Stochastic Processes

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Module - 4
Discrete-time Markov Chain
Lecture - 5
Limiting Distributions, Ergodicity and Stationary Distributions

Good morning. This is module four, lecture five – limiting distributions, ergodicity and stationary distributions. In the last four lectures, we have discussed the discrete-time Markov chain starting with the definition transition probability matrix. Then in the second lecture we have discussed the Chapman-Kolmogorov equations. Then we have discussed the one-step transition probability matrix followed by that, we have discussed the n-step transition probability matrix. In the lecture three, we have classified the states of the discrete-time Markov chain as a recurrent, that is a positive recurrent and null recurrent; transient states, absorbing state and periodicity. Then we have… In the fourth lecture, we have given simple examples. In the fifth lecture, we are going to discuss the limiting distributions, ergodicity, stationary distributions.

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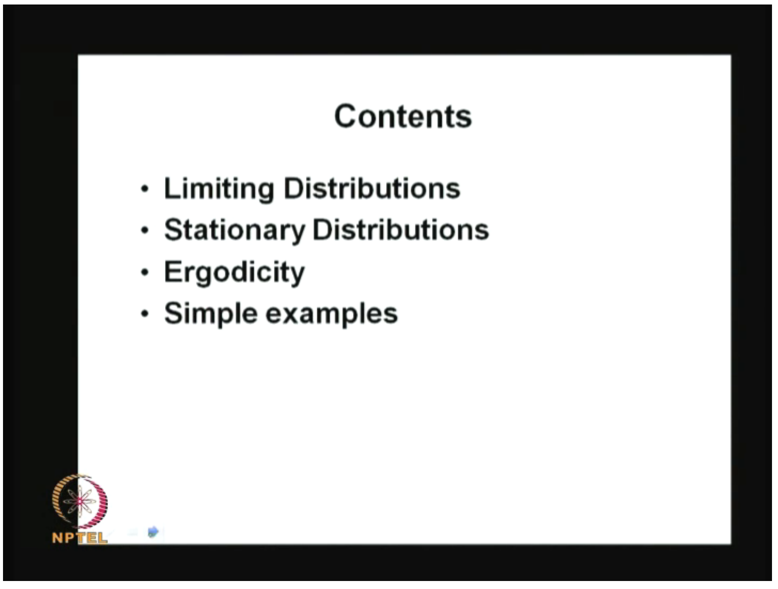

If I am not able to complete limiting distribution and the ergodicity, then I will discuss the stationary distribution in the next lecture. And followed by the limiting distribution and the ergodicity, I am going to give some simple examples also.

The introduction – what is the meaning of limiting distribution. It is very important concept in time-homogeneous discrete-time Markov chain. And the limiting distribution is going to give some more information about the behavior of the discrete-time Markov chain. And before I move into the limiting distribution, let me discuss some of the important results; then I am going to give the limiting distribution.

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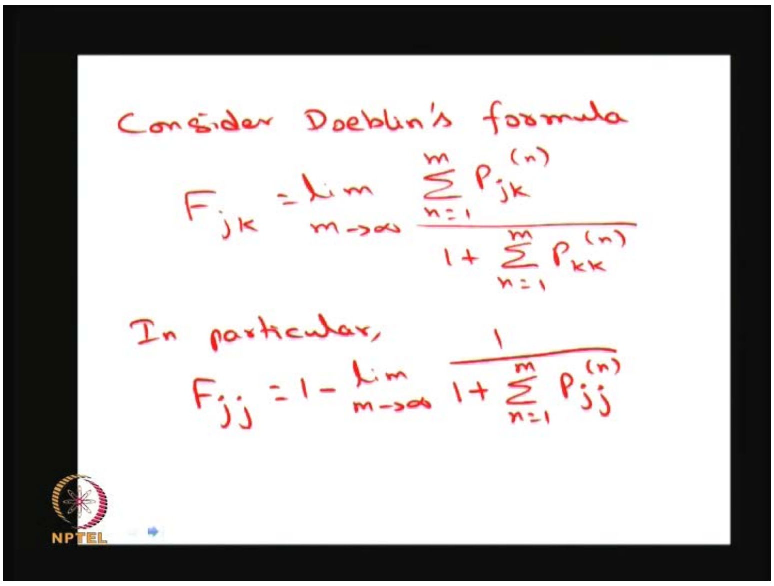

Consider the Doeblin's formula; that is F j k in terms of limit m tends to infinity of summation.

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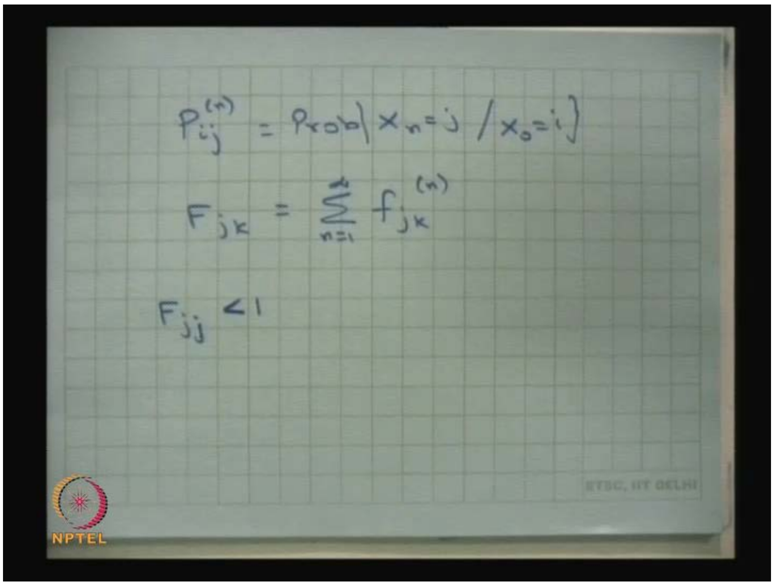

We know that, the P i j of n is nothing but what is the probability that the system will be in the state j given that the system was in the state i; whereas, the capital F j k can be written as in terms of f j k n; where, n is running from 1 to infinity. Here the small f j k of n is nothing but the first visit to the state k starting from the state j in nth step. And all the combinations of n steps – that will give capital F j k. So, now, you see the capital F j k is nothing but the limit m tends to infinity the summation divided by 1 plus the summation. In particular, we can go for k equal to j. So, that is nothing but 1 minus this.

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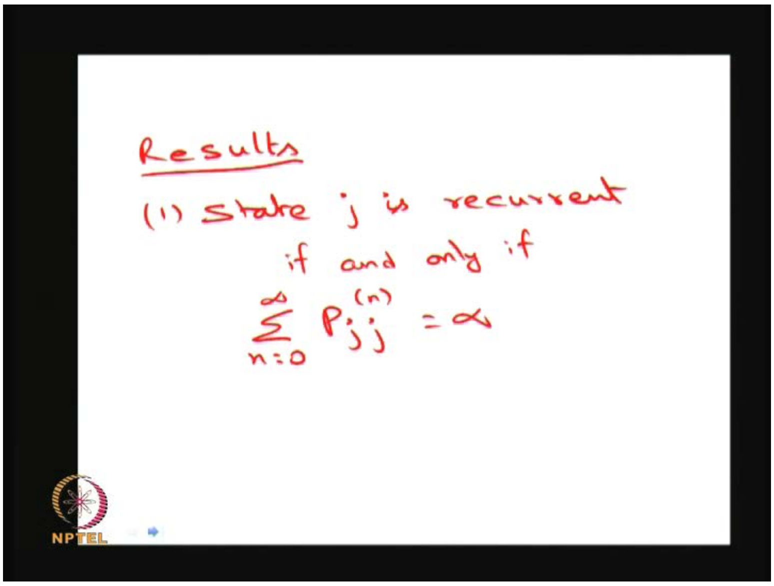

Now, based on the state is a recurrent transient and so on. I can discuss the further results. The first result – the state j is going to be a recurrent if and only if the summation of p j j of n has to be infinity. The if and only if means if the state is recurrent, then you can come to the conclusion this summation of the probability, not the first visit, starting from the state j to j in n steps; that summation is going to be infinity. If for any state j, the summation is going to be infinity, then that state is going to be recurrent.

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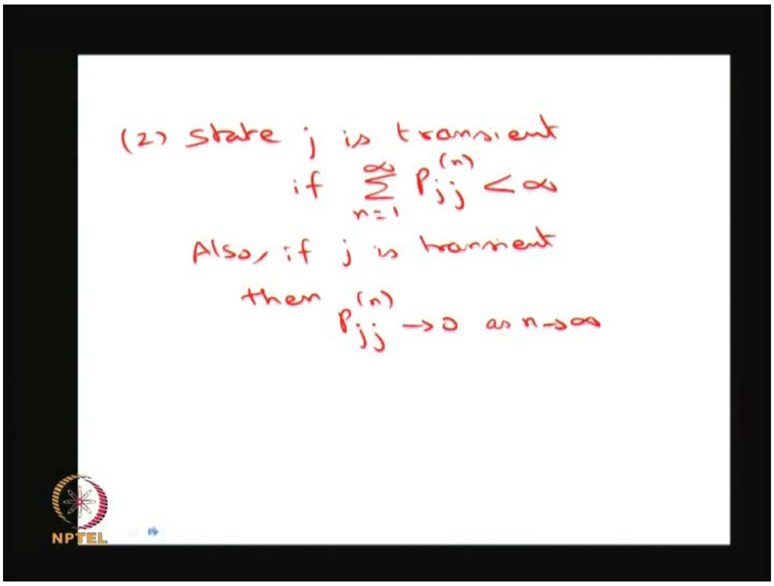

The second result – suppose the state is transient; then you can have the p j j of n tends to be infinity as n tends to infinity. This you can conclude easily. If the state is a transient, then you know that, the F j j is going to be less than 1. The probability of the system coming back to the state is going to be less than 1. Therefore, the p j j of n tends to infinity as n tends to infinity for the transient state.

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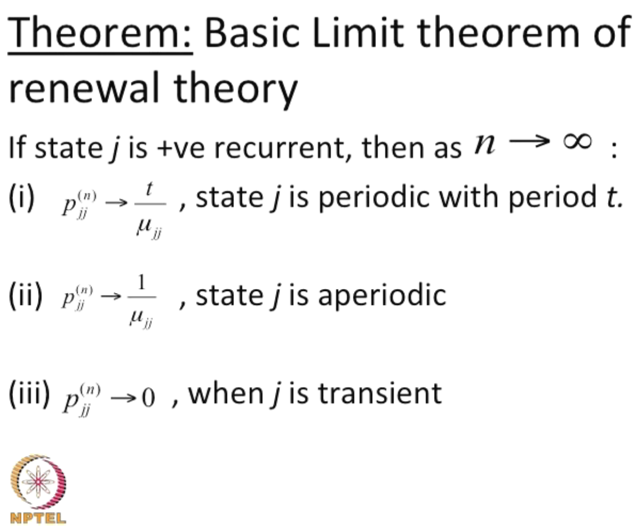

And also, if the summation is going to be a finite quantity, then you can conclude the state is going to be transient. Based on this, I am going to give the next theorem, that is, basic limit theorems of renewal theory. I am not giving the proof here; I am just only stating the theorem. If the state j is a positive recurrent; that means the state is going to be a recurrent as well as it satisfies the positive recurrent property; that means the mean recurrence time is going to be a finite value for that state j.

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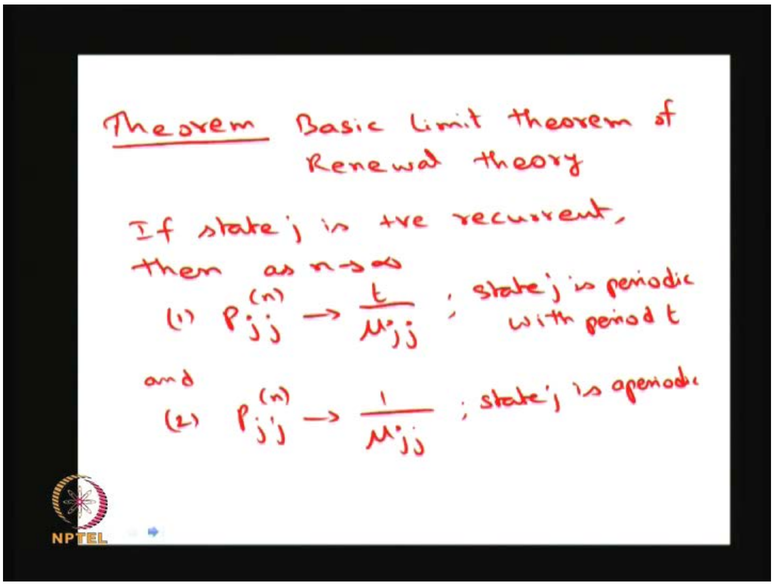

Then, the p j j of n – that will tend to t divided by mu j j; where, mu j j is nothing but the mean recurrence time for the state j; and the t is nothing but the periodicity for the state j. If the periodicity is going to be 1; then as n tends to infinity, the p j j of n – that is nothing but what is the probability that the system starts from the state j and reaches the state j in n steps, will tend to 1 divided by the mean recurrence time for positive recurrent state with aperiodic. If state j is transient, then limit p j j as n tends to infinity is 0.

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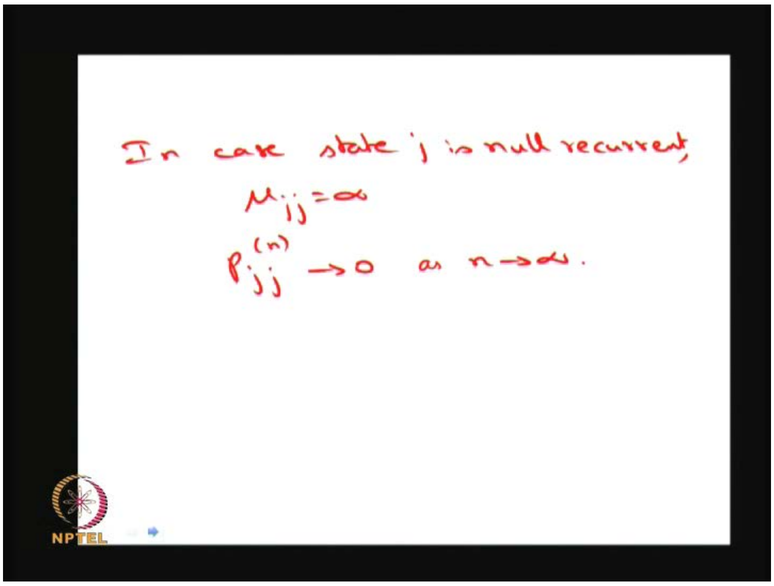

In a case of null recurrent; if the state j is a null recurrent; then you know that, for a null recurrent, the mean recurrence time is going to be infinity. Therefore, as n tends to infinity, the p j j of n will tend to 0.

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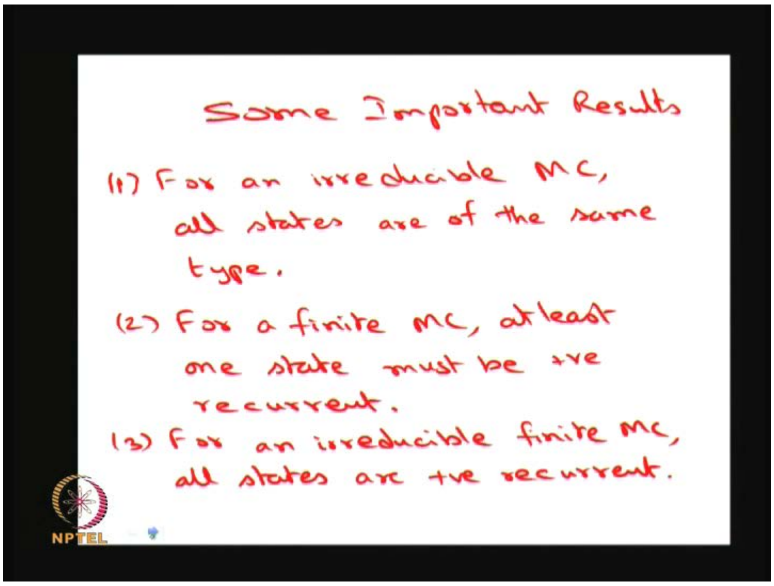

Now, I am going to give some more important results for a discrete-time Markov chain. Here I am considering a time-homogeneous discrete-time Markov chain only. For an irreducible Markov chain, all the states are of the same type; that means if the Markov chain is going to be irreducible; that means each state is communicating with each other state; then only, the Markov chain is going to be called as an irreducible Markov chain. That means for an irreducible Markov chain, all the states are of the same type; that means if one state is going to be a positive recurrent, then all the states are going to be positive recurrent. If one state is going to be a null recurrent, then all the states are going to be null recurrent.

The second result – for a finite Markov chain – the discrete-time Markov chain with the finite state space, at least one state must be a positive recurrent. This can be proved easily. But here I am not giving the proof. At least one state must be a positive recurrent, because it is a finite Markov chain; that means it has finite states. Therefore, the mean recurrence time – that is nothing but on average time spending in the state starting from the state j and coming back to the state j; that means recurrence time – that is going to be always a finite value at least for one state.

Now, I am combining the result one and two, gives the third result; that means the finite Markov chain has at least one positive recurrent state. And the first result states that, if the Markov chain is irreducible, then all the states are of the same type. Therefore, the third result is for an irreducible finite Markov chain; that means it is a time-homogeneous discrete-time Markov chain with the finite state space; and all the states are communicating with all other states. That is irreducible; then all the states are going to be a positive recurrent.

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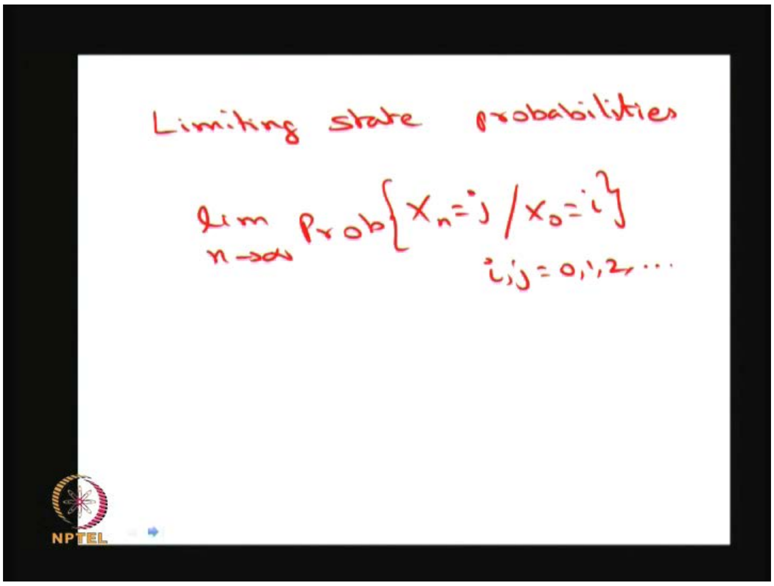

Now, I am describing the limiting distribution. The limiting distribution means what is the probability that, the system starting from the state i and reaches the state j as nth steps as n tends to infinity. So, this is nothing but… This is the definition of limiting state probabilities. We are only considering a time-homogeneous discrete-time Markov chain. So, if this limit is going to exist, then it is going to be unique. So, what is the limiting state probability for any time-homogeneous discrete-time Markov chain? Whether it will exist? If it exists, what is the value? That is what we are going to discuss in this class in this lecture.

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Suppose the limiting probabilities are independent of the initial state of the process. Then:

$$v_j = \lim_{n\to\infty} p_{ij}^{(n)}, v = [v_0 v_1 \ldots]$$

Suppose limiting probability is independent of initial state of the process; p naught vector suppose; I am just making the assumption, if the limiting probability is going to exist as well as if it is independent of an initial probability distribution, we can write as v j, because that is nothing to do with i. So, v j is nothing but what is the limiting state probability of system being in the state j as n tends to infinity. That is nothing but limit n tends to infinity p i j of n. So, now I can write a vector v consists of v naught, v 1. So, those entries are nothing but the limiting state probabilities.

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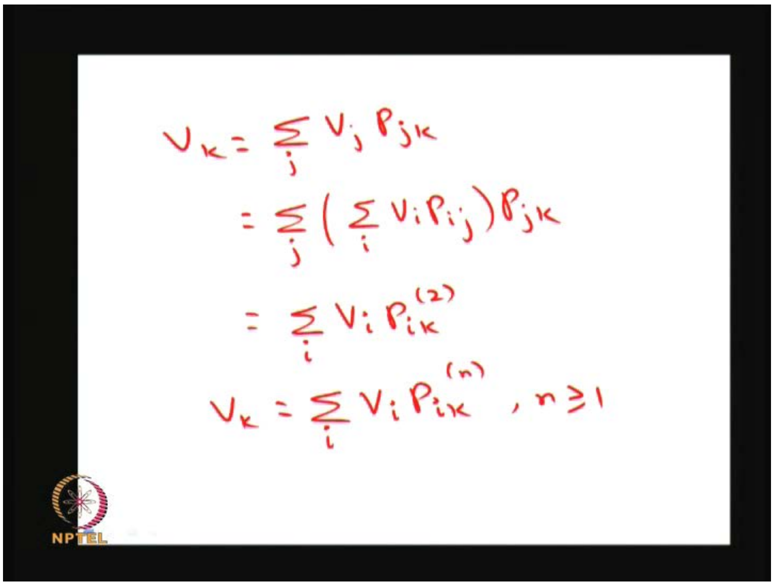

This I can compute as v k is equal to summation j v j p j k; that means the p j k is nothing but the one-step transition probability. So, that possibility summation will give v k. Now, I can replace v j by again the summation over i v i p i j. I can do simple calculation. It will land up – v k is equal to summation i v i p i k of 2. Again, I can repeat the same thing for v i. So, I will get v k is equal to summation over i v i p i k of n for n is greater than or equal to 1; that means this is the entry of n step transition probability matrix having the probability. That is the probability of system is moving from the state i to k in n steps for n is equal to 1, 2, 3 and so on.

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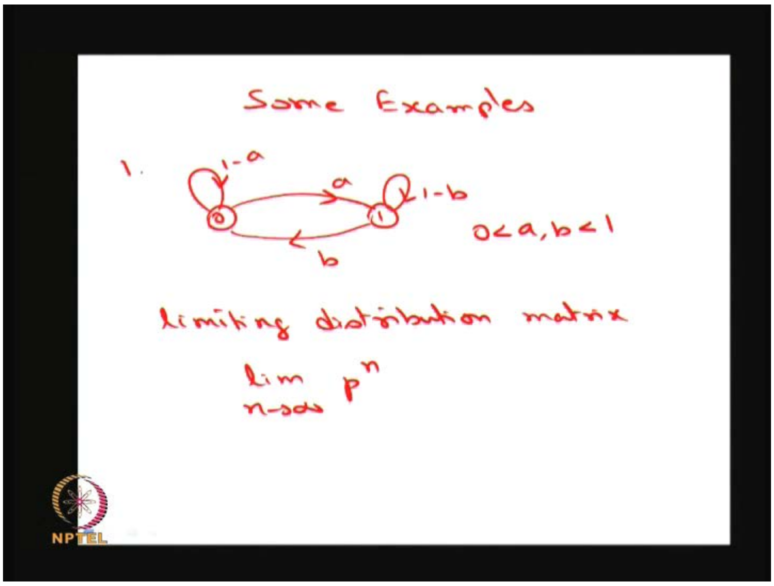

Now, I am going to discuss the simple situation in which, how we can get the limiting state probabilities. This is a simple model in which we have only two states. And this two-state model is the very good example in the sense; this can be interpreted as the many situations. For example, you can think of weather problem in which 0 is for rainy day and 1 is for the sunny day; and what is the probability that the next day is going to be a sunny day? From the rainy day, that probability is a; and from rainy day to sunny day, it is going to be the probability b; and the next day is going to be the same thing; whether it is the rainy day or sunny day according to the probabilities 1 minus a and 1 minus b. And you can assume that, both the probabilities a and b lies between open interval 0 to 1. In this case, this is a very simple two-state model. Like this we can give many more applications can be interpreted with the two-state model with the transition probability.

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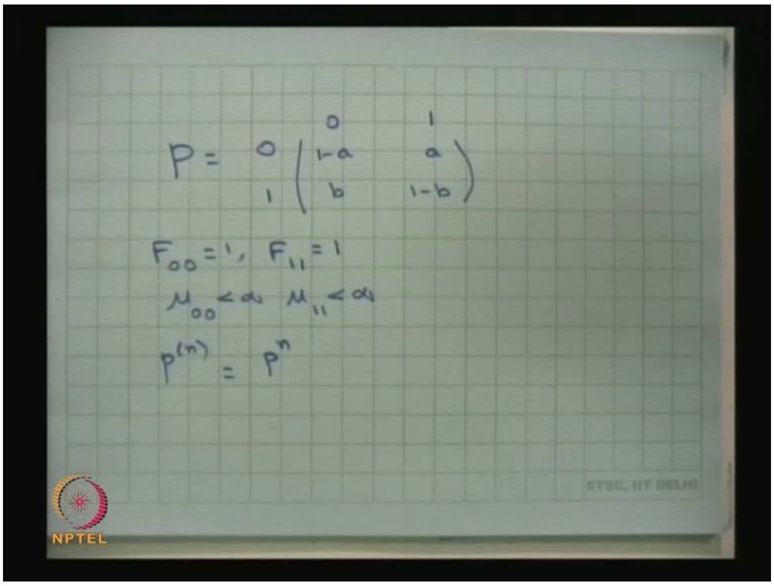

This is a one-step transition probability with the p matrix; that is, the p matrix is the state 0 and 1; 0 and 1. So, 0 to 0 – 1 minus a; 0 to 1 – that probability a; and 1 to 0 – the probability is b; and 1 to 1 – that is probability 1 minus b. So, this is a one-step transition probability matrix. And from this model, you can see that, since a and b is open interval 0 to 1, this is going to be an irreducible Markov chain. And with the finite state space; therefore, using the result, we can conclude all the states are going to be a positive recurrent. That can be verified from the classification of the states also. You can verify the first one is a recurrent state; that means you can find out the probability of F 0 0; that is going to be 1. And similarly, you can find out F 1 1; that is also going to be 1. So, you can conclude both the states are going to be a positive recurrent. And you can find out mu 0 0 – that is going to be a finite quantity; as well as mu 1 1 – that is also going to be a finite quantity. Therefore, you can conclude it is going to be a positive recurrent.

Now, our inertest is what is the limiting distribution; that means you find out what is the limiting distribution matrix; that is nothing but a limit n tends to infinity p power n; where, p power n is nothing but the n-step transition probability matrix. That is same as the one-step transition probability matrix power n; that means you have to find out what is p power n for any n. Then you have to find out what is the p power n matrix as n tends to infinity. So, you can use either eigenvalues and eigenvector method or you can use by induction method; that means you find out p power 2, then p power 3 and so on. Then you find out what is p power n by mathematical induction. Or, you find the eigenvalues or eigenvectors. Then you find out the p power n.

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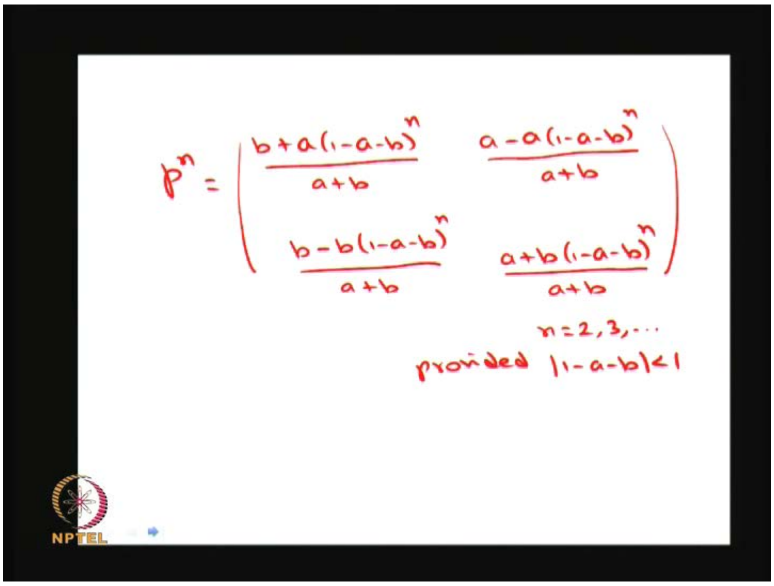

Here I am directly giving the p power n values matrix. So, this consist of four elements with function of a, b and n. This will exist provided the absolute of 1 minus a minus b is less than 1; otherwise, this p power n would not exist.

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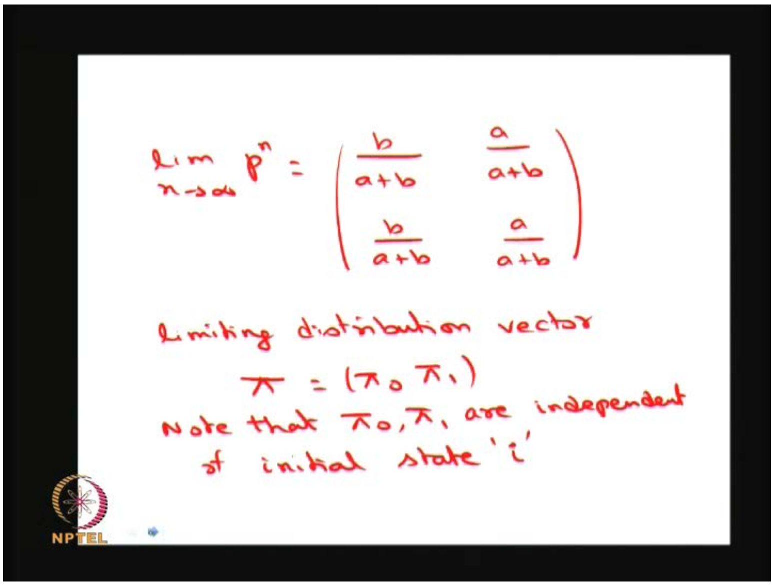

Now, we are going for as n tends to infinity, what is the matrix; that is, limit n tends to infinity the p power n is that matrix is going to be… Again, it is going to be a stochastic matrix, because the row sum is going to be 1 and all elements are greater than or equal to 0. Therefore, if the limiting probability matrix exists, then it is going to be unique. The limit exist means it is unique. And the row values are all the rows are going to be identical; that you can visualize. So, that vector is going to be pi; that is, pi 0 and pi 1. So, the pi 0 is nothing but b divided by a plus b; and pi 1 is nothing but a divided by a plus b. These are all the limiting state probability; that means in a longer run, the system will be in the state 0 or in the state 1.

And, the system will be in the state 0 in a longer run with the probability b divided a plus b. In the longer run, the system will be in the state 1 with the probability a divided by a plus b. Note that, these probabilities are independent of initial state i; that means whether you start at time 0 in the state 0 or 1 does not matter; in a longer run, the system is going to be in the state 0 or 1 with these probabilities. So, this is the situation for a time-homogeneous discrete-time Markov chain with the finite state space and irreducible Markov chain. Therefore, all the states are positive recurrent. And we are getting the limiting state probabilities, which are all going to be independent of initial state. So, this information is going to be useful later.

Based on this, I am going to distinguish three different probabilities distribution: the one is the limiting distribution; the next one is the stationary distribution; the third state is the steady state or equilibrium distribution. In general, all these three distributions are different; that is, the limiting distribution, stationary distribution and steady state or equilibrium distribution. All three are different in general. But there are in some situations; that means for a special case of discrete-time Markov chain, all these three results are going to be same. So, for that, this example is going to be an important one.

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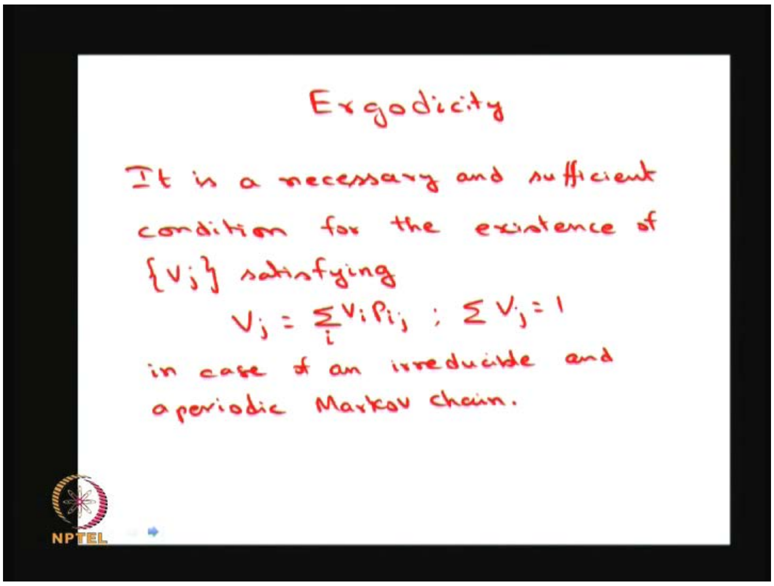

Now, I am going to discuss the ergodicity. This is a very important concept in any dynamical system. But here we are discussing the Markov process or we are going to discuss the time-homogeneous discrete-time Markov chain. But the ergodicity is an important concept for any dynamical system. So, I can give the easy definition; that is, it is necessary and sufficient condition for existence of v j's. That is nothing but some probability – state probabilities. If that is satisfying, v j's are going to be summation v i p i j; and the v i's are going to be summation, is going to be 1 for j in case of irreducible aperiodic Markov chain. Then we are going to say the system is an ergodic system; that means whenever the system is irreducible and aperiodic Markov chain; and then that system is going to be called as an ergodic Markov chain. This process is called the ergodicity. That means if you have an irreducible and aperiodic Markov chain, the ergodicity property is satisfied.

What is the use of ergodicity property in the Markov chain? Since it is irreducible and aperiodic, these limiting distributions – these probabilities are going to be independent of initial state. Therefore, this is used in the discrete event simulation; that means if you want to find out what is the proportion of the time the system being in some state in a longer run; that you can compute by finding the… That is nothing but the limiting probability. This limiting probability is same as this probability – v j's can be computed in this way using the one-step transition probability matrix. And that probability is going to be always independent of initial distribution; that means whatever the seed you are going to provide in the discrete event simulation, that does not matter; and you are interested only in the longer run, what is the proportion of the time, the system being in some state. So, that can be easily computed for an ergodic system; that means before you use an ergodic property in any dynamical system, you have to make sure that, that system is irreducible aperiodic. Then you can use the ergodicity concept. And I am going to discuss the ergodicity property some more when I am discussing the problem.

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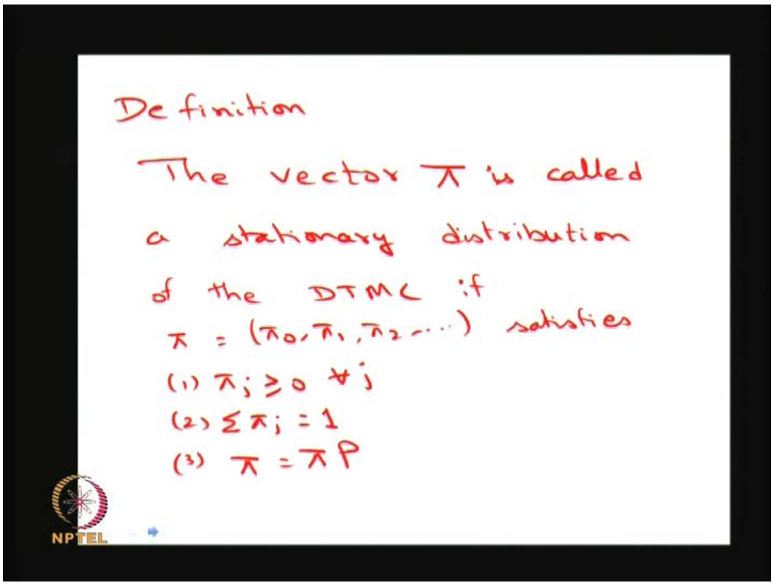

Now, I am going to move to the stationary distribution. The stationary distribution also is a very important concept in the Markov chain. And as such, first I am going to give the definition of a stationary distribution. The vector pi is called a stationary distribution of a time-homogeneous discrete-time Markov chain if that vector satisfies the first condition. All these values pi j's are greater than or equal to 0 for all j. And the summation over the pi j's – that is going to be 1. In the third condition, pi is going to be same as the pi times p; where, p is the one-step transition probability matrix. So, any vector pi satisfies these three conditions, then that vector is going to be called as a stationary distribution. This is nothing to do with the limiting distribution, the one I have discussed earlier. But for an irreducible aperiodic Markov chain, the limiting distribution is same as the stationary distribution. That is also going to be same as the equilibrium or a steady state distribution. All these three distributions are going to be same for an irreducible aperiodic Markov chain. But in general, all these three things are going to be different. So, here I am giving the definition of a stationary distribution by satisfying these three properties.

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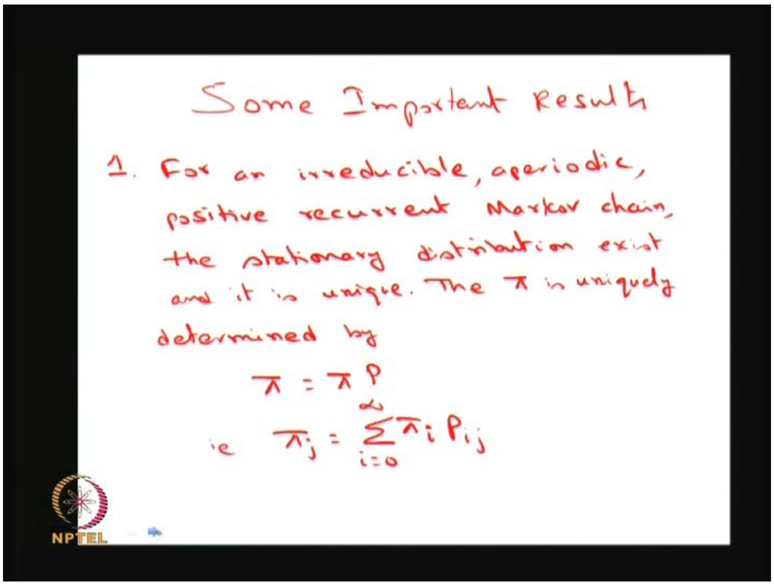

Now, I am going to give some important results for that. The first result is for irreducible aperiodic positive recurrent Markov chain, the stationary distribution exists and it is unique. The one definition I have given earlier – I have discussed aperiodic irreducible; I have to include positive recurrent also, because these three things are important for an irreducible aperiodic positive recurrent Markov chain; all these three distributions: limiting distribution, stationary distribution, steady state or equilibrium distribution – all three are same; I have to include the positive recurrent also.

So, what I am giving in this result; then pi is uniquely determined by solving this equation pi is equal to pi p with summation of pi's are going to be 1. So, if I solve pi is equal to pi p along with summation of pi is equal to 1, that will give a unique pi and that pi is going to be a stationary distribution for an irreducible aperiodic positive recurrent Markov chain. Irreducible means all the states are communicating with all other states. Aperiodic means the periodicity for a state is 1. The greatest common divisor of a system coming back to the same state; all the possible steps – that greatest common devisor is 1. The positive recurrent means it is a recurrent state; that means with the probability 1, the system starts from one state and coming back to the same state; that probability is 1. The positive recurrent means the mean recurrence time – that is going to be a finite value. If these three conditions are going to be satisfied by any time-homogeneous discrete-time Markov chain, then the stationary distribution can be computed using pi is equal to pi p and the summation is equal to 1. That is going to be a unique value.

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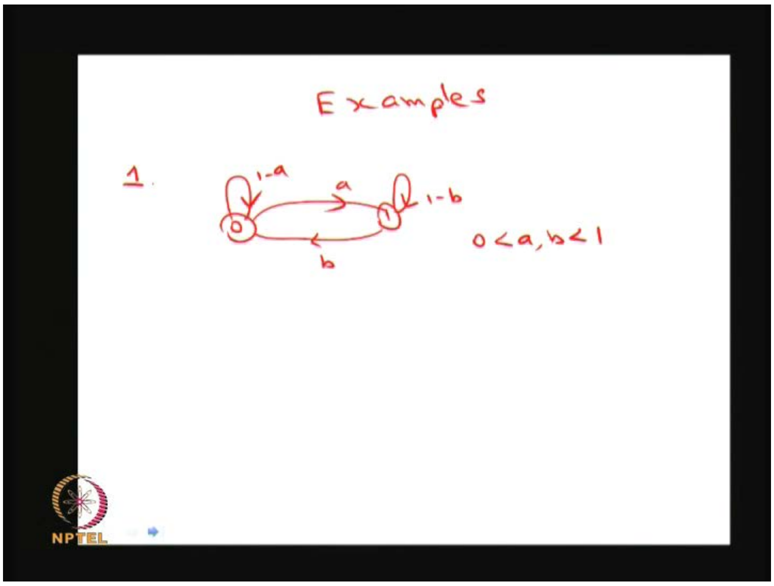

I am giving the same example; that is, the two state model with states 0 and 1 with the probabilities self loop 1 minus a and self loop 1 minus b. And system going from the state 0 to 1 in one step; that is a. And the system is going from the state 1 to 0; that probability is b. So, I am giving a very simple two state model. And you can solve pi is equal to pi p and the summation is equal to 1 and you will get the probabilities. And these probabilities are same as the probabilities you got it in the limiting state probability. If you solve the two-state model with pi is equal to pi p, you will get the probabilities that pi naught is going to be b divided by a plus b and pi 1 is going to be a divided by a plus b. And it satisfies the summation of pi a is equal to 1 and it also satisfies pi is equal to pi p. That means in this model, it is irreducible aperiodic positive recurrent model. Therefore, the limiting distribution is same as the stationary distribution also.

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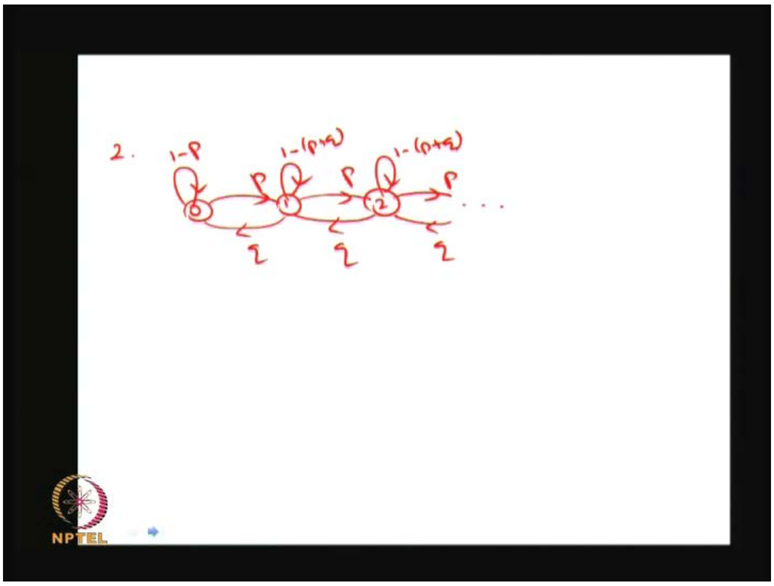

The second example – that is with the infinite state. Here the number of states are going to be countably infinite. I can start with to find out the stationary distribution. Before that, I have to cross check whether it is going to be an irreducible aperiodic positive recurrent Markov chain. It is irreducible, because the way I have given the probabilities…

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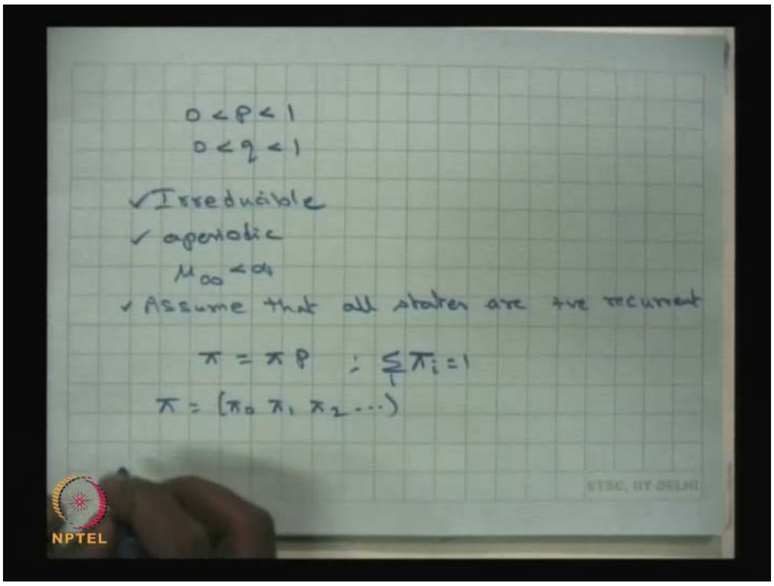

I make the assumption, the probabilities lies between 0 to 1. And the probabilities of the q also lie between 0 to 1. Therefore, each state is communicating with each other state. Therefore, it is going to be irreducible. The second one – it has to be aperiodic. Aperiodic means the periodicity for each state, because the greatest common divisor is going to be 1, because the coming back to the state is via self loop or going to some other state, and coming back; and there also has a self loop. Therefore, it is going to be – all the states are going to be aperiodic. Therefore, the Markov chain is aperiodic. The third one – positive recurrent; since it is an infinite state model, you cannot come to the conclusion, whether this mu 0 0 is going to be a finite quantity unless otherwise substituting the value of p and q. So, what I will do; I will make the assumption.

Assume that all states are positive recurrent. Then later, I will find out, what is the condition to be a positive recurrent. So, I make the assumption. Even I do not want to make the assumption for all the states are going to be positive recurrent; I can make the assumption for only one state is going to be a positive recurrent. And since it is an irreducible Markov chain and all the states are going to be of the same type; therefore, it will come to the conclusion, all the states are going to be positive recurrent. So, I make the assumption, one state is going to be a positive recurrent. Therefore, it will land up – all the states are going to be positive recurrent.

Now, once I made an assumption of all the states are positive recurrent; therefore, it satisfies all the results of the first result, that is, irreducible aperiodic positive recurrent Markov chain with the infinite state space. Therefore, I can come to the conclusion, the stationary distribution exists and it is going to be unique. And that can be computed by solving the equation, pi is equal to pi p with the summation of pi i is equal to 1; where, pi is the vector and p is the one-step transition probability matrix. That one-step transition probability matrix can be created using the state transition diagram, which I have given.

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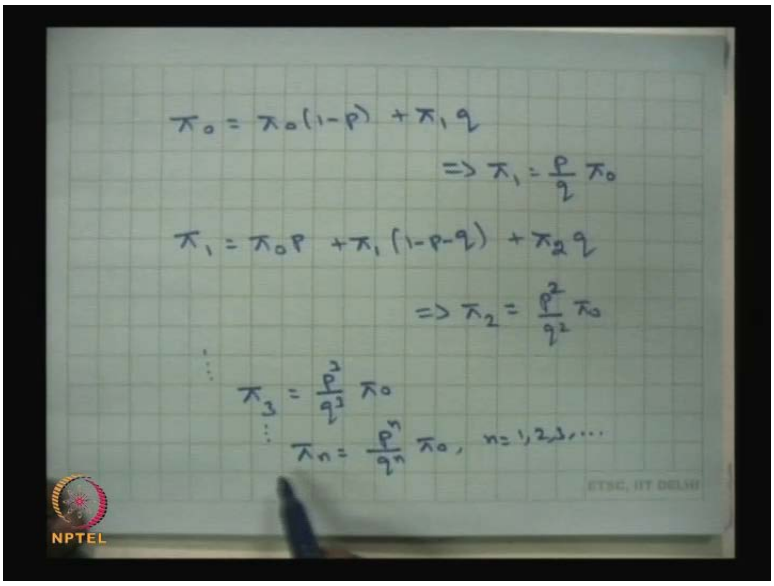

If I find out what is the first equation from this vector pi is equal to pi naught, pi 1, pi 2 and so on; here also, this and p is the matrix. Therefore, I will get the first equation as pi naught is equal to pi naught times 1 minus p plus pi 1 times q. So, this is the first equation in the matrix form pi is equal to pi p. So, the first equation is pi naught is equal to pi naught times 1 minus p plus pi 1 times q. So, from this equation, I can get pi 1, because I can take this pi naught this side and I can cancel. So, I will get pi 1 is equal to p divided by q times pi naught. From the first equation, we get the relation, pi 1 in terms of pi naught.

Now, I will take the second equation from pi is equal to pi p. So, that will give pi 1 is equal to pi naught times p plus pi 1 times 1 minus p minus q plus pi 2 times q. So, this equation have pi naught, pi 1 and pi 2. So, what I can do, I can write pi 1 in terms of pi naught. And I can simplify this equation. If I simplify, I will get pi 2 is same as p square by q square times pi naught, because I am substituting pi 1 in terms of pi naught in this equation. Therefore, I will get pi 2 in terms of pi naught; that is, pi 2 is equal to p square by q square times pi naught. Similarly, if I take the third equation and do the same thing; finally, I will get pi 3 is equal to p cube by q cube pi naught. The same way I can go further. Therefore, I will get pi n in terms of pi naught for n is equal to 1, 2, 3 and so on. So, this is the way, I can solve this equation, pi is equal to pi p; that is a homogeneous equation; we have to be very careful with the homogeneous equation. So, the trivial solutions are going to be 0. But we are trying to find out the non-trivial solution.

Therefore, we are using the normalization, that is, the summation of pi i is equal to 1. Till now, I have not used. So, I have just simplified that pi is equal to pi p; I am getting pi n in terms of pi naught.

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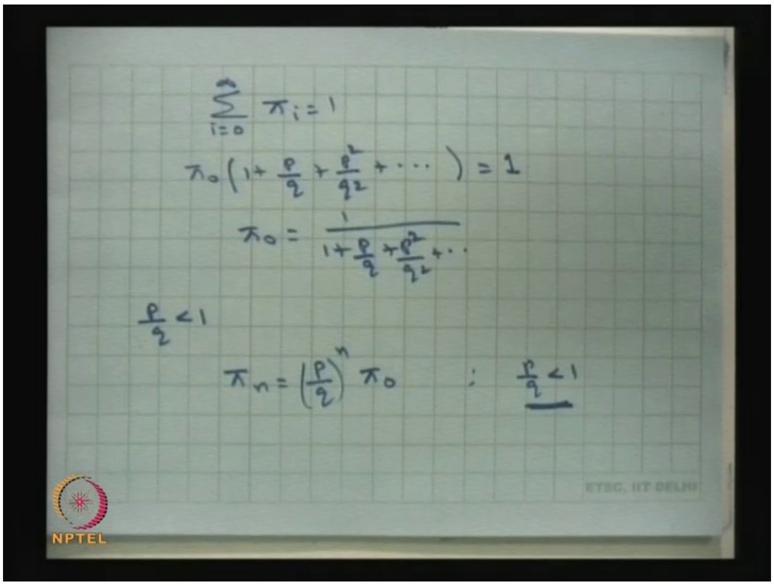

Now, I have to use summation of pi i is equal to 1 starting from pi is equal to 0 to infinity. Therefore, the pi naught will be out; 1 plus p by q plus p square by q square and so on. That is equal to 1. Therefore, the pi naught is going to be 1 divide by 1 plus p by q plus p square by q square and so on. That is pi naught. Since it is infinite terms in the denominator; as long as this converges, we will get nonzero value for pi naught; in turn, you will get pi i is equal to p by q power n times pi naught provided this denominator is going to be converges. Then the denominator is going to be converges. In this situation, as long as p by q is going to be less than 1; if p by q is less than 1. Earlier condition is p lies between 0 to 1 and q lies between 0 to 1.

Now, I am making the additional condition p by q is less than 1. That will ensure the denominator converges. Therefore, the pi naught is going to be a nonzero value. Therefore, the pi n's are going to be p divided by q power n times pi naught; where, pi naught is written, 1 divided by 1 plus p by q plus p by q whole square and so on provided p by q is less than 1. If you recall, we made the assumption, the states are going to be positive recurrent. If this p by q is less than 1; then you can conclude the mean recurrence time is going to be a finite value. If you make the assumption p by q is less than 1; that will ensure the mean recurrence time for any state is going to be a finite value. Therefore, all the states are going to be positive recurrent and then the stationary distribution exists. Therefore, this is the condition for a positive recurrent state for this model.

And, the stationary distributions – that is going to be pi n is equal to p by q power n times pi naught. This is nothing but in a longer run, what is the probability that the system will be in the state n; that probability is p by q power n times this pi naught; and pi naught is given in this form. And in this example, we have taken each state for… The p by q is same for all the states. We can go for in general situation, the system going from 0 to 1 could be p naught; the system going from the state 1 to 2 may be p 1 and so on. Therefore, need not all the p's need not be the same and q's also need not to be same.

So, you can generalize this model. And this model is nothing but one-dimensional random walk. And here this 0 is a barrier. The system is not going away from the 0 in the left side. Therefore, 0 is a barrier. And this is one-dimensional random walk in which the system is keep moving into the different states in subsequent steps. And there is a possibility the system will be in the same state with the positive probability of 1 minus p plus q in this model. In general, you can go for the p naught, p 1, p 2, and so on. And similarly, q 1 q 2 q 3 and so on also.

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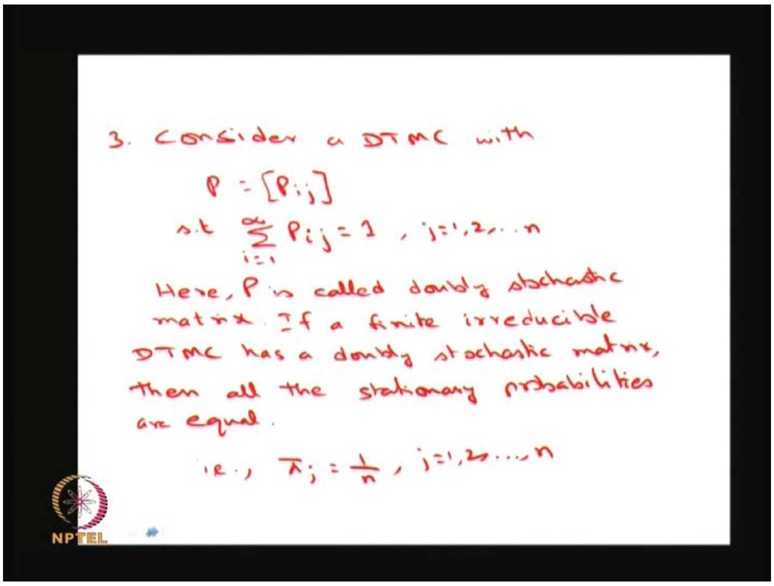

The third example – I am considering a discrete-time Markov chain. Obviously, it is a time-homogeneous discrete-time Markov chain. With the one-step transition probability matrix satisfies the additional condition; that is, the column sum – that is also going to be 1. Obviously, the stochastic matrix means the row sums are going to be 1. And here I am making the additional condition along with the row sum; the column sum is also going to be 1 for a finite Markov chain. In this model, in this situation, this stochastic matrix is going to be called as a doubly stochastic matrix; that means it is a stochastic matrix; that means each entities lies between 0 to 1 and row sum is going to be 1. Along with the row sum, the column sum is also going to be 1. Then that matrix is going to called as a doubly stochastic matrix.

If you have discrete-time Markov chain with the finite and the doubly stochastic matrix and also it is irreducible; I am making an additional condition. If it is a finite irreducible with the one-step transition probability matrix is a doubly stochastic matrix; then the stationary probability exists as well as that stationary probabilities are going to be uniformly distributed; that is, that values are 1 divided by n; where, n is the number of states of the discrete-time Markov chain. To get this result, you can use all the previous results also. It is an irreducible Markov chain, therefore; and also, it is a finite. So, for a finite irreducible Markov chain, all the states are going to be a positive recurrent. You can use the previous result; only the aperiodicity is missing. But since it is a doubly stochastic matrix, that aperiodicity is taken care. Therefore, the stationary probabilities exist.

Now, if you compute the stationary probabilities for a doubly stochastic matrix situation, then the pi is equal to pi p if you solve with the summation of pi i is equal to 1 since the matrix is going to be a doubly stochastic; that means its column sums are going to be 1. Therefore, it is going to be boils down or the simplification is boils down to the state probabilities are going to be 1 divided by n.

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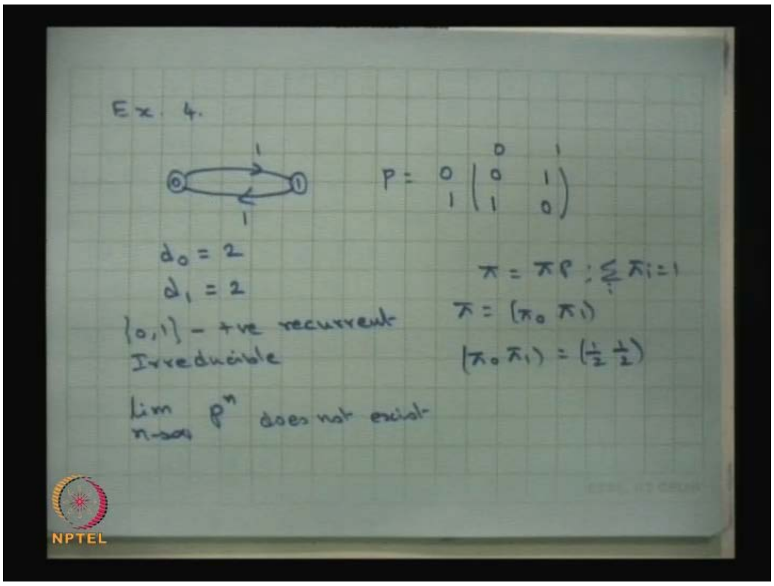

I am not going to give the derivation for that. That can be worked out. That is example 4. You consider a two-state model. The system is going from the state 0 to 1 in one step; that probability is 1. The system is going from the state 1 to 0; that probability is also 1. Therefore, the p matrix – one-step transition probability matrix – 0 to 0 is 0; 0 to 1 – that is 1; 1 to 0 – that probability is 1; and 1 to 1 – that is 0. So, this is the one-step transition probability matrix. And if you see that this is a finite state model, irreducible; it is not aperiodic because there is no self loop. So, if you find out the periodicity for the state 0, the greatest common divisor of system starting from the state 0 coming back to 0 in how many steps; you find out the greatest common divisor of that. And since it can come back in 2 steps or 4 steps and so on; therefore, the greatest common divisor is 2.

Similarly, since it is a finite state model; if one state is of periodicity, then all other states are also going to be same periodicity as long as it is irreducible. Therefore, the periodicity for the state 1 – that is also going to be 2. Or, you can compute it separately coming back to the state 1 starting from the state 1; that is going to be either 2 steps or 4 steps or 6 steps and so on. Therefore, the greatest common divisor is 2. Since it is an irreducible model, all the states are going to be of the same type. Since it is finite, one is going to be a positive recurrent. Therefore, both the state are going to be positive recurrent; and periodicity 2 and irreducible. Note that, the example which I have formulated; the column sum is also 1. Therefore, we use a doubly stochastic matrix. Therefore, you can use the previous result – the example which I have given – finite irreducible doubly stochastic. Therefore, the stationary distribution exists. So, if you solve pi is equal to pi p with the summation pi i's, is going to be 1; where, pi is nothing but pi naught, pi 1 vector. So, if you solve pi is equal to pi p with the summation of pi i is equal to 1, you will get pi naught, pi 1; that is same as 1 by 2, 1 by 2. So, this is a stationary distribution that exists. And that value is state probabilities; stationary state probabilities are going to be 1 by 2, 1 by 2; that means in a longer run, the system will be in the state 0 or 1 with the probability half.

Whereas, if you try to find out the limiting state probabilities or limiting distribution; that means the limit n tends to infinity p of n; that means find out the n-step transition probability matrix. Then you make n tends to infinity. This does not exists for this model. If you see the result, which I have given the limiting distribution; it is going to be exist and unique and so on. There I have not discussed the periodicity. There I have made aperiodic. So, here it is a period 2 model. So, whenever you have an irreducible positive recurrent state; if the periodicity is not 1; that means it is not an aperiodic model. There is a possibility the limiting distribution would not exist, but still the stationary distribution exists. So, this is the example in which the limiting distribution does not exists; whereas, the stationary distribution exists. But if the model is irreducible aperiodic positive recurrent, then the stationary distribution exists as well as the limiting distribution exists; and both are going to be same.

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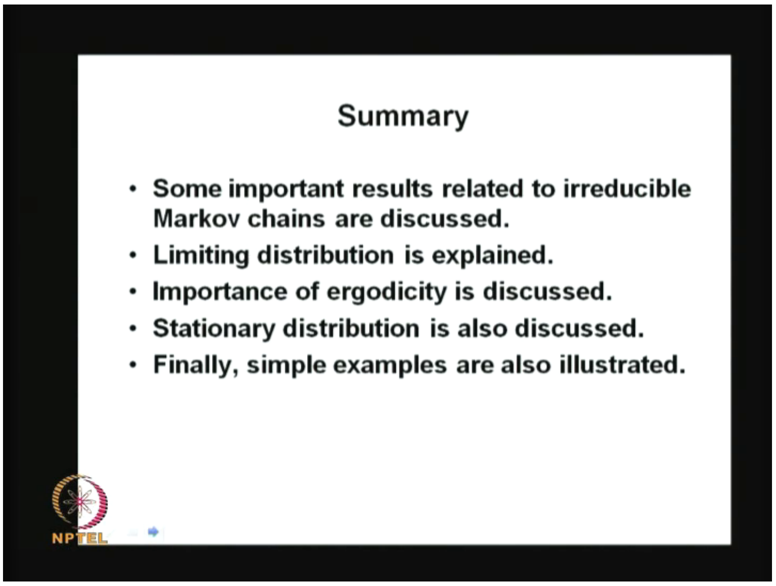

Now, I am going to give the conclusion. In this talk, we have discussed some important results for the irreducible Markov chain. Then I have discussed what the meaning of limiting distribution is. And I have given one example of how to compute the limiting state probabilities. Then I discussed the ergodicity. Then I have discussed the stationary distribution and how to compute the stationary distributions for an irreducible aperiodic positive recurrent whether it is a finite state or infinite state Markov chain. I have given few examples. And I have given an example in which the stationary distribution exists; whereas, the limiting distribution does not exists. And I have given some examples also. With this I complete today’s lecture.

Thanks.