Stochastic Processes
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Module - 4
Discrete-time Markov Chain
Lecture - 3
Classification of States and Limiting Distributions

In this, we are discussing a discrete-time Markov chain. And we have finished already two lectures on this module, and this is the third lecture – classification of states and limiting distributions.

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In this lecture, I am going to give the information about the classification of the states for the time-homogeneous discrete-time Markov chain. Then I am going to give the definition of a limiting distribution. Then I am going to discuss a few simple examples, so that we can understand the classification of states as well as the limiting distribution. Why do you need classification of states? Whenever we study the time-homogeneous discrete-time Markov chain our interest is to find out the limiting distribution of the random variable X n. To study the limiting distribution or a stationary distribution, later we are going to use the word called equilibrium distribution. All those things, you need the classification of a state. Without the classification of states, we cannot come to the conclusion, whether the limiting distribution exists, whether that is going to be unique and so on. So, for that we need classification states.

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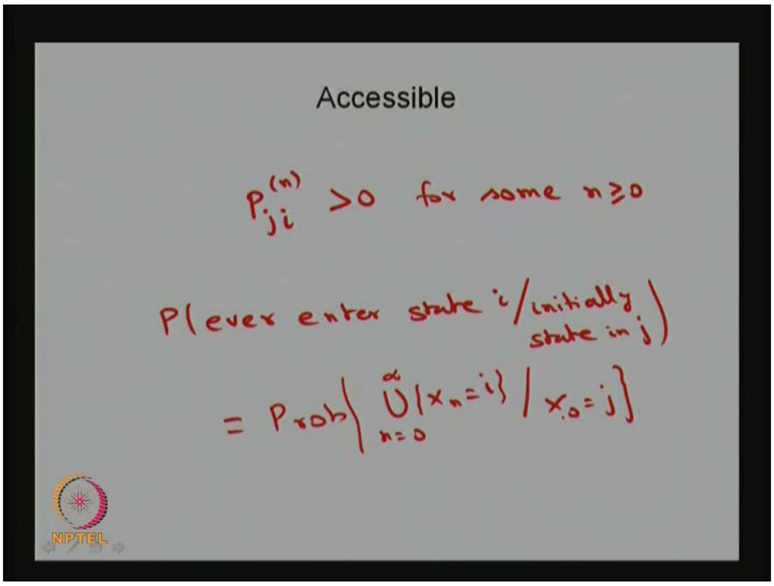

Before moving into the classification of states we need some concepts, so that using those concepts, we can classify the states. The first concept is called accessible. When we say the state i is said to be accessible from the state j; whenever the P suffix j to i in n steps has to be greater than 0 for some n, which is greater than or equal to 0. We are including n is equal to 0 for the safer side. Whenever we say the state i is set to be accessible from the state j, if the P j comma i in n steps, has to be greater than 0; that means this is the transition probability from the n step transition probability matrix. And if that element is going to be greater than 0, then we say, the state i is set to be accessible from the state j.

Using these, we can write down, what is the probability that ever enter state i given that, initially, the system is in the state j. You can find out what is the probability of the system ever enter to the state i given that, initially, it was in the state j. That is nothing but the union of all the events corresponding to the X n takes a value i given that it was in the state j initially. We can find out, what is the probability that, ever entering the state i given that, initially, the system is in the state j. That is the union of the probability of union of X n is equal to i given that, X naught is equal to j.

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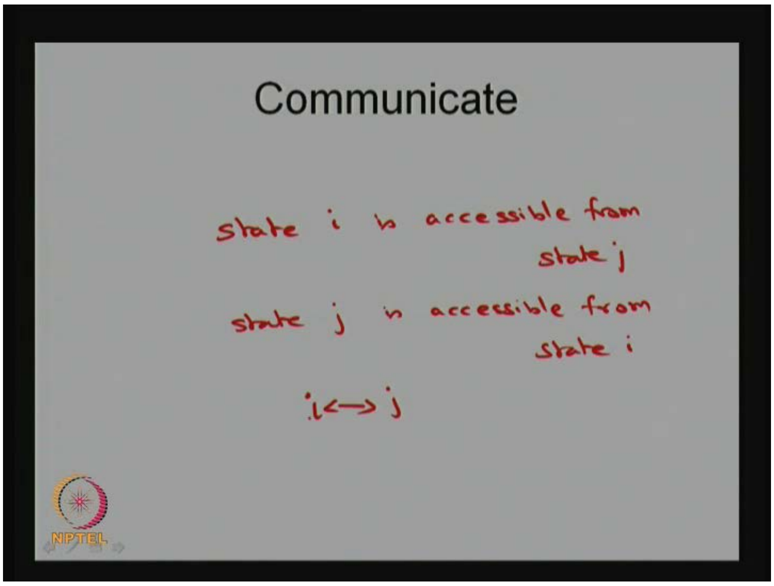

Now, I am going to define… Now, I am going to give the next concept called communicate using the accessible. Two states are set to be communicate; that means the state i is accessible from the state j as well as the state i that is accessible from state i. Whenever the state i is communicate with the state j; that means state i is accessible from state j as well as the state j is accessible from state i. In notation, we can use the notation i arrow in both sides with j. State i is communicating with the state j; it means state i is accessible from state j as well as the state j is accessible from state i.

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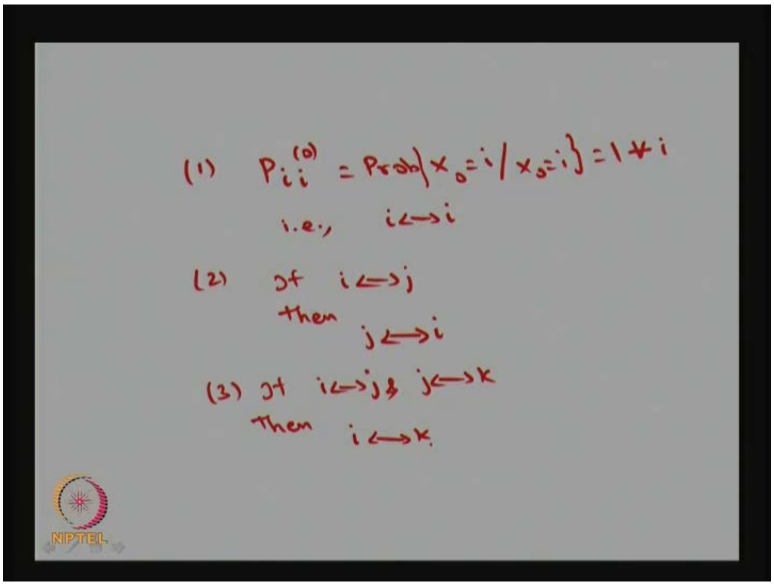

Since I use the concept of access to define communicate, it is going to satisfy few properties. The first property – any state communicate with itself; that means the P i i of 0; that is nothing but what is the probability that X naught is equal to i given that, X naught is equal to i; that is going to be 1 for all I – any state communicate with itself. The second one… That means in notation, i communicates with the i itself. The second property – if state i communicate with the state j, then the state j communicate with the state i also; that means it is a symmetric property; that means if i communicates with j, then j communicates with i. The communicate satisfies the symmetric property. The third one – if i communicate with j and j communicate with k, then we can conclude, i communicates with k. This relation is called transitive.

So, the communicate – that property satisfies itself; and it satisfies the symmetric property as well as the transitive property; that is, if i communicates with the state j and the state j communicates with the state k, then the state i communicates with the state i. Communication is an equivalence relation on the set of states. And hence, this relation partitions the set of states into communicating classes. I am not giving the proof here. So, one can easily prove; using the one-step transition probability and n-step transition probability matrix and the accessible concept, one can prove these three properties.

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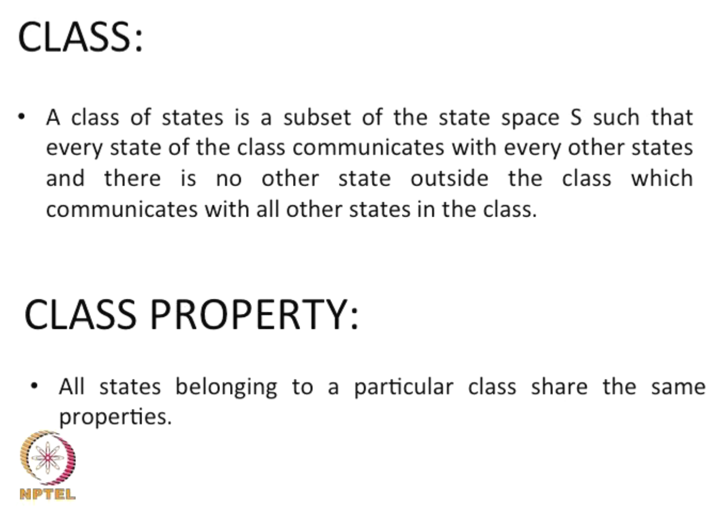

Now, I am going to define the next concept called class property. What is the class property? A class of state is a subset of the state space S such that every state of the class communicates with every other states and there is no other state outside the class which communicates with all other states in the class; instead, the time-homogeneous discrete-time Markov chain. Since it is a discrete-time Markov chain, you have a state space; the state space may be a finite number of elements or countably infinite number of elements. So, that is the state space S. In the state space S, you are going to create a subset. That is going to be called as a class, if within the subset of that collection, it satisfies the communicate; that means each state inside the class has to be communicate with each other state.

And also, it has to satisfy the second property, that is, no other state outside the class, which communicates with all other states in the class; that means (( )) you can start with one element, then you can include one more element and you can include one more element. Once this property is satisfied; that means you cannot make including one more state and make it as a class; then you have to stop framing the class. So, the subset will be created by including one more state, one more state, one more state in the state space as long as this property is satisfied. So, once the second property violates; that means we should stop with creating the subset and that is going to be the class. We are going to discuss this – how to create the class via simple examples. So, that I am going to do it later.

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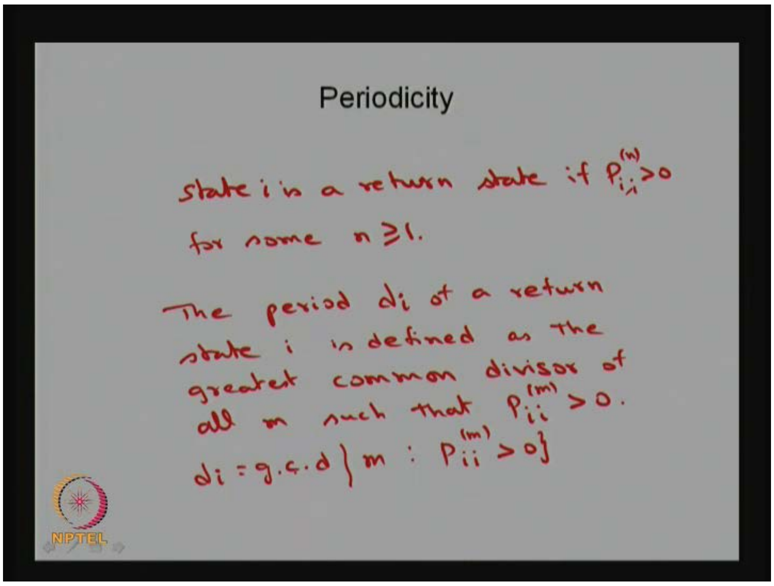

Next concept is periodicity. The definition of periodicity goes like this – the state i is a return state if the P i comma i of n, which is greater than 0 for some n, which is greater than or equal to 1. First, I am defining what the meaning of return state is. Here any state is going to be called as a return state, if the probability of starting from the state i coming to the same state in the nth step if that is greater than 0, then we say it is a return state.

Now, I am going to define the periodicity only for the return state. The period – in notation, it is d suffix i – suffix i – i is for the state – of a return state i is defined as the greatest common divisor of all m such that P suffix i comma i of m, which is greater than 0. So, the period of a return state is going to be an integer. And that integer is computed by using the greatest common divisor of all the possible m such that the P i comma i of m should be greater than 0; that means we find out how many steps you will take to come to the same state if you start from the state i. You collect all the possible number of steps; you will come back to the state with the positive probability; and you find out the greatest common divisor of those integers – those positive integers. Then that number is going to be the period or periodicity of the return state or the period of the state; that means we can write down in short, d suffix i is the greatest common divisor collection of m such that the P i i of m should be greater than 0.

If the greatest common devisor of collection of m such that greater than 0; if this d i is going to be 1, then we say that state is a periodic state; otherwise, if it is greater than 1, and whatever be the integer you are going to get; and that is going to be the period of the state i. If the period is going to be 1, then we call it as a periodic state. Note that, whenever you have a class in which we have more than one state; if one state has the period some number, then the other states of the same class also going to have the same period. That can be proved easily. So, within the class, all the states will be having the same period.

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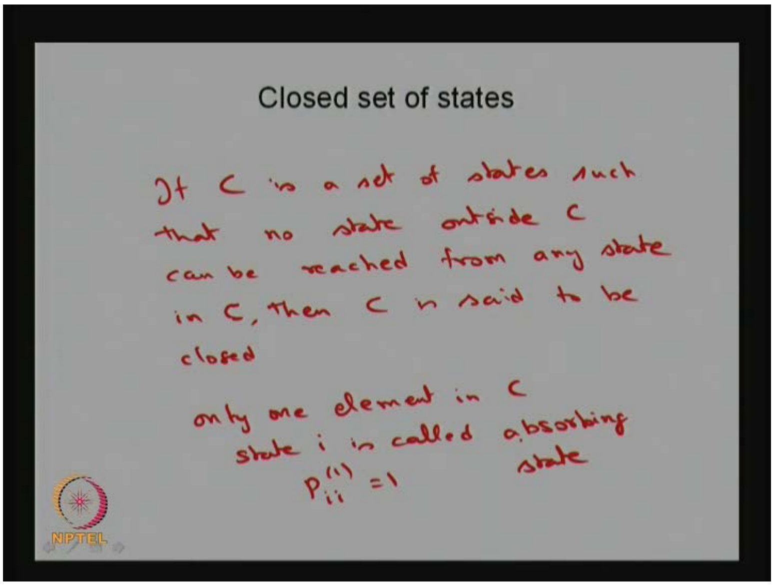

Now, I going for the next concept called closed set of states. Closed set of states – if C is a set of states such that no state outside capital C can be reached from any state in capital C, then we say then the collection or the set C is said to be closed. So, whenever you create a collection of states; and that set we call it as a capital C. If it satisfies this property, then we say that set is called the closed set. So, we can combine the class property with the closed set property. If both the properties are satisfied; that communicates with each other as well as the close property is satisfied, then you can say that, the closed communicating class. So, any subset in the state space S – if it satisfies each element within the set is communicate each other and satisfies this property, then we say that collection is going to be a closed communicating class.

There is a possibility in a set; you can have more than one element, more than one state in the collection. The class may have only one element or it may be more than one element. If any closed set or the close communicating class has only one element; that means you cannot include one more state and to make it as the closed or communicating class; then that closed set is called… or that state is called only one element in capital C. Then the state i is called absorbing state. A state i is said to be absorbing state; then it is going to form a close communicating class, which has only one element in that class. There is a possibility more than one element, is also possible in the closed communicating class.

So, we can define the absorbing state through the closed communicating class; or, we can make it in the same absorbing state using the definition, P i i in steps one that is going to be 1; that means if you see the one-step transition probability matrix, the diagonal element of that corresponding state – the corresponding role – the element is going to be 1; that means the system starting from the state i; and in one step, the system moving to the same state i; that probability is 1. If this probability is 1, then we say that state is going to be absorbing state. In the other way round, we can go for defining the absorbing state via closed communicating class, has only one element also. So, there are two ways we can say the absorbing state.

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Irreducible

- If a Markov chain does not contain any other proper closed subset of the state space S, other than the state space S itself, then the Markov chain is said to be an irreducible Markov chain.

- The states of a closed communicating class share same class properties. Hence, all the states in the irreducible chain are of the same type.

Using these concepts, I am going to develop the next concept called irreducible Markov chain. We are discussing time-homogeneous discrete-time Markov chain; whereas, this concept called irreducible – that is valid for the discrete-time Markov chain as well as the continuous-time Markov chain. So, that we are going to discuss later.

Now, I am defining the irreducibility for a time-homogeneous discrete-time Markov chain. If the Markov chain… Since the irreducible concept comes for the discrete-time Markov chain and the continuous-time Markov chain; who use the word called Markov chain; that is valid for both. If the Markov chain does not contain any other proper closed subset other than the state space capital S, then the Markov chain… In short, we can use the word MC for Markov chain; then the Markov chain is called irreducible Markov chain. Whenever the state space cannot be partitioned into more than one closed set – the proper set; that means you can have only one closed set and that is same as the capital S. All the elements in the state space is going to form only one closed set; in that case, that Markov chain is going to be called as irreducible; irreducible means you cannot partition the state space.

If more than one closed proper closed subsets are possible from the state space, then that Markov chain is going to be called as a reducible Markov chain. If more than one or we can able to make the partition of the state space into more than one closed set as well as few transient states and so on; that I am going to discuss later. So, whenever you are not able… If you are able to partition the state space, then that is going to be a reducible Markov chain. If you are not able to partition the state space and the whole state space is going to be only one proper closed subset, then that Markov chain is going to be called as a irreducible Markov chain. In this case, all the states belonging to that class is going to form a one class. And since it is going to have only one class, all the states going to have… If one state has the period something, then all the other states are also going to have the same period. Because you are not able to partition, you have only one class. Therefore, if one state has the period some number – some integer, then that same period will be for all other states also. So, the Markov chain, which are not irreducible are said to be reducible or non-irreducible Markov chain.

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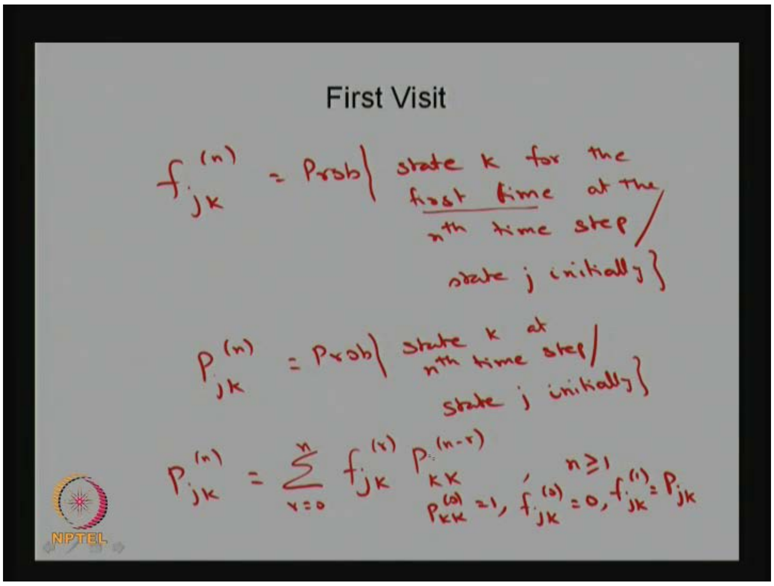

Now, I am going to give the next concept called first visit. We did not come to the classification of a state. Before that, we are developing few concepts. Using these concepts, we are going to classify the states. The next concept is called first visit. What is the meaning of first visit? I am going to define the probability mass function as the f suffix j k with superscript n; that means what is the probability that the system reaches the state k for the first time; that is important. For the first time at the nth time step given that the system starts the state j initially. This is the conditional probability mass function of a system moving from the state j to k and system reaching the state k at the nth time step for the first time. That is important. So, this is the first time, the system reaches the state k at the nth step – exactly, at the nth step.

And, this conditional probability mass function – that I am going to write it as the f suffix j k of n. This is different from the P j k of n. This is also conditional probability. Whereas, this probability is defined, what is the probability that the system reaches the state k at the nth time step given that it was in the state j initially. This is also conditional probability. The only difference is the first time; that means there is a possibility the system – here the P suffix j k of n means there is a possibility the system would have come to the state k before nth step also. So, that probability is included. Whereas, the f suffix j comma k at the nth step means this at only the nth step, it reaches the state k. Therefore, the way I have given the first time conditional this probability and this is not necessarily the first time; this is also conditional probability; I can relate the f suffix j comma k with the P suffix j comma k.

Both are in the n-step transition probability. But one is for the first time; the other one is not necessarily. I can relate both in the form of P j suffix k superscript n, that is, the n step. That is same as f suffix j k of r steps and P suffix k k of n minus r steps; and r can vary from 0 to small n for n is greater than or equal to 1. This means if the system is moving from the state j to k in n-step, not necessarily the first time; that can be written as the union of mutually exclusive events for different r in which the system moves from the state j to k in r steps for the first time.

And, the remaining n minus r steps – there is the possibility the system would have moved from the state k to k not necessarily the first time. And possible r can be 0 to small n. And this n can be vary from 1 to infinity. Obviously, we can make out the… I can give the P k k of 0 step; that is going to be 1. And similarly, we can make out f suffix j of k, that is, 0 steps also 0; and f j k of one step – that is nothing but the P j k. The first time, the system is moving from the state j to k in one step. That is same as the one step transition probability. The first time and one-step transition probability is same. Whereas, for n is greater than or equal to 1; then it is going to be the combination of the first time with not necessarily the first time n minus r step transition probability. That all possible events – that will give altogether final probability. So, here we have use the total probability root as well as the Chapman-Kolmogorov equation for the time-homogeneous discrete-time Markov chain to land up giving the relation between the P j k with the f j k.

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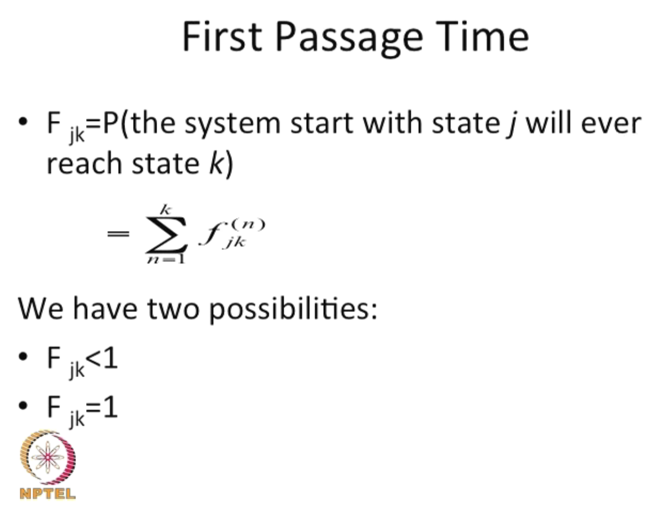

Now, I am going to give the next concept called first passage time distribution. First passage time distribution – that is written in the F suffix j k; that is nothing but what is the probability that, the system start with the state j will ever reach state k. So, this probability I am writing as F suffix j comma k. Therefore, this is same as… There is a possibility it would have gone to the state k in n steps first time. And all the possible steps for the first time – that union will give F j comma k. What is the conditional probability that, the system is starting from the state j and ever entering into the state k; that is, all the possible of first time to reaching the state n and all possible n – that will give the probability of ever visiting the state k starting with the state j.

Now, we have two issues or two cases. One is what is F j k, which is less than 1? What is the situation corresponding to this probability is going to be less than 1? The other case of interest is when F j k is equal to 1; that means with the probability 1, you will be ever visiting the state k by starting from the state j with the probability 1; or, whether this probability is going to be less than 1. If it is less than 1, then it is not the correct one; that means with the 1 minus of this probability, there is a possibility you would not ever visit the state k if you start from the state j – the first case.

The second case says, with the probability 1, you will always reach the state k, whatever be the number of steps starting from the state j. So, our interest is both less than 1 as well as equal to 1. So, the F j k – equal to 1 that will give the probability distribution; and that distribution is called the first passage time distribution. So, this case is our interest and this will give the first passage time distribution, because whenever the system is starting from the state j; whatever be the number of steps; if we are reaching the state k with the probability 1; that means you have the whole mass is 1 and this is going to be the distribution of the first passage time. Using this, I am going to give the next concept called mean first passage time or mean recurrence time.

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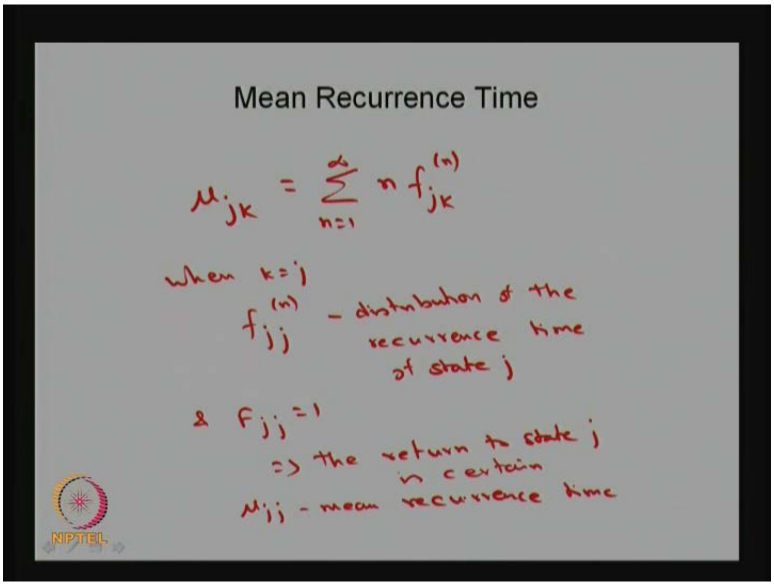

Mean first passage time is same as the mean recurrence time. That is defined as mu suffix j k. That is nothing but what is the average first passage time or average recurrence time. Whenever the system starts from the state j to the state k; that is, how many steps you have taken and what is the probability that starting from the state j to k in n steps. And for all possible values of n, that summation is going to give the mean first passage time or mean recurrence time. Then our interest will be when k equal to j; return to the same state so; that means f j j of n – that will give the distribution of the recurrence time of the state j.

And, if F j j equal to 1, this corresponding f suffix j j n is going to be the distribution. So, correspondingly, F j j is going to be 1. This implies the return to the state j; whenever the system starts from the state j; that is certain, because that probability is 1. Whenever F j j is 1; that means with the probability 1, if you start from the state j, you will definitely come to the state j. Therefore, that is corresponding to F j j is equal to 1. And the mu j j – that will give what is the mean recurrence time. mu j j – the mu j j will give mean recurrence time for the state j. So, we are considering the second case in which F j j is equal to 1. So, that is nothing but the return to the state j whenever the system starts from the state j is certain. And the small f j j of n will give the distribution of the recurrence time. And our interest is also for the mean recurrence time. That can be calculating by using mu j j.

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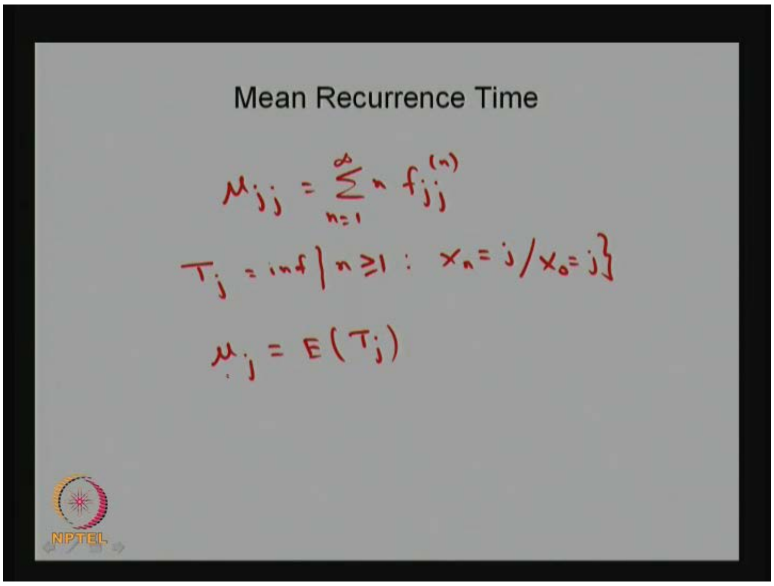

Earlier, we have given mu j j; that is same as n times f j j of n. By knowing f j j of n, we can find out the mean recurrence time for the state j. The same thing can be obtained by using another concept by introducing the random variable, that is, T suffix j; that is nothing but inferior of n greater than or equal to 1 such that the X n is state j given that X naught was state j. This is the random variable denoting the first return time to the state j. The first return time – time here it is the nth step. And you find out what is the first time you return to the state j starting from the state j reaching the state j. So, whatever be the first number – that integer; and that is going to be the T j; and this is going to be the random variable. So, using this random variable also, you can give the definition of a mean recurrence time.

Now, I can define the mean recurrence time – mu j; you do not want two suffix j comma j; one suffix is enough. So, mu suffix j is nothing but what is the expected or expectation of the random variable T j. So, the T j will give the step that denotes the first return time. Therefore, the expected first passage time – that you can write it as the mu suffix j. So, this mu suffix j and mu j j – both are one of the same. And here you are finding the distribution. And using the distribution, you are getting. And here you are finding the time and finding the average time using the expectation of T j. So, in both ways, one can define the mean recurrence time.

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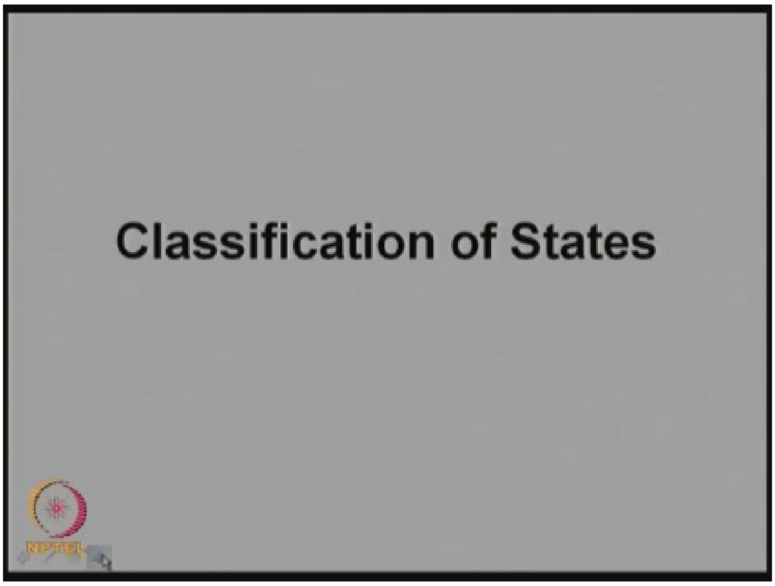

Now, we are going for the actual classification of a state using the concept of accessible, communicate, closed set, then communicating class. Then we have defined a first visit; then we have defined the mean passage time or mean recurrence time or mean first passage time. So, using these concepts, we are going to classify the states.

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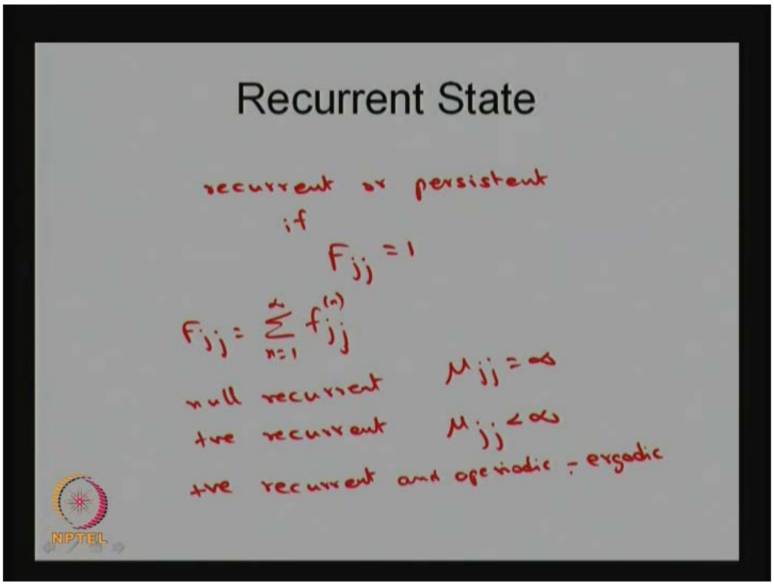

The first definition is recurrence state. A state j is set to be recurrent or the other word called persistent if the F suffix j j equal to 1. If you recall what is F j j; F j j is the probability of ever enter to the state j given that it was in the state j. So, the F j j – I have given in the summation form of a small f j j of n using the first visit. So, if you recall, the F j j is nothing but what are all the possible ways the system can reach the state j as a first visit. You add all the combinations, all the probabilities; that is going to be the capital F j j. So, if capital F j j… That means the probability of returning to the same state j, if that probability is certain; that means if the probability is 1, then that state is going to be the recurrent state.

We can classify the recurrent state into two forms: one is called null recurrent; the other one is called the positive recurrent based on the mean passage time value. So, based on the capital F j j, that is, a probability, we classify the state is going to be a recurrent state. Now, based on the first passage time distribution – the mean first passage time, we are going to classify that, recurrent state is going to be a null recurrent or positive recurrent. Accordingly, the mu j j – if it is a finite value, then we say that, recurrent state is going to be the positive recurrent state. If mu suffix j j is going to be an infinite value; that means on average, the first passage time is going to be infinite; then that corresponding recurrent state is going to be called as a null recurrent state. So, whenever any state is going to be called as a recurrent state, if the probability of ever entering into the state j starting from the state j, it is certain; or, the probability is 1, then that is a recurrent state.

And, the recurrent state is going to be called as a null recurrent, if the mean first passage time or mean recurrence time or mean return time is infinity. If that is going to be a finite quantity, then the recurrent state is going to be called as a positive recurrent state. If any state is going to be a positive recurrent as well as a periodic, then that state is going to be called as a ergodic state. Any state is going to be called as a ergodic, whenever that state is a positive recurrent as well as a periodic. A periodic means the periodicity of that recurrent state is 1; that means the greatest common devisor of all possible steps in which the system coming to the same state – that value is this one. If the period is 1 and as well as the positive recurrent; it should be a recurrent as well as positive recurrent; that means the mean recurrent time is going to be a finite quantity. Then it is going to be called as a ergodic state.

In a Markov chain, if all the states are going to be ergodic one; that means all the states are going to be positive recurrent as well as periodic, then we call that Markov chain itself as ergodic Markov chain; that means there is a possibility, the Markov chain may be irreducible; that means you will land up with only one class in which all the states are going to form a one close communicating class. Suppose each one state is going to be positive recurrent and periodic, then all other states are also going to be of the same type and same period. Therefore, all the states are going to be the ergodic states. Then that Markov chain is going to be called as an ergodic Markov chain.

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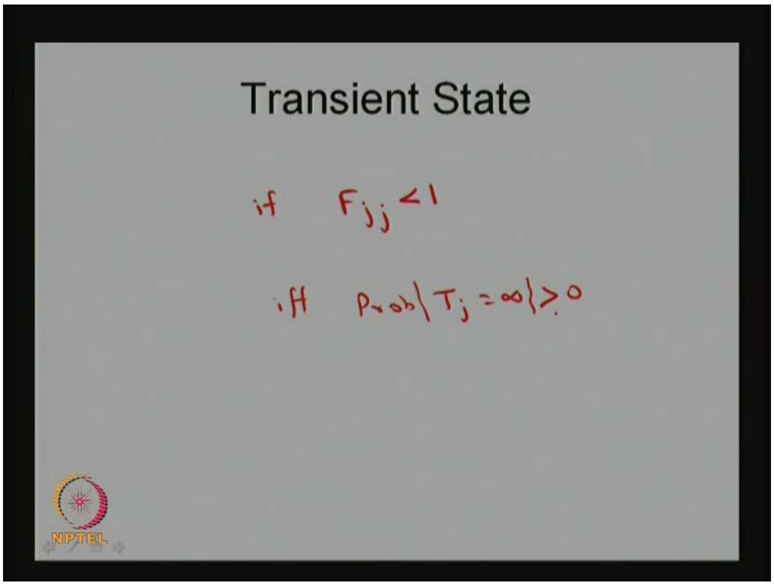

Now, I am going to classify the state as a transient state whenever the F j j value is less than 1. If you recall, we have considered only two cases: whether the F j j is less than 1 or F j j is equal to 1. Equal to 1 land up recurrent state; and F j j is less than 1 – that gives the transient state; that means the probability of returning to the state j starting from the state j is not certain; that means 1 minus of this probability with that much probability, the system may not return to the same state j if the system starts from the state j; that means with some positive probability, because 1 minus this value is less than 1. Therefore, 1 minus of F j j is going to be greater than 0. So, with some positive probability, the system may not return to the same state if it starts from the state j. Then that corresponding state is going to be called as a transient state.

By seeing the one-step transition probability matrix or by seeing the state transition diagram of discrete-time Markov chain, you can easily come to the conclusion the state is going to be a recurrent state or a transition state. Whenever it is going to be a finite number of states, it is easy to come to the conclusion. If it is infinite number of states, then we need some work to be needed to come to the conclusion whether it is a positive recurrent or null recurrent. But easily, you can make out the given state is going to be a transient state. That you can make out from the state transition diagram or one-step transition probability matrix.

The conclusion of the state is going to be the transient state; that can be given via the random variable T j also. So, the state j is transient if and only if the probability of that T suffix j is equal to infinity and that if this probability is strictly greater than 0. The probability of the mean, the probability of the system return to the first passage; the first passage return time – that is infinity, if that probability is greater than 0; that means there is a certainty over the system return to the state j with the infinite amount of time going to take. If that event is going to be with the positive probability, then that state is going to be the transient state. So, there are through two ways we can conclude the given state is going to be the transient state: either F j j is less than 1 or the probability of T j equal to infinity, which is greater than 0.

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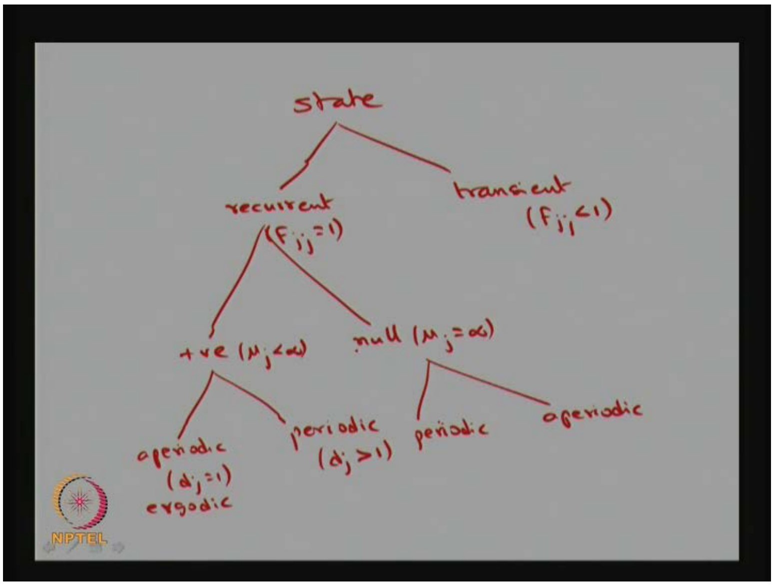

Based on this, I can come to the conclusion any state could be recurrent or transient; that means this is corresponding to F j j is less than 1 and this is corresponding to F j j is equal to 1. I can classify the recurrent state into two forms: either it could be a positive recurrent or null recurrent. Positive recurrent corresponding to the mu j or mu j j – both are the one of the same; that is going to be finite value. Or null recurrent is corresponding to mu j is equal to infinity; that means based on the mean recurrence time, you can conclude whether it is a positive recurrent or null recurrent.

Again, I can classify the positive recurrent into two: one is aperiodic and other one is periodic. Periodic means that corresponding positive recurrent state – that period is greater than 1. Aperiodic means that T j is 1. So, the aperiodic – a positive recurrent state – that is going to be called as a ergodic state. Similarly, I can classify the null recurrent state into two: one is periodic and other one is aperiodic. The absorbing state is a special case of positive recurrent state, where the transition probability from a state to itself is 1. So, this is the way you can classify the state is a recurrent state or transient state, positive recurrent state, null recurrent state. Again, each one could be aperiodic or periodic state.

In this lecture, we started with the few concepts of accessible, then communicate, then closed set. Then we have discussed communicating class. Then we have discussed what is the meaning of first visit. Then we have given the first passage time. Then we have given the mean first passage time distribution or mean recurrence time distribution. So, based on those concepts, we have classified the state as a recurrence state or transient state. So, this is related to the probability; whereas, conclusion of the positive recurrent or null recurrent is related to the average time. So, here only it involves the probability that whether (( )) certain probability, the system will come to the same state with the probability 1. Whereas, here there is uncertainty; the system may not come to the state j, if the system starts from the state j. If there is uncertainty of returning; that means with some positive probability, the system would not be back; then that state is going to be called as a transient state. So, this you can easily visualize in the state transient diagram of any discrete-time Markov chain.

You can see it whether the… By seeing the state transition diagram, you can come to the conclusion whether the state is going to be the transient or recurrent. But through these diagrams, you cannot come to the conclusion whether it is going to be a positive recurrent or null recurrent unless otherwise you will evaluate this quantity; mu j is going to be n times f j j of n So, you find out that summation. So, based on the summation value, it is going to be a finite one or infinite one. Accordingly, that means, whether the mean recurrence time or mean return time or mean first passage time is going to be a finite quantity or infinite quantity. Accordingly, you can conclude whether that recurrent state is going to be a positive recurrent or null recurrent. So, here you need a computation. Whereas, by seeing the state transition diagram; sometime you can come to the conclusion whether it is a transient state or recurrent state.

Now, the issue of periodicity; the periodicity is important to conclude whether the limiting distribution exist or not, whether that is going to be unique. So, you need to find out the aperiodic or periodic. So, if the period is going to be 1, then that state is going to be called as aperiodic. If the period is greater than 1, then it is a period with that integer. When it is going to be a null recurrent, then also we can come to the conclusion whether it is periodic or aperiodic. Whenever you have a Markov chain with the finite number of states, then it is easy to find out whether it is going to be a positive recurrent or transient.

So, a quite good exercise is needed whenever the Markov chain have infinite number of states; then you need some work to be done to come to the conclusion it is a null recurrent and so on. In today’s lecture, with this classification, I stop here. And all the simple examples and the limiting distribution – that I will explain in the fourth lecture. Thanks.