from one, two and so on.

This is a stochastic process, this stochastic process is going to be a embedded Markov chain. The reason is, this is a stochastic process and also satisfies the Markov property. That means what is the probability that the system will be in some state in future given that, you know the state at the present as well as the past. That is same as what is the probability that the system will be in the future in some state given the present, not the past.

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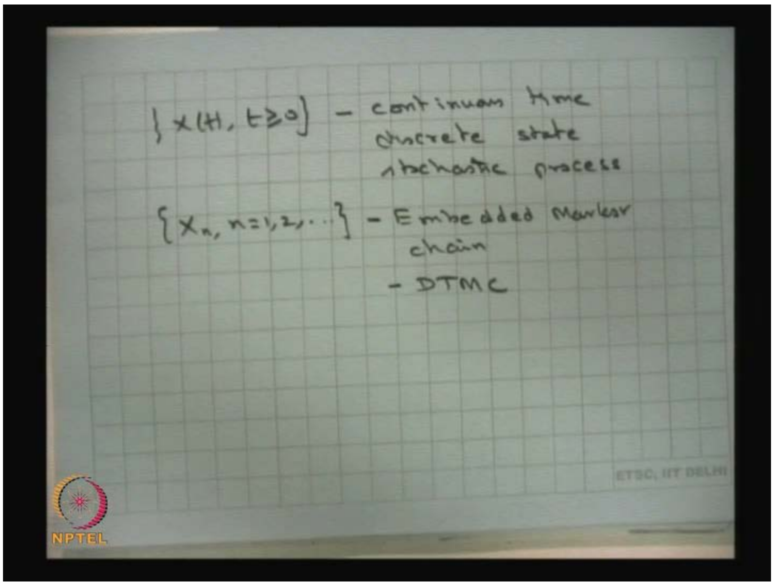

So, that Markov property is going to be satisfied for this discrete time, discrete state stochastic process. Therefore, this is going to be a Markov chain. So, the x n is a DTMC. Since it is embedded in x of t this is called a embedded Markov chain. I am not giving the Markov property, how it is satisfied, but one can analyze, because the inter arrivals, the arrivals are i i d random variables.

Whenever how many customers are entering into the, during the one’s haircut, whether 1 customer is entering or 0 customer is entering or 1 customer is entering or 2 customer is entering during the 1 person’s haircut. If you assume all the arrivals are i i d random variables then whenever you see the completion time, and you collect those completion time those random variable, and that random variable satisfies the Markov property. Therefore, this is called DTMC and also call it as a embedded Markov chain.

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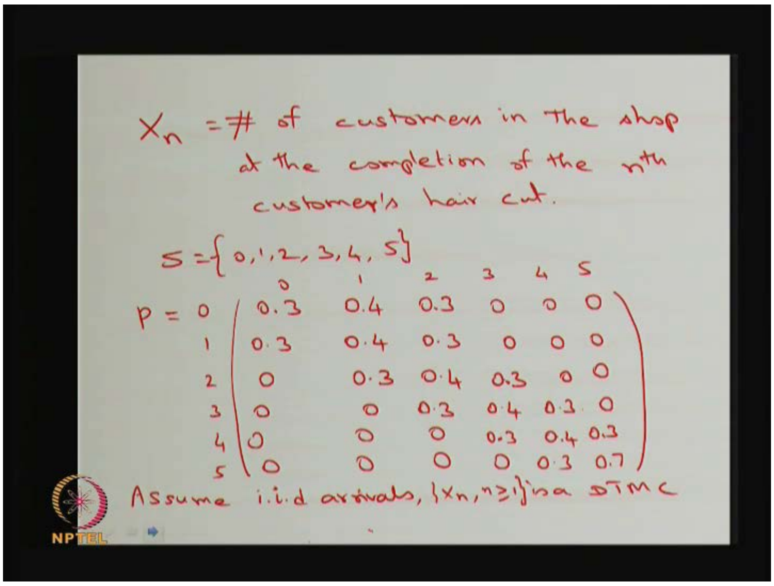

Now, you can find out what is the one step transition probability for this embedded Markov chain that i have presented here that is a p matrix because the possible states are 0 1 2 3 4, and 5 therefore, I have written the six cross six matrix. The entries are first let me explain what is the probability that, what is the probability that the system was in the state zero. When, so before explaining the p matrix let me give the trace then after giving the trace or sample path then I will come back to the one step transition probability matrix.

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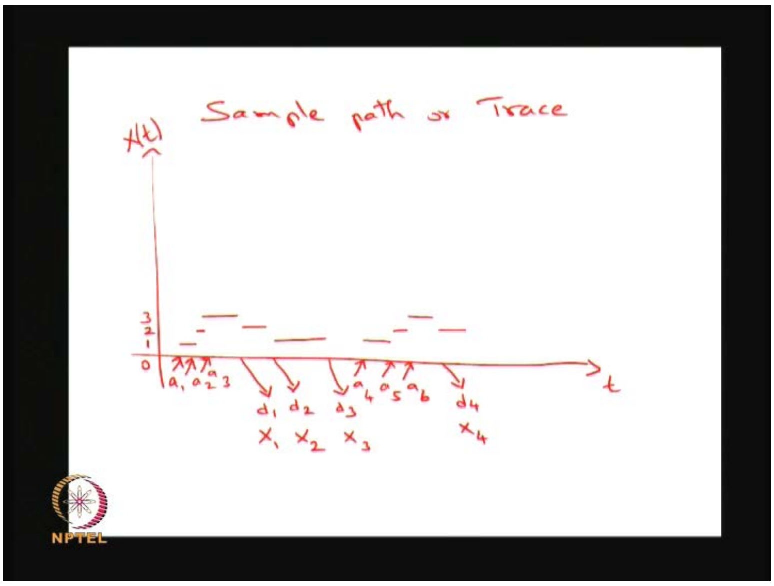

Because, through the sample path or trace one can easily visualize how you can get the entries of one step transition probability matrix. See this is a sample path, you know the meaning of sample path, this is the one possibility like that you can go for many possibilities. This is called the sample path or the trace, so over the time over the times, so I have made it wrongly. So, this is over the time, and this is number of customers in the system at time t.

So, the, a suffix 1 denotes the first arrival, the a 2 denotes the second arrival with the probability point 3, the possibility of no arrival, or with the probability point 4 one arrival, with the probability point 3 two arrivals. So, here the a 2 denotes the second arrival, but that is the two arrivals together, therefore the system goes from 1 to 3. Whereas the service is one by one. So, the d 1 denotes the time at which the first customer leaves the system that is the first departure.

So, the departures are with the notation d’s and a’s for the arrival. At the time of d 1 you are observing the system, therefore that is x suffix 1. So, how many customers in the system at the time point d 1 that is how many customers in the system at the completion of the first customer leaves that is going, that denotes the x 1. So, since there are 3 customers in the system, initially 1 then 2 more customers entered, therefore the total number is 3 and 1 leaves, so when he leaves he will see 2 customers in the system. So, x

1 is 2, x 2 is the second customer departure, when second customer departure already one will be there, so x 2 is equal to 1.

During the second customer, see during the first customer there are 2 customers enter into the system, whereas during the second customer’s service no one arrived. Therefore, the d 3 is the third customer leave the system, and x 3 is the number of customers behind him when the third customer leaves. So, that is going to be 0. You would not bother when the a 3 comes and a 4 comes, and so on. The next the embedded Markov chain that state that random variable is x 4, that is when the fourth customer leaves, so you would not bother when the a 3 third arrival comes or the fourth arrival comes.

Whereas, here the d 4 is, now there are again 2 customers enter into the system. Therefore, first 1 customer enter into the system, and during his service another 2 customers enter, therefore 3, therefore x 4 is going to be 2, because when he leaves he will find 2 customers in the system. So, this is the possible values of x 1, x 2, x 3, x 4, this is going to be the possibilities. So, this is a sample path, now you have explain, now you do not known how to, what is the value of x n’s or how to get these values. Now, I am going to explain how you can get the entries of the p matrix.

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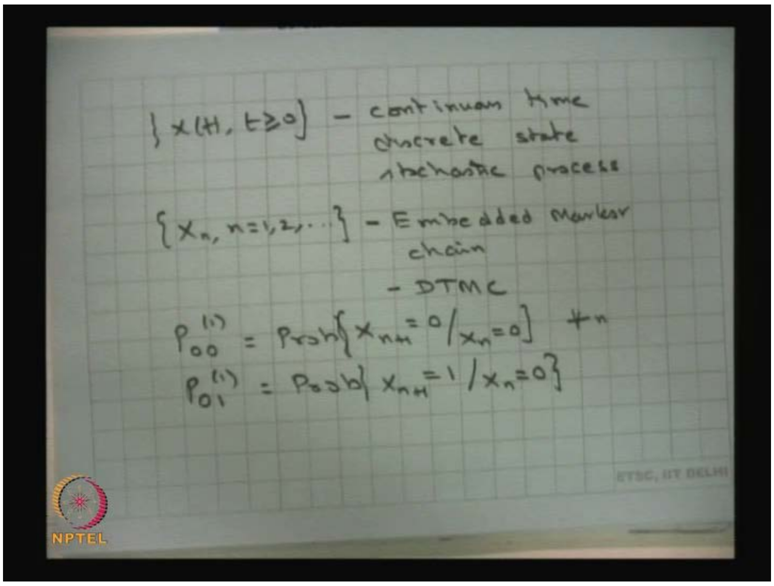

What is the meaning of 0 to 0 p 0 0? So p 0 0 is nothing but, what is the one step transition probability values. So, the p 0 0 in one step that is nothing but what is the probability that the x n plus 1 is equal to 0 given that x n was 0. So, this is the one step transition probability for all n, for all n what is the probability, what is the transition probability of system is moving from the state zero to zero from the n th step to n plus one th step.

That means when n th person leaves, when n th person leaves he did not see anyone in the system that means the system was empty after he leaves. When the n plus one th person leaves, he also sees no one in the system behind him, therefore x n plus 1 is equal to 0. So, what is the transition probability of the system is moving from the state zero to zero at the end of n th completion time, n th customer completion time, and at the end of n plus one th customer's completion time.

That means during the n plus one th customer's arrival time, no one enter into the system. You do not want to bother what time the n plus one th customer enter into the system that is immaterial, because we are considering only the completion time, or the time epochs at the completion time. Therefore, this is nothing but this situation is nothing but no one enters during the n plus one th customer's haircut. That probability you know that is the given, so that probability is a 0.3 that means 0 customers enters during one person's haircut. Here we assume all the, all the person's are identical, all the arrivals, and so on. All are i i d random variables, therefore here whether it is a n, and n 1, and so on for all n it satisfies, therefore this probability is .3.

Similarly you can go for what is the probability of the system is moving from the state zero to one in one step, that is nothing but what is the probability y that x n plus 1 will be in the state one given that x n was 0. That means when n th person leaves he did not find anyone in the system the system was empty.

When n plus one th person leaves, he find one behind him. That means when the n th person leaves no one in the system, therefore the n plus one th person would have arrived at sometime, during his service there is a possibility of one customer's entry into the system. Therefore, when he leaves he finds the person who enters during his haircut. That means what is the probability that one customer's enters during one person's haircut that probability is 0.4, so that is this one.

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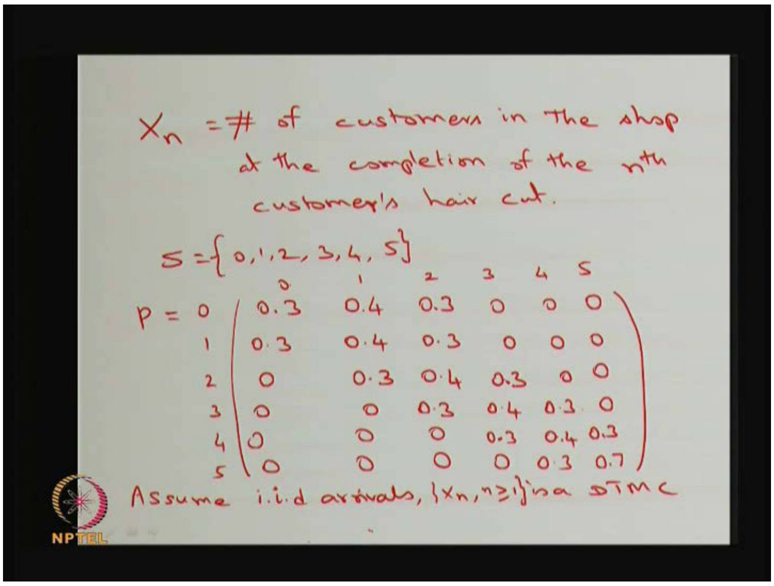

Similarly, you can go for finding out what is the entity of p 0 2, p 0 2 that is nothing but the system is moving from the state zero to two in one step. That is nothing but two persons would have arrived during the n plus one th customer's haircut time. So, that probability is 0.3. The other possibilities are not possible, because we have either 0 customer or 1 customer or 2 customers can enter into the system, therefore the other probabilities are 0.

Now, you move into the second row. What is the probability that the system goes from the state one to zero, one to one, one to two, and so on? This also can be discussed in the same way which I have explained in the first row. Here when he leaves he has seen one person, when the n th person leaves he has seen 1 person, therefore the system was in the state one.

When the n plus one th customer leaves, what is the, what is the, what is the probability that he sees no one in the system. That means during the time no one arrived, that is during the haircut time no one arrived. Therefore, that probability is 0.3. Now, when the n th person leaves he saw one, 1 customer in the system. When n plus one th customer leaves he also saw 1 person in the system. That means during his haircut time one is arrived, so that probability is 0.4. Similarly, you can go for the system going from state one to two in one step that is going to be the 0.3. And, all other possibilities are not there therefore, that probabilities are 0.

Now, we will come to the state that is corresponding to p 2 0, 2 0 means when someone leaves he saw 2 people, when the next person leaves he saw 0 people. That is not possible because one by one we are taking a random variable x n x n plus 1, and so on. Therefore, there is no way when the next person leaves he cannot see 0 people in the system, because after the service is over then only he can leave the system. Therefore, the system going from two to zero that is not possible that probability is zero.

Whereas the system is going from the state two to one that is possible. That means when n th person leaves he saw 2 people, when n plus one th person leaves he saw 1 person, that means during his time no one arrived, so that probability is point 0.3. Then 0.4 and 0.3 the same way you go for the argument for the other rows, the last row is again important. The system going from five to four, that means one decrement, that is because of no one arrived, therefore that probability is 0.3. Whereas, the system is going from the state five to five, when the n th person leaves he saw 5 people in the system, that means the system was full during a the n th person's haircut time. Therefore, the since the capacity is 6, that means maximum 5 people can wait, and 1 person is under service.

Therefore, when the system is in the state five at the completion of n th person's haircut, that means the system is full. So, if during his haircut time if 1 people or 2 people arrive, if the 2 people arrive then the still the system will be in the system 6 only. Not more than that, therefore the possibility of the 1 person arrive, and the possibility of 2 person arrive, still the system will be with the maximum capacity 6. Therefore, when the n th person leaves he see 6 people, when the n n plus one th person leaves he saw he will see again 5 people, either because of 1 person arrived during his haircut or 2 people arrived during his haircut. So, you have to include both, therefore 0.4 plus 0.3, therefore it is 0.7.

You can verify all the row sums that is going to be one, and the entities are zero to one. So, this is the one step transition probability matrix corresponding to this problem. This problem has an embedded Markov chain. The original stochastic process is number of customers in the system at any time t and that is a continuous time discrete state stochastic process that is a non-Markovian model. But, when you are making embedded, that means we are finding the time epochs in which the Markov property satisfied, and those time epochs at the completion of every haircut time the Markov property is satisfied. Therefore, you land up by embedded Markov chain, and this is the, this is the example or this is the application of discrete time Markov chain in the queueing model.

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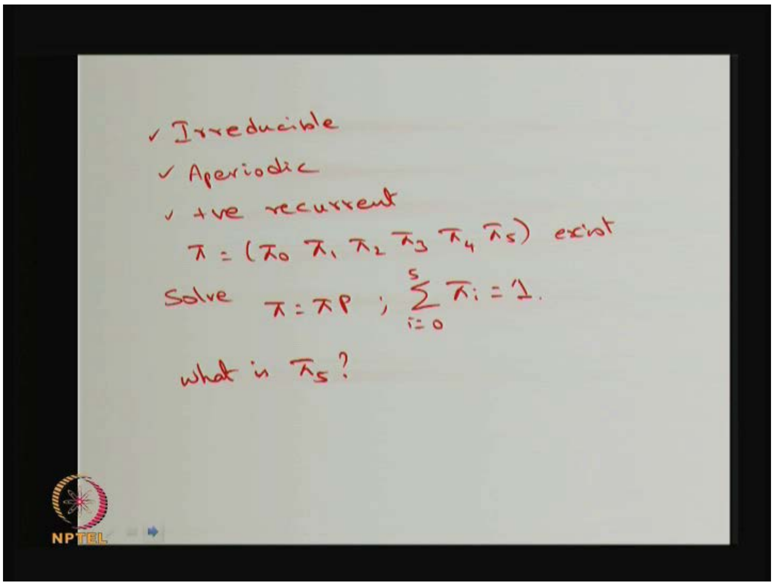

This is the DTMC because of the assumption of i i d arrivals. Now, you can verify whether it satisfies the irreducible property. Because, the way you have Markov, the one step transition probability matrix from that you can make out this is a, each state is communicating with each other state, all other states or you can draw the one state transition diagram, and you can conclude it is a irreducible Markov chain. That means you have a one closed communicating class and each state is communicating with all other states, and that is same as the state space.

So, the whole state space is same as the, the one closed communicating class, therefore this is a irreducible Markov chain. Next aperiodic, you can find out this is a finite model. It is a positive recurrent state, you can find out the periodicity of the recurrent state since it is, it has the self loop with the positive probability, and so on. You can conclude the greatest common divisor of system coming back to the same state that numbers that number is going to be one. Therefore this is a aperiodic irreducible, and also you can verify it is a positive recurrent, it is a finite model irreducible, therefore it is going to be a positive recurrent.

You know the results for any irreducible aperiodic positive recurrent Markov chain, the stationary distribution exist, and that is going to be unique and you can find out by solving pi p pi p is equal to pi with summation of pi i is equal to 1. So, you solve this equation, because this a homogenous equation used in this normalizing condition summation of pi i is equal to 1, you will get the vector pi with the entries values pi 0 to pi 5. Once you get the pi values, you can go for what is the question is asked.

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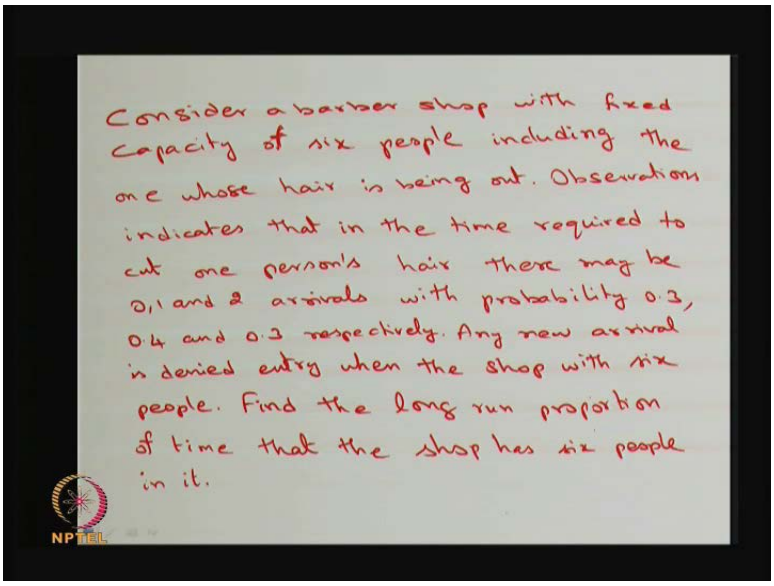

The question is find the long run proportion of the time the shop has the 6 people in it. So, here this is nothing but in a long run what is the probability that 6 people will be in the system? Suppose you analyse the system with the stochastic process x of t then you would have got the long run proportion of the time, but that is same as we are using the result, that is same as if you study the embedded Markov chain, and that embedded Markov chain the steady state probability that is nothing but, if you want to find out the long run what is the probability that the system has the 6 people in it, that is same as in a longer run what is the proportion of the time the shop has 5 people at the end of the completion time.

So, the steady state probability of 6 people in it, there is a long, long run proportion time that is same as what is the stationary distribution in a, or the limiting distribution the 6 people in it that probability is same as at the time epochs the completion time epochs. What is the probability that in a longer run 5 people in the system at the time epochs, therefore that is same as the pi 5.

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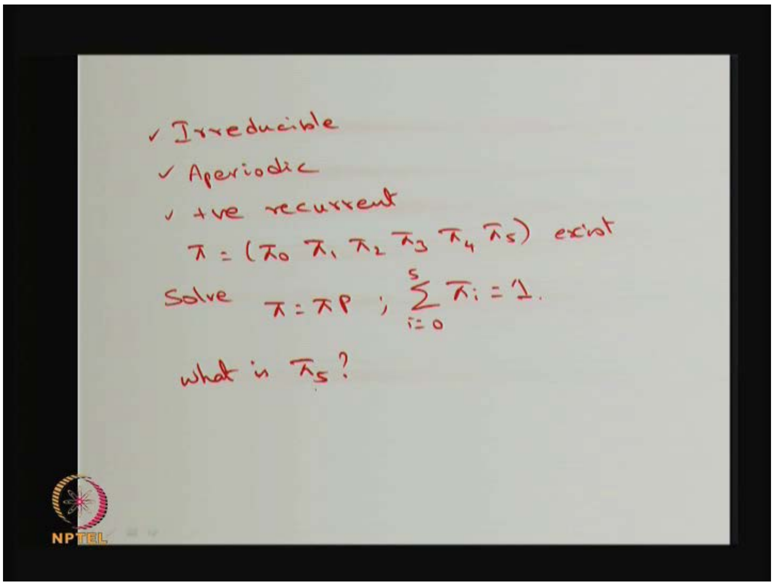

Because, this is the pi 5 is the stationary probability that 5 customers in the system at the completion of the time epoch in a long run. So, whatever you are getting the probability values for the pi 5 that is going to be the proportion of the time the system has a 6 people in it in a longer run.

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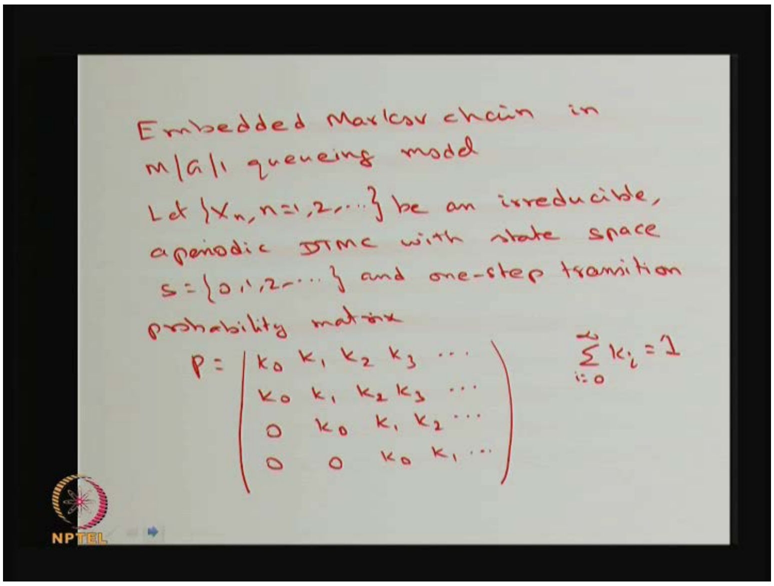

So, this same example can be extended with the model called m g 1 queueing model, because the, this example is the m g 1 6 model. Whereas, you can go for m g 1 queueing model, and in this model also you have a embedded Markov chain, that is related with the random variable x suffix n. That is going to be the same thing what is the or the number of customers in the system at the time epochs of completion of n th customer service. So, that is nothing but that is the x n.

That will form a discrete state discrete time stochastic process, even though the service is going to be a non-exponential distribution whenever the service is completed. You see how many customers in the system, because the customers inter arrival time is exponential distribution which is independent of the service time. Only 1 customer in the, only one server in the system. First come first out service policy with the infinite capacity in the system, the Markov property is going to be satisfied at the service completion time epochs.

Therefore, how many customers in the system after the service is completed, after the time at which the n th n th customer's service is completed. If you find out that random variable and collect those random variable over the customers x n for n is equal to one two and so on. That random variable is going to, that stochastic process is going to be satisfies the Markov property. Therefore, that is called a discrete time Markov chain, and this is also embedded Markov chain, because that is embedded in the stochastic process x of t.

If you assume that, this is going to be a irreducible aperiodic, and this also has the one step transition probability matrix p of the form with the entries k naught, k 1, k 2. Here the k naught, k 1, k 2, k threes are similar to the number of arrival, arrival of customers in the system during the 1 person's service. So, that is so the k i is nothing but what is the probability that i customers arrived during the any n th any n th customer's service.

Therefore, since it is the embedded one, so the first row and the second row will be one and the same. Then you will get the other entries, therefore the way I have explain the previous example that has the finite capacity model, the finite state discrete time Markov chain. Whereas, this is a, it is a infinite state, infinite state space DTMC. Therefore, I have made it with the p matrix in this form.

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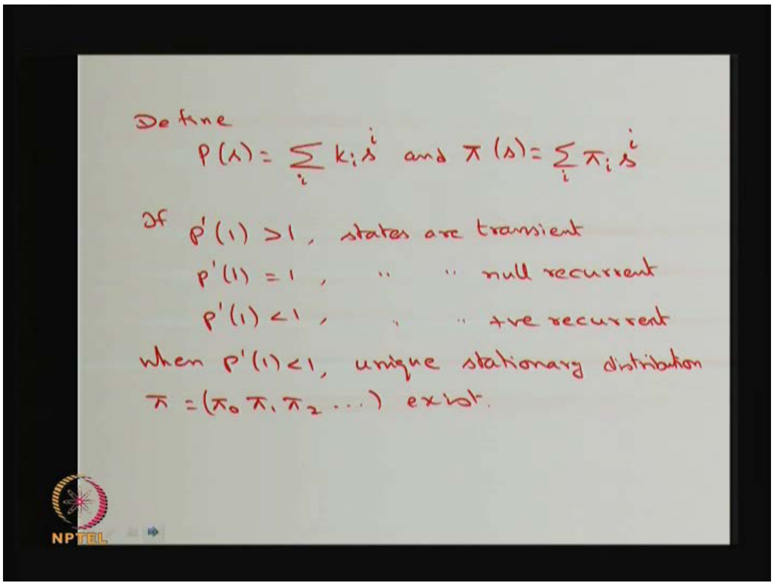

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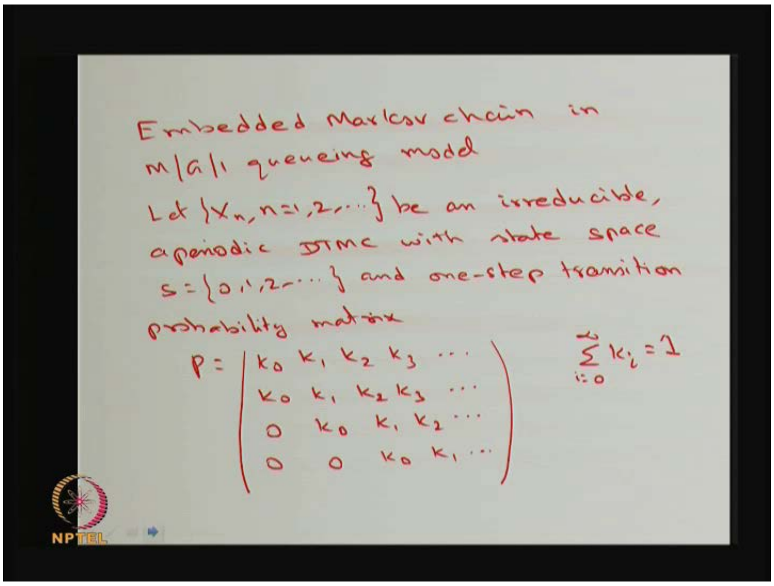

Our interest is to find out what is the stationary distribution. For that I am introducing a two generating functions one is a p of s, that is a related with the number of customers arrived during the n th customer, the other one is the stationary distribution that is pi of s. If you see I did not make the assumption whether it is a positive recurrent or null recurrent or transient, and so on. So, here I made the assumption only it is irreducible. So, if the probability generating function p of s if you differentiate then substitute s is equal to 1, if that value is going to be greater than 1 then all the states are going to be transient.

If that value is going to be 1, then all the since it is a irreducible all the states are going to be a null recurrent. If the p dash of 1 is less than 1, then all the states are going to be a positive recurrent. You can cross check using the simple way of checking the positive recurrent null recurrent by mean recurrent time is going to be a finite, the infinite also. So, if this is the condition then you can classify the states as a transient null recurrent or positive recurrent. Now, you make the assumption it is going to be a positive recurrent by satisfying the condition p dash of 1 is less than 1.

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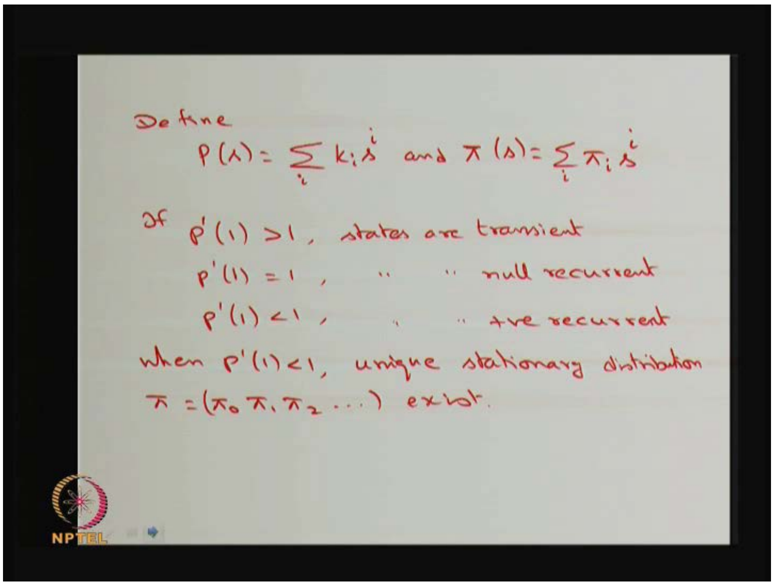

So, since it is a positive recurrent irreducible, and also I have taken a aperiodic, if it is not aperiodic then it is a different thing. So, I made the assumption it is irreducible aperiodic, and satisfying the condition p dash of 1 is less than 1. Therefore, it is going to be a positive recurrent. So, the unique stationary distribution exists.

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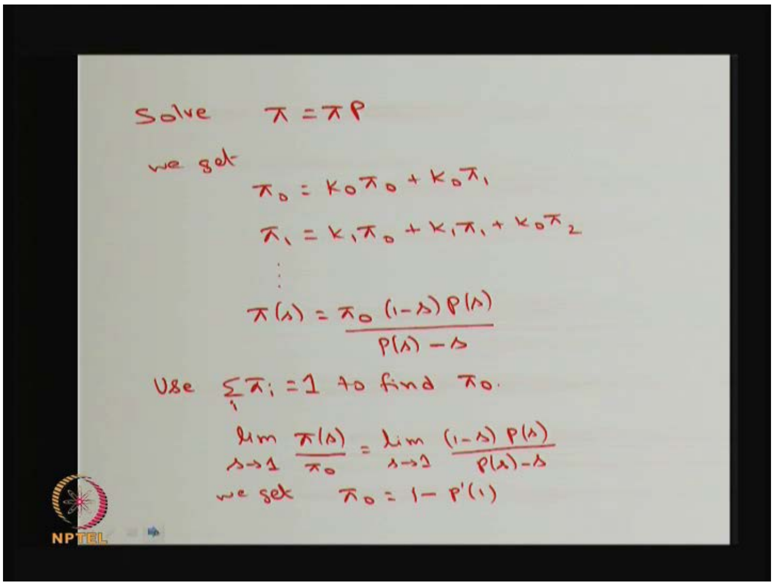

Now, will explain will see how to find out the stationary distribution. You have to solve pi is equal to pi p, you know the p matrix, so you take the first equation pi naught is equal to k naught pi naught plus k 1 pi 1. From these you will get pi 1, and the second equation you can get pi 2 in terms of pi naught, and so on. Instead of these, you can use the generating function that means you multiply s, pi 1 s the second third equation you can make multiplication of s square, and you add you will get the generating function. So, you will get the pi of s is equal to in terms of pi naught with p of s.

Use summation of pi i is equal to 1 to get the pi naught. So, you can find out pi naught by using this concept a limit s tends to 1, the pi of s divided by pi naught is same as limit s tends to 1, 1 minus s times p s divided by p s minus 1. So, you can get pi naught that is same as 1 minus p dash 1, and already we made the assumption p dash of 1 is less than 1, therefore this quantity is going to be a positive quantity pi naught.

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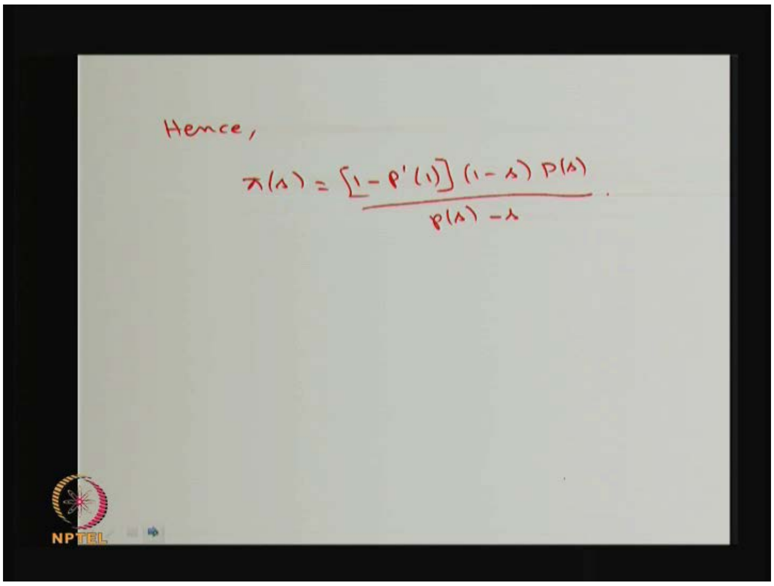

Hence you substitute pi naught in the pi s expression, therefore you will get generating function for a stationary distribution. By taking the once you know the generating function you can get the probabilities. This stationary distribution in terms of p s, you know what is p s also, because the number of arrivals that probability is given, therefore you know the probability generating function for the arrivals also. So, you know p s, using the p s you can get the generating function for the stationary distribution, therefore you can get the probabilities.

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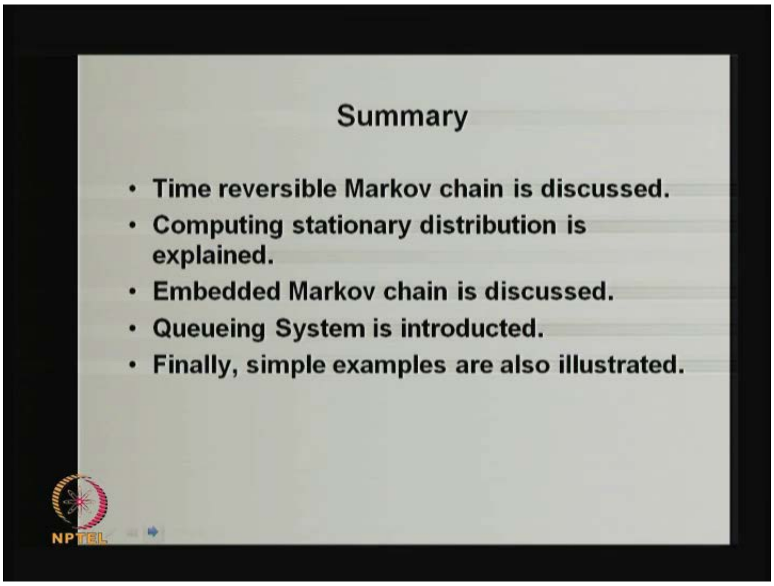

As a summary we have finished the time reversibility Markov chain, and we have explained embedded Markov chain also. I have explained the queueing system, I have also given the simple examples. And these are all the reference books for this lecture.

Thanks.