Stochastic Processes

Prof. Dr. S. Dharmaraja
Department of Mathematics
Indian Institute of Technology, Delhi

Module - 4
Discrete-time Markov Chain
Lecture - 2
Chapman-Kolmogorov Equations

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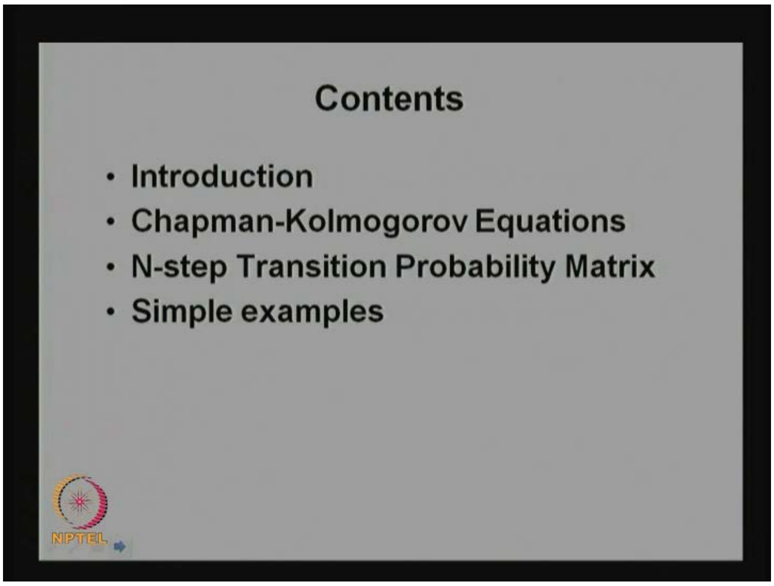

Good morning; this is a stochastic process module 4, discrete time Markov chain, and this is the lecture 2. And in this lecture, we are going to discuss about the Chapman Kolmogorov equations. Then we are going to discuss N step transition probability matrix, and then we are going to discuss the few more examples in this lecture.

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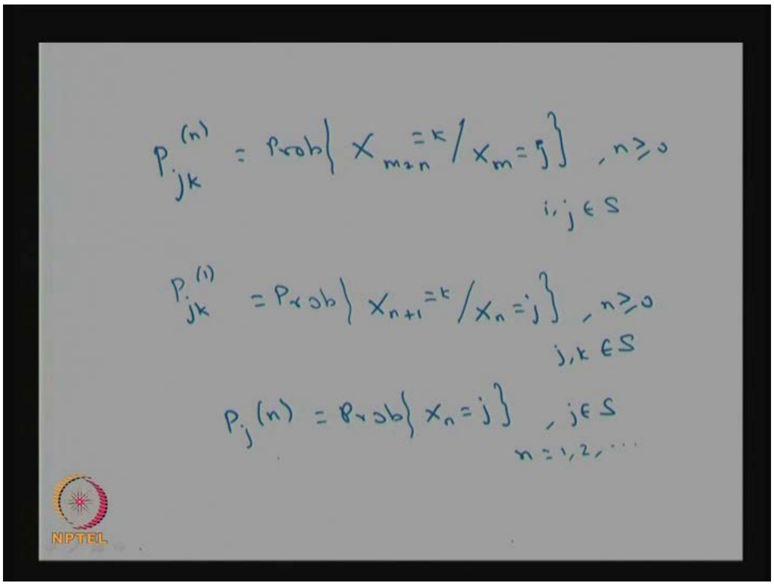

In the last class we have discussed, the transition probability of j to k in n steps, as the probability that, the X m plus n takes a value k given that X m was j, for n is greater than or equal to 0, and i comma j belonging to s. Since the underlining DTMC is the time homogenous, this is the n step transition probability of system is moving from the state j to k in n steps. So, this we denoted as the conditional probability of p j, k in n step transition probability, where i comma j is belonging to s, where s is the state space, and n can take the value greater than or equal to 0. Also we have discuss in the last class, what is the one step transition probability of p j k, it is we can write it within the bracket 1, or we can remove the bracket 1 in the superscript also; that is nothing but what is the probability that the system will be in the state k in n plus oneth step, given that it was in the state j in the nth step. Here also j, k belonging to capital S.

So, this is the one step transition probability. So, our interest is to find out, what is the distribution of X m, whenever the sequence of random variable X n is time homogenous DTMC, our interest is to find out the distribution of X n. So, it has the probability mass function, the p j of n; that is nothing but what is the probability that, the system will be in the state j at the nth step. So, the j is belonging to s, and n can be 1 or 2 and so on, because you know the distribution of n is equal to 0; that means, you know the initial probability vector of x 0. So, our interest is to find out what is the distribution of X n, or n is equal to 1 2 3 and so on. So, how you are going to find out this distribution?

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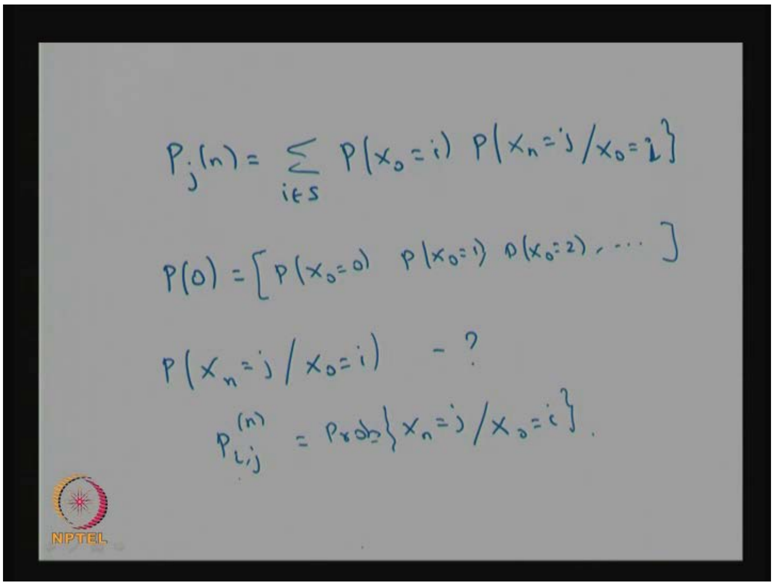

So, this distribution can be written, using the p j of n is nothing but the summation over i belonging to s; such that the system was in the state i at zeroth step, and multiplied by what is the probability, that it will be in the state j given that it was in the state i at nth, at zeroth step. So, this is nothing but what is the probability that, the system will be in the state j; the nth step, that is same as what are all the possible ways, the system would have been moved from the state i from the zeroth step, to the state j in the nth step. So, this is the product of one marginal distribution and one conditional distribution, for all possible values of i that gives the distribution of a X n in the nth step. So, for that you need to compute this distribution of X n, you need n step transition probability, as well as the initial distribution vector, or initial probability vector, or the distribution of X naught. So, the distribution of X naught can be given as the vector p of 0.

This is a vector p of 0, with consist of the element what is the probability that X naught takes the value 0, what is the probability that X naught takes the value 1, what is the probability that X naught takes the value 2 and so on. So, this is the initial probability vector, why we have taken the state 0 1 2 and so on. Unless otherwise as mention the set of the state space; that is going to be the possible values of 0 1 2 and so on, in this unless, otherwise it is assume it, you can take always this values. So, this is a initial probability vector, or initial distribution vector. So, what we need, what is the n step transition probability of the system will be in the state j, given that it was in the state i at the zeroth step, this is what you want to find out.

What is the conditional probability mass function of n step transition probability vector. So, that we can write it in the form of p i, j of superscript n; that is nothing but the probability of, the system will be in the state j, given that the system was in the state i at the zeroth step. That is we need to compute the n step transition probabilities; that is p i j of n. So, this can be computed by using the method called Chapman Kolmogorov equations. So, this Chapman Kolmogorov equation, provide a method, or computing this n step transition probability probabilities. So, how we are going to derive this Chapman Kolmogorov equation, that I am going to do it in the do it now. So, we are going to derive the Chapman Kolmogorov equations, for the time homogenous discrete, time Markov chain.

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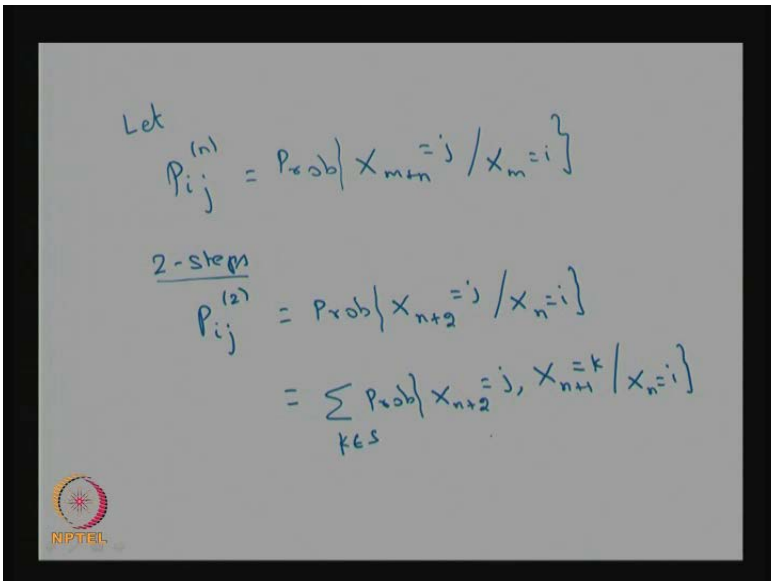

So, let the p i j of superscript n; that is nothing but what is the probability that the X m plus n takes the value j, given that X m was i. Since the discrete time Markov chain is a time homogenous, so this is the transition probability of system moving from the state i to j, from the mth step to m plus nth step. Therefore, this transition is the n step transition probability matrix, for the time homogenous discrete time Markov chain. Let us start with the two step; the two step is nothing but what is the probability that, system is moving from the state i to j in two steps. So, n plus 2 takes the value j given that x n was i, this for all n it is true, because the d t m c is discrete time homogenous. So, this probability we can write it as; this two steps transition probability of system moving from i to j, the state i to the state j in two steps, that you can write it as what are all the possible ways, the system is moving from the state i to j by including one more state, in the first step the state is k, given that the system was in the state i in then the step, for all possible values of k belonging to s. I can write this conditional two step; conditional probability mass function from the nth step to n plus second step; that is same as. I can include one more possible state of k in the n plus oneth step. Now I can expand this as, that is same as for possible values of k.

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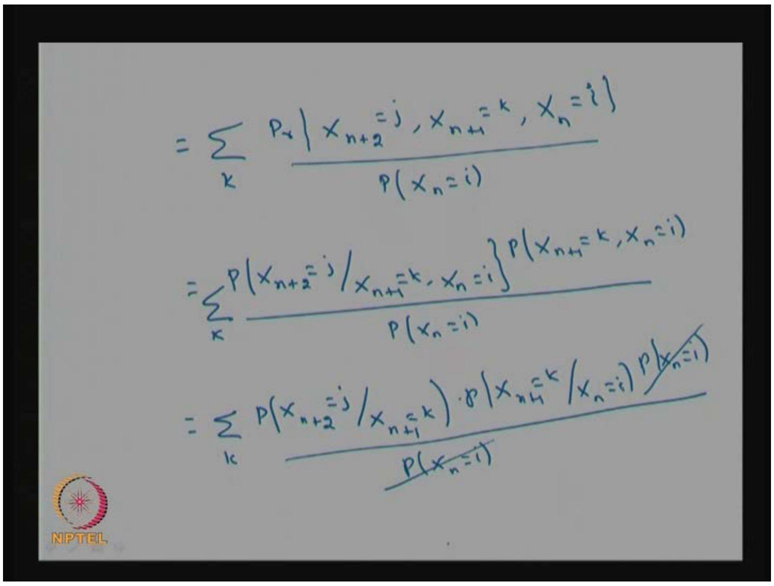

And what is the probability that, the system was in the state j in the n plus second step, and the system was in the state k in the n plus oneth step, the system was in the state i in the nth step, divided by what is the probability that in the nth step, it is in the state i. The numerator joint distribution of this three state, this three random variable, that I can write it as in the form of conditional, what is the conditional probability that, the X n plus 2 takes the value j, given that X n plus 1 takes the value k, and x n takes the value i product of X n plus 1 takes a value k, X n takes a value i divided by what is the probability that X n takes a value i and here the summation is over the k. So, basically I am writing the numerator, joint distribution of this three random variable, as the product of the conditional distribution with the marginal distribution of those two random variable.

Since the X i's are time homogeneous Markov chain, this conditional distribution by using the Markov property, is same as the conditional distribution of X n plus 2 takes the value j given that only the latest value is important, the latest value is a needed not the previous is history, therefore because of the memory less property, X n takes the value i is free mode. Therefore, this conditional distribution, is the conditional distribution with only X n plus 1 with X n plus 2. And similarly I can apply the joint distribution of this two random variable X n plus 1 and X n. I can again write it as the probability of X n plus 1 takes the value k, given that X n takes the value i and probability of X n takes a value i, whole divided by probability of X n takes the value i. So, this and this get cancelled, so this is nothing but the conditional probability. This is nothing but the one step transition probability of system moving from k to j, and the second term is, the one step transition probability of system is moving from i to k.

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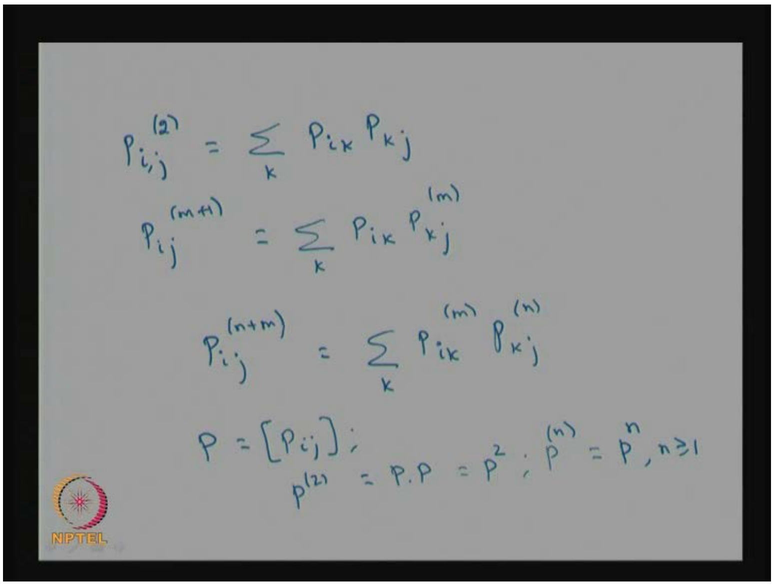

Therefore, the left hand side we have, what is the two step transition probability of i to j, is same as all possible values of k. What is the one step transition probability of system is moving from i to k, and one step transition probability of k to j. So, this product will give, the two step transition probability of system is moving from the state i to j; that is same as, what is the possible values of k, the system is moving from the state i to k, and k to j, so this is for the two step. Similarly by using the induction method one can prove, i to j of m plus one step; that is same as what is the possible values of k, the system is moving from one step from i to k, and m steps from k to j, this is the two step. So, this one step from i to k, and one step from k to j, by induction I can prove the m plus one step will be i to k, and k to j in n steps. Similarly I can make it in the other way round also, it is i to k in m steps, and k to j in one step also.

That combination also land up the m plus one step, the system is moving from i to j. In general, we can make the conclusion, the system is moving from i to j, in n plus m steps; that is same as the possible values of k, of probability of system is moving from i to k in m steps, and the n step, the system is moving from k to j; that will give for all possible values of k, that will give the possibility of system is moving from i to j in n plus m steps. So, this equation is known as Chapman Kolmogorov equation for the time homogeneous discrete time Markov chain. So, whenever you have as, stochastic process is time homogeneous discrete time Markov chain, then that satisfies this equation, and this equation is known as the Chapman Kolmogorov equations. In the matrix form, you can write the capital P is the matrix, which consist of the element of one step transition probability one step transition probabilities. In that case, if you make m is equal to 1, and n is equal to 1.

Then the matrix of p of superscript 2; that is the in matrix form of two step transition probability; that is nothing but, you put n is equal to 1 and m is equal to 1, you will get p into p, and that is going to be p square. So, the right hand side p superscript within bracket 2 means, it is the two step transition probability matrix, and the right hand side the p square; that is the square of the p matrix, where p is the one step transition probability matrix. So, in this form, in general you can make, the n step transition probability matrix is nothing but p of n, for n is greater than are equal to 1, for n is equal to 1 it is obvious, for n is equal two onwards, the p power n that is same as the n step transition probability matrix. Hence, now, we got the n step transition probability is nothing but, the p power n, where p is the one step transition probability matrix.

Therefore, in matrix form, I can give the p of n, the p of n is nothing but in the matrix form of the distribution of X n, or this is nothing but the vector, which consists of the nth step, where the system will be. So, this is nothing but what is the probability that, in the nth step, the system will be in the state 0, or in the nth step, the system will be in the state 1, and in the nth step, the system will be in the 2 and so on, this is the vector. So, the p of n you can find out in the matrix form, by using the above equation.

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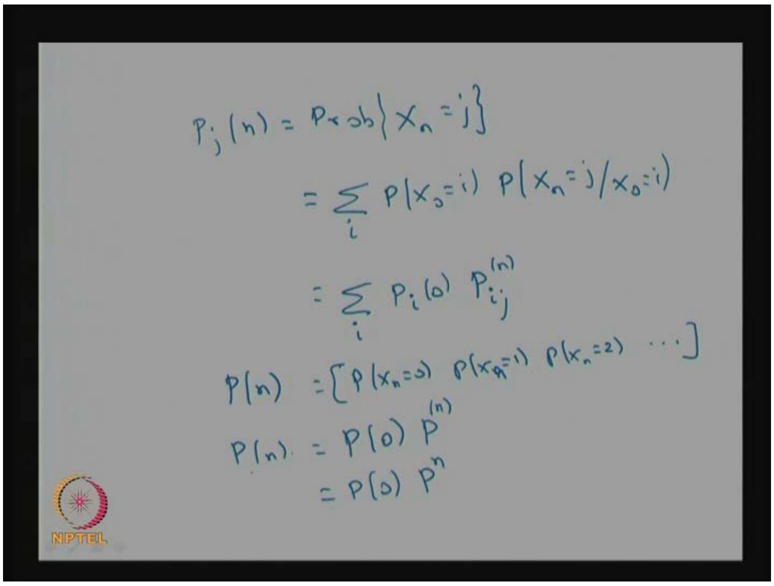

It is going to be p of 0; that is also vector, initial probability vector multiplied by p power p of within bracket n; that is n step transition probability matrix, but the n step transition probability matrix is nothing but the p power n. Therefore, this is same as the p of 0 into p power n. In the last slide, we got p of a superscript within bracket n; that is the n step transition probability, matrix is same as the one step transition probability with the power n.

Therefore, this is going to be the distribution of X n in the vector form; that is same as the p 0 multiplied by the p power n, where the p is nothing but the one step transition probability matrix; that means, if you want to find out the distribution of X n for any n, you need only the initial probability vector, and one step transition probability matrix. Because the discrete time Markov chain is time homogeneous, we need only the one step transition probability matrix, and the initial probability vector, that gives to find out the distribution of X n for any n. So, with the help of one step transition probability matrix and the initial probability vector, you can find the distribution of X n for any n. Now, we are moving into simple examples, using the n step transition probability matrix, and the one step transition probability vector, how to find the distribution of X n for some simple example.

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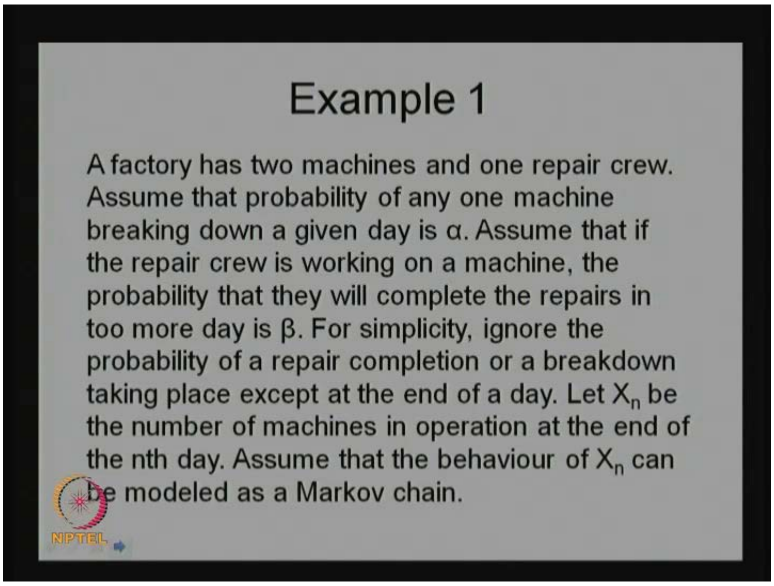

The first example which I have discussed in the lecture one. This is the very simple example in which, the underlying stochastic process is the time homogeneous discrete time Markov chain, with the state space S is 0 1 and 2.

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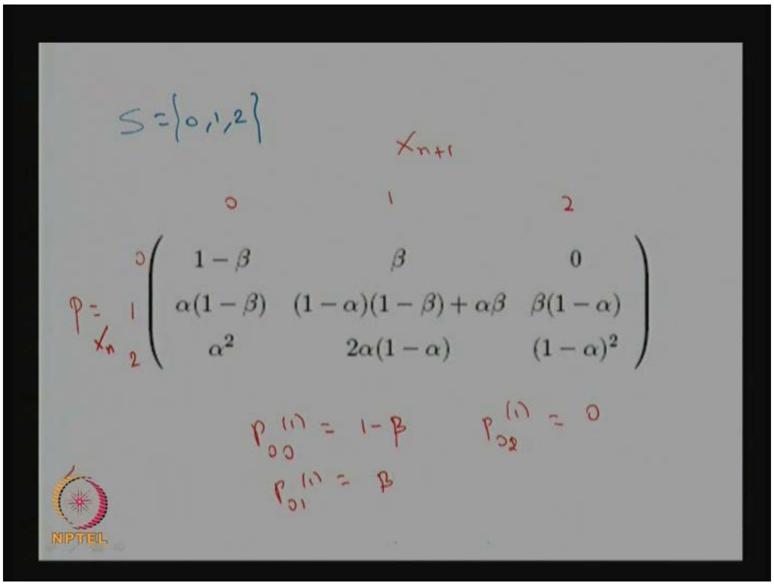

So, this is the state space, and the information which we have based on that we can make a one step transition probability matrix; that is nothing but what is the possible probability, in which the system is moving from the state i to j in one step that you can fill it up. So, this exercise we have done it in the lecture one.

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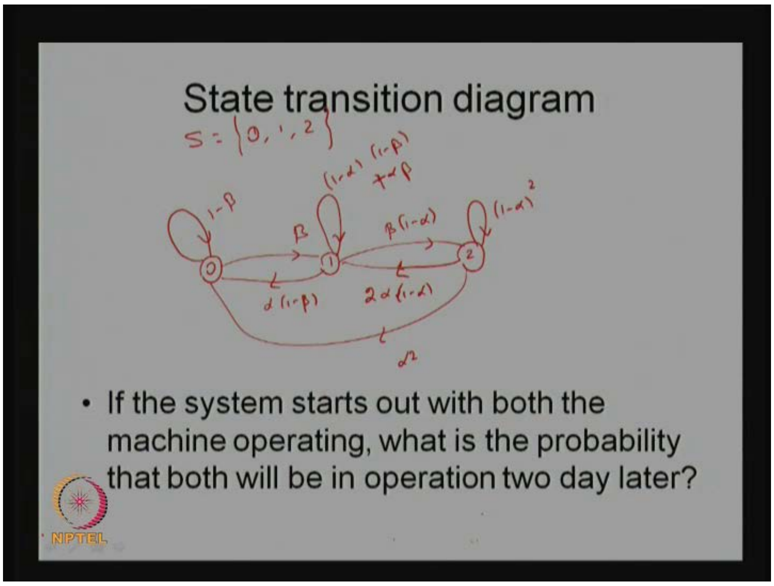

And now our interest is, to find out, and also we have made the state transition diagram, which is equivalent to the one step transition probability matrix, and we have got the state transition diagram. Now, the question is, if the system starts out with both the machines operating, what is the probability that, both will be operation two days later. So, if you recall what is the random variable, X n be the number of machines in operation at the end of the nth day. So, the random variable is, how many machines are in the operation at the end of nth day. So, here the clue, is at time 0, or the zeroth step, both the machines are operating; therefore, x 0 is equal to 2 with the probability 1. So, the given information with the probability 1, the both the machines are working at zeroth step.

So, this can be converted into the p X 0 takes a value 2, that probabilities. Or we can make it in the initial probability distribution, or initial probability vector, as what is the probability that at X 0 the system was in the state 0, at the zeroth step the system was in the state 1, so this is the initial probability vector. So at time 0 the system was in the state 2; therefore, that probability is 1 and all other probabilities are 0. So, this is the given information about the initial probability vector.

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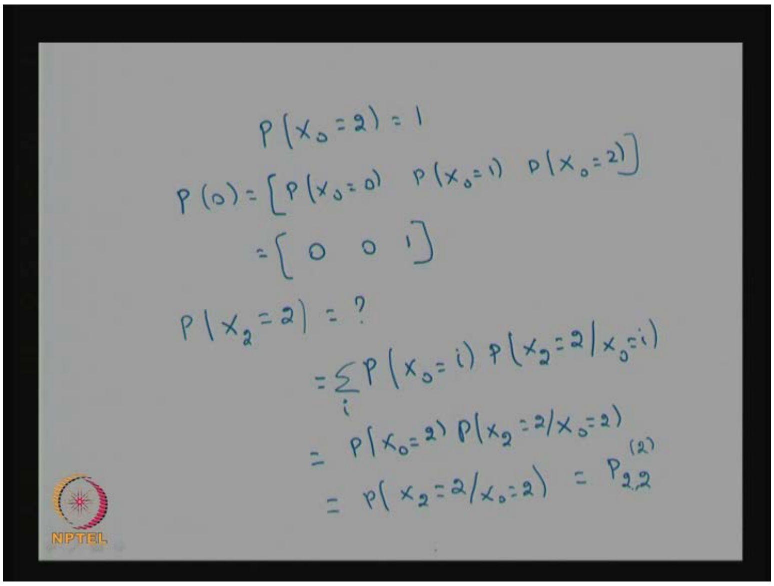

Now, the question is, what is the probability that both will be operation two days later; that means, what is the probability that, you can convert this into, what is the probability that, X 2 in the second step, the system will be in the state 2, given that the system was in the state 2 at the zeroth step. So, this is what you have to find out, what is the conditional probability. If the system starts with the both the machines, what is the probability that both will operation in two days later. So not even this is a conditional probability, the question is what is the probability that, the system will be in the state.

So, to find this, you can make a what is the probability that, so with the given information is there, X 2 is equal to 2 given that X naught is equal to i, for all possible values of I, and this is same as; since the initial probability vector is going to be 0 0 1, so this is land up what is the probability that, the X naught is equal to 2 multiplied by what is the probability that X 2 is equal to 2, given that X naught is equal to 2, and all other probability are 0, therefore 0 in to anything is going to be 0. Therefore, the same as what is the probability that X 2, X naught is equal to 2 and the conditional probability, and X naught is equal to 2 is 1; therefore, this is same as what is the probability that, X 2 is equal to 2 given that X naught is equal to 2. So, this is the same as what is the probability that 2, 2 in two steps. This is nothing but the system was in the state 2 at zeroth step, and the system will be in the state 2 after the two steps. So, this is the two step transition probability of system moving from the state 2 to 2.

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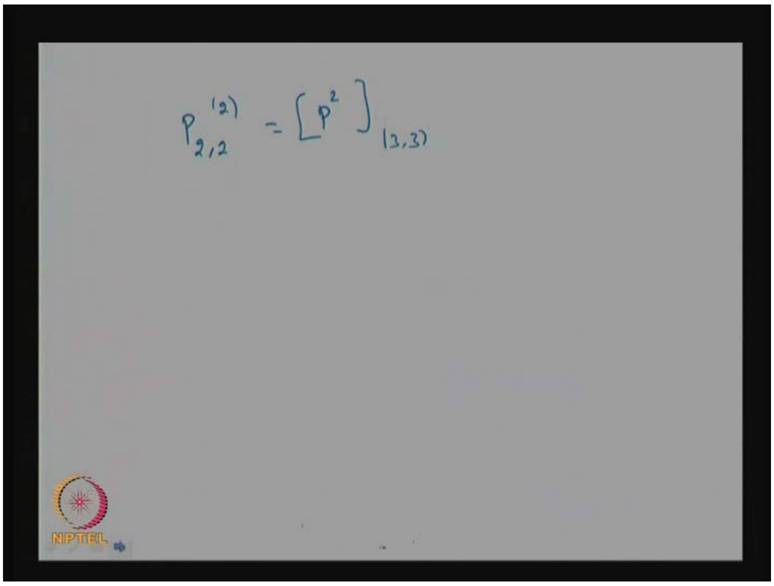

This is same as, you find out the p square matrix, and from the p square matrix, this is nothing but the 2, 2 that is going to be the last element. Out of that nine elements, the third row third column element; that is going to be the element for this probability. So, what you have to find out is, find out the p square. So, we have provided the p. So, this is the p matrix. So, from the p matrix you find out the p square. So, the p square is also going to be a 3 cross 3 matrix. So, from the p square 3 cross 3 matrix, you take the third row third element, third column element, and that is going to be the probability for two step transition of system moving from the state 2 to 2; that is going to be the answer, for the given question, what is the probability that both will be operation into two days later. Similar to this, we can find out the probability for any day, for any finite day by finding the p power n matrix, then pick the corresponding element, and that is going to be the corresponding probability.

Now, we will move into the next example, this is the abstract example, in which the X n be the discrete time Markov chain. The default discrete time Markov chain is always, it's a time homogeneous. So, this is the time homogeneous discrete time Markov chain, with this takes place 0 1 and 2. And also, it is provided the initial probability vector; that is a p 0; that is a vector, that is the one fourth of one fourth. So, the summation is going to be 1, therefore, this is the initial probability vector; that means the system can start from the nth zeroth step, with the probability one fourth from the state 0.

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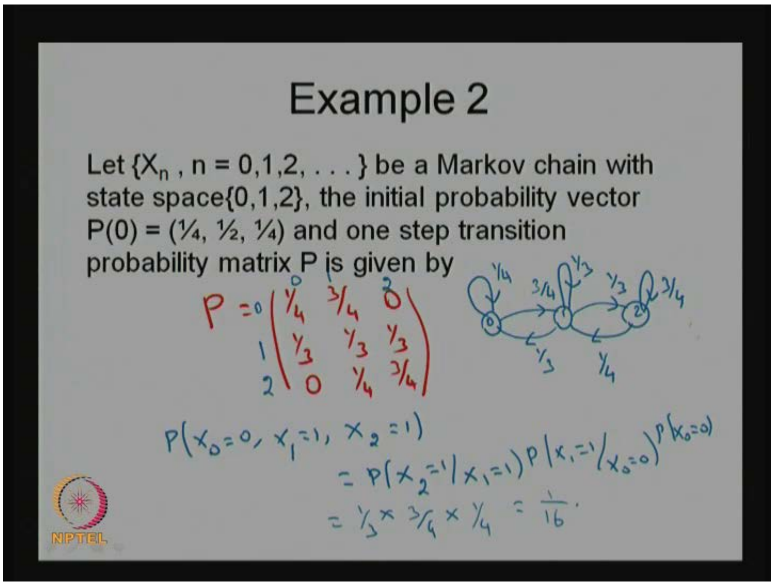

From the state 1 with the probability of, with the probability 1 forth it can start from the state 2. And also it is provided the one step transition probability matrix. From the one step transition probability matrix, you can draw the state transition diagram also, because the state space 0 1, 2. Therefore the nodes are 0 1 and 2, and this is the one step transition probability; therefore, 0 to 0 that probability one step of one step, the system is moving from the state 0 to 0; that is one fourth, and the system is moving from the state 0 to 1 in one step; that is the three fourth, and there is no probability from the going from the state 0 to 2, therefore, we should not draw the arc.

From 1, the one step transition probability of 1 to 1, is one third, and this is one third, and similarly this is one third. From the state 2, 2 to 0 is 0, and 2 to 1 is one forth, and 2 to 2 is three forth. This diagram is very important to study the further properties of the states. Therefore we are drawing the state transition diagram, for the discrete time Markov chain. So, this is the one step transition probability matrix, and this is the state transition diagram. Our interest is to find out, the few quantities; that is what is the probability that, X naught is equal to 0, and X 1 is equal to 1, and X 2 is equal to 1. What is the probability that, the system was, it is a joint distribution of these three random variable, X naught is equal to 0, and X 1 is equal to 1, and X 2 is equal to 1. So, this is same as the joint distribution, the same as, you can write it in the product of the conditional distribution.

And the conditional distribution again you can write it using the Markov property, the conditional probability of only one step. Therefore, this is going to be, by using the probability theory, you apply the joint distribution is same as the product of conditional distribution, by using the Markov property you reducing into the another conditional distribution. So, this is same as what is the probability that X 2 is equal to 1 given that X 1 is equal to 1 multiplied by X 1 is equal to 1 given that X naught is equal to 0, and probability of X naught is equal to 0. So, this is the first term is nothing but the one step transition of system is moving from 1 to 1, and this is nothing but the system is moving from the state 0 to 1. And this is the initial, you take the probability from the initial probability vector of X naught is equal to 0. that

Now we are going to label the one step transition probability matrix, with the 0 1 2 and 0 1 2, from this you can find out. This is the one step transition probability of system moving from 1 to 1. So, 1 to 1 is one third, into this is the system probability of system moving from 0 to 1. So, 0 to 1 is the three forth, and system started from the state 0 from the zeroth step, so that you can take it from the first element; that is one forth. So, if you do the simplification you will get 1 by 16. So, this is the joint distribution of, the system was in the state 0 at zeroth step, the system was in the state 1 at the first step, and the system was in the state 1 at the second step that probability is 1 by 16.

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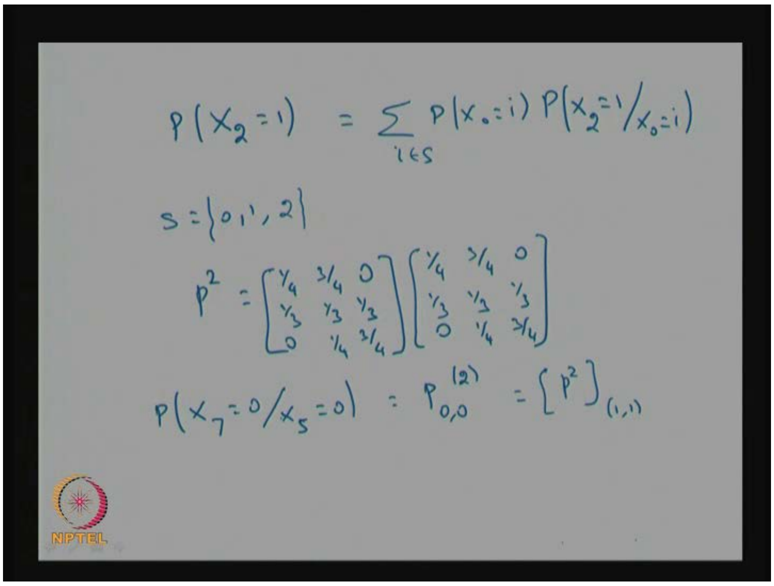

Similarly, you can find out the other probabilities also; that is a, suppose our interest is what is the probability that, at the end of a second step, the system will be in the state 1; that is nothing but what are all the possible states in which the system would have been started from the state i, and what is the two step transition of system is moving from the state i to 1, so the i is belonging to s. So, here the s is 0 1, 2. So, already we have given the initial probability vector; that is the one fourth of and 1 forth. Using this and you need two step transition probability; that means, you need to find out what is the p square. So, the p square will give the two step transition probability matrix; therefore, the p is provided to you.

So, the p is 1 forth 3 forth and 0, one third, one third, one third, 0, 1 forth, 3 forth. So, this is the p, so you multiply the same thing one third, one third, one third, 0, 1 forth, 3 forth, you find out the p square. So from the p square, you pick out the element of X 0 is equal to for all possible values i, then multiplied this and this, that multiplication will give probability of X 2 is equal to 1. So, I am not doing to do the simplification. So, once you know the p square, you can find out probability of X 2 is equal to 1. Similarly one can compute the other conditional probabilities also. Suppose our interest, find out what is the probability of X 7 is equal to 0 given that X 5 was 0.

This is same as, what is the probability that, the system was in the state 0 if the 5th step, given that what is the probability that, the system will be in the state 0 in the 7th step. That is same as what is the probability of two, what is the probability of sorry what is the probability of 0, 0 in two steps; that means, you find out the p square. From the p square the 0, 0 is the nothing but you take the first row first column element; that is going to be the probability of X 7 is equal to 0 given that X 5 is equal to 0. Similarly, you can find out all other different conditional probability, and what you have to do is, always you have to convert, because of the given DTMC; say time homogeneous. So, you converted into find out the n step transition probability, and n step transition probability is same as, the p power n. So, you pick the corresponding element, to find out the conditional probability.

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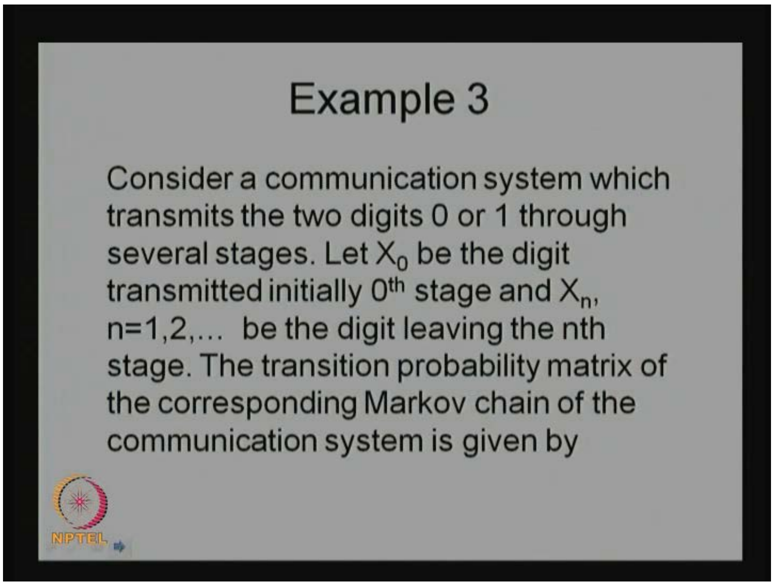

Now, you are moving into the third example. This example, talk about the communication system, in which the whenever the transmission takes place, with the digits 0 and 1 in the several stages. Now we are going to define the random variable X naught, be the digit transmitted initially; that is a 0th step. Either the transmission digit will be 0 or 1, therefore only two possibilities can be takes place at any nth step transition, either 0 or 1, like that we are making the transmission over the different stages. Therefore, this X n over the n will form a stochastic process, because you never know, which digit is transmitted in the nth stage. So, each stage is going to be a one random variable, and you have a collection of random variable over the n stages.

Therefore it is sequence of random variable, so this is going to form a stochastic process. And this stochastic process is nothing but the discrete time, discrete state stochastic process, because the possible values of X n is going to be 0 or 1, therefore the state space is 0 or 1. And it’s the discrete time discrete state stochastic process, the way the subsequent transmission takes place, depends only on the last transmission, not the previous stages; therefore, you can assume that this follows a Markov property. Therefore this stochastic process is going to be, call it as discrete time Markov chain. Now, our interest is to find out, so now I will provide, what is the one step transition probability for the Markov chain, or let me give the transition diagram for that.

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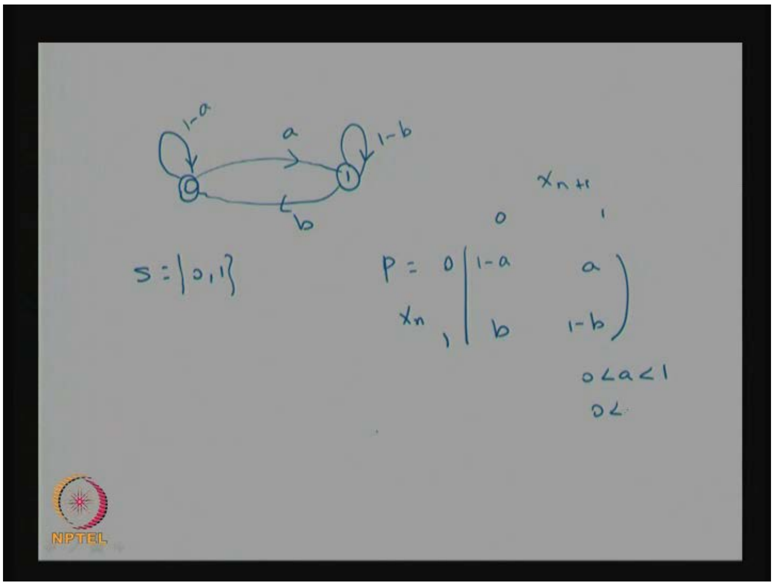

So, state transition diagram the possible states are 0 or 1, because the state space is 0 and 1, and the probability that in the next step also, the transmission is 0 with the probability 1 minus a. This is the conditional probability of the nth stage, the transmission was 0, the n plus 1th stage is also the transmission 0, with the probability 1 minus a. The one step transition probability of system is moving from 0 to 1, that probability is a; that means, the nth stage the transmission was a digit 0, the n plus oneth stage is also the trans the n plus oneth stage the transmission will be the digit 1, with the probability 0. Similarly I am going to supply the one step transition probability of 1 to 1; that is 1 minus b, and this is b. That means this 1 to 0, that probability is 1 to 0, is b, and 1 to 1 is 1 minus b. Obviously, this a is lies between 0 to 1, and b is also lies between 1.

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$$P^{(n)} = \begin{array}{c} 0 \\ 1 \end{array} \overset{\begin{array}{cc} 0 & \quad\quad\quad 1 \end{array}}{\begin{bmatrix} \dfrac{b+a(1-a-b)^n}{a+b} & \dfrac{a-a(1-a-b)^n}{a+b} \\ \dfrac{b-b(1-a-b)^n}{a+b} & \dfrac{a+b(1-a-b)^n}{a+b} \end{bmatrix}}$$

$$for \ |1-a-b| < 1$$

So this is the, a is the probability that, the system is transmitting from the zeroth step, sorry nth stage with the digit 0, and the n plus oneth stage with the probability to the stage with the digit 1, that probability is a; therefore the negation is one minus a, because the system can transmit either 0 or 1. So, once you say that the one step transition probability of 0 to 1 is a, then 0 to 0 will be 1 minus a. Similarly 1 to 0 is given has a probability b, and the other digit transmission will be 1; therefore, it is going to be 1 to 1 will be 1 minus distribution .

So, this is the state transition diagram, and this is the one step transition probability, for a given time homogeneous discrete time Markov chain. Our interest is to find out what is the distribution of X n for n, for that you need what is the n step transition probability matrix. Since the one step transition probability matrix is given, you can find out the p square, p power three and so on. By induction method you find out the p power m, but using the p power m, you can find out the p m plus n. Therefore, you can come to the conclusion, what is the n step transition probability of system is moving from 0 to 1 and 0 to 0 and so on.

So, this is nothing but I am just giving the only the result, b plus a times 1 minus a minus b power n divided by a plus b, and this is nothing but a minus a times 1 minus a minus b power n divided by a plus b. Similarly, if you find out the n step transition probability of system moving from 1 to 0; that is b minus d times 1 minus a minus b power n divided by a plus b. This is nothing but a plus b times 1 minus a minus b power n divided by a plus b. So here I am just giving the n step transition probability in matrix form, by given p, you should find out the p square p power q by induction, you can find out the p power n, and this is valid, provided one minus a minus b which is less than 1, because you are finding the p power n matrix. So, here it needs some determinate also, unless otherwise the absolute of 1 minus a minus b, which is less than 1, this result is not valid. So, provided this condition the p of n; that is the matrix, so that is same as p power n also, p of n is same as p power n.

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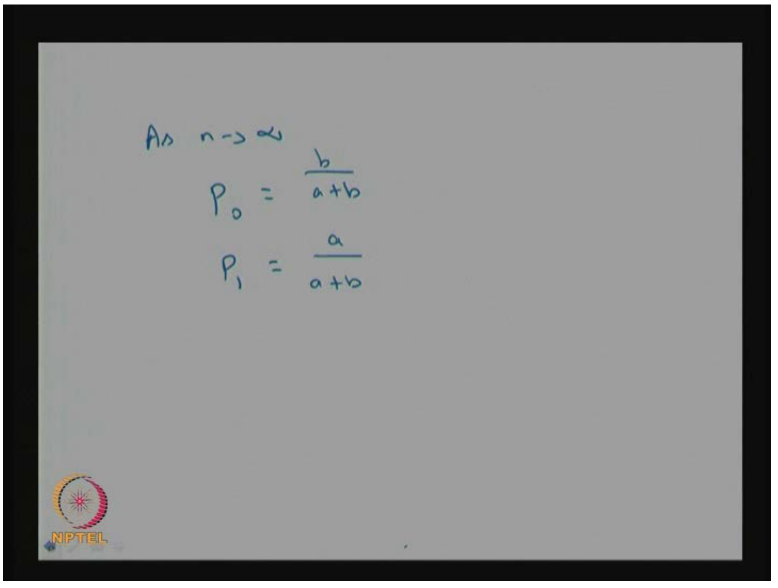

So, as n tends to infinity, as n tends to infinity you can come to the conclusion, what is the probability that, the system will be in state 0; that is same as b divided by a plus b. And similarly, what is the probability that, the system will be in state 1 as n tends to infinity, that will be a divided by a plus b. This kind visualize from the state transition diagram easily, whenever the system is keep moving into the state 0 or 1 with the probability a b, and with the self loop 1 minus a and 1 minus b. The subsequent stages the system will be in any one of these two states. So, with the proportion of b divided by a plus b, the system will be in the state 1. Similarly, with the proportion a divided by a plus b, the system will be in the state 1, to the proportion b divided by a plus b, the system will be in state 0 in a longer. The interpretation of as n tends to infinity, this probability is nothing but in a long run, in a long run with this proportion the system will be in the state 0 or 1. So, this state transition diagram, will be useful to study the long run distribution, or where the system will be as n tends to infinity, to study those things, the state transition diagram will be useful.

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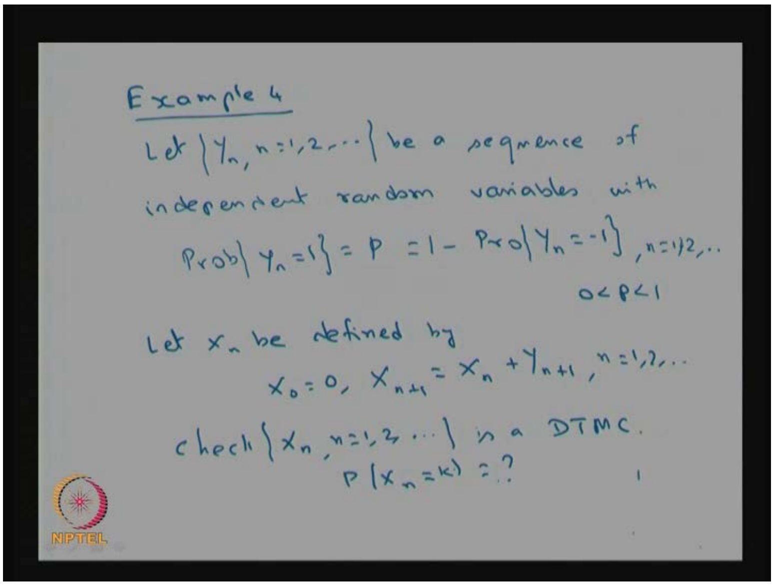

Now, we will move into the next problem; that is example four. Let so sequence of random variable, be a sequence of independent random variables, with condition the probability of Y n takes a value 1; that probability is p; that is same as one minus the probability of Y n takes a value minus 1. We have a stochastic process, and each random variable is the independent random variable, and the probability mass function is provided with this situation. The probability of Y n takes the value 1 is p, you can assume that the p takes the value 0 to 1; that is same as one minus of probability of Y n takes the value minus 1 for all n. Now, I am going to define another random variable, let X n be defined by X naught is equal to 0. Whereas a X n plus 1 onwards that is going to be X n plus Y n plus 1, for n is equal to 1 2 and so on.

So, we are defining a another random variable X n with X naught is equal to 0, and X n plus 1 is equal to X n plus Y n plus 1. Now, the question is, check X n that stochastic process, is the DTMC. If it is a DTMC, also find out what is the probability of X n takes a value k. We started with one stochastic process and we define the another stochastic process with the earlier stochastic process, and check whether the given the new stochastic process is a discrete time Markov chain; that is a default one; that is a time homogeneous discrete time Markov chain, if so, then what is the probability of X n takes the value k; that is nothing but find out the distribution of X n. So, how to find out this, the given or the X n is going to be the DTMC.

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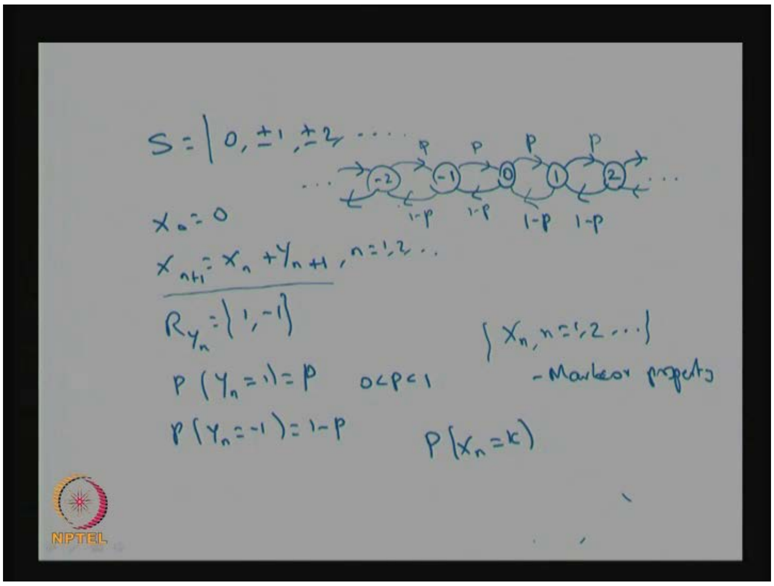

Since Y n takes the value 1 with the probability p, and Y n takes a value minus 1, with the probability 1 minus p, you can make out the possible values of Y n, is going to be 0 or plus r minus 1 plus r minus 2 and so on, because the relation is the X naught is equal to 0, and X n is the X n plus Y n plus 1, and the range of Y n with the range of Y n is 1 comma minus 1; therefore the range of X n that inform a state space, and X naught is equal to 0; therefore, X n that relation is the X n plus 1, sorry X n plus 1 is equal to X n plus Y n plus 1, so n takes the value 1 and so on. Therefore, the possible values of X n will be 0 plus r minus 1 or plus r minus 2 and so on; therefore, that it will form a state space. Now, the given clue is a probability of Y n takes the value 1 is probability p, and probability of Y n takes value minus 1; that is 1 minus p, and the probability p is lies between 0 to 1. So, using this information you can make a state space of the Y n; that is going to be 1 2 and so on, minus 1 minus 2 and so on.

Now, you can fill up what is the one step transition of system is moving from 0 to 1; that means, the X 1, 0 to 1. Suppose you substitute 0 here, then suppose it takes the value 1, then the system can move from the state 0 to 1 in one step. Suppose, you put the value X naught X n is equal to 0, suppose you put X n is equal to 0, and Y n plus 1 takes the value 1, with the probability p. Then the X n plus 1 value will be 1, with the probability

p. Now you can go for what is the state transition probability of 1 to 0. Suppose X n value was 1, suppose Y n plus 1 value was minus 1. Then the X n plus 1 value will be 0. So, the one step transition of a system moving from a 1 to 0, because of happening probability of Y n plus 1 is equal to minus 1. That probability is 1 minus p, so this is going to be. So, whenever the system is moving from one step forward, that probability will be the probability p, and one step backward, that probability will be 1 minus p.

So, this is the way it goes forward step, and this is the way it goes to the backward steps, so you can fill up all other probabilities; forward probability with the probability p, and the backward probability with the 1 minus p. Also we can come to the conclusion, the way we have written X n plus 1 is equal to X n plus Y n plus 1, and all the Y i's are independent random variable. The X n plus 1 going to take the value, depends only on X n, not the previous X n minus 1 or X n minus 2 and so on. Therefore the conditional distribution of X n plus 1, given that X n X n minus 1 till X naught; that is same as the conditional distribution of X n plus 1 given X n. That means, the X n, is going to satisfy the Markov property, because of this relation, because of X n plus 1 is equal to X n plus independent random variable. Therefore, the X n, n is equal to 1 2 3 and so on.

This stochastic process is going to satisfy the Markov property; therefore, this discrete time discrete state stochastic process is the going to be the discrete time Markov chain, because of the Markov property is satisfied. Once it is Markov property is satisfied, by using the Chapman Kolmogorov equations, you can find out what is the distribution of X n takes the value k; that is nothing but where it started at time 0, and what is the conditional distribution of n step transition probability, and n step transition probability is nothing but the element from the p power n, and is from here you can find out the one step transition probability matrix. From the one step transition probability matrix, you can find out p, p square, p cube and so on, and you can find out the p power n, and that element is going to be the n step transition probability, using that you can find out the distribution. Since we do not know the value of p, where p is lies between 0 to 1. it is

I am not going to discuss the computational aspect of finding out the distribution, this is left as an exercise and the final answer is provided. The difference between the earlier example and this example, and this example the state space is going to be a countably infinite; therefore, the p is not going to be a easy matrix, which is going to be matrix with the many elements in it. Therefore finding out the p square and the p power n, it's going to be little complicated than the usual square matrix. Also hence the conclusion is, by knowing the initial probability vector, and the one step transition probability matrix, or the state transition diagram; we can get the distribution of X n for any n. There is a small mistake, the running index for X n plus 1 value is equal to X n plus Y n plus 1; that is starting from 0, 1, 2 and similarly the previous slide X n plus 1, is equal to X n plus Y n plus 1, and n is a running from 0, 1, 2 and so on.

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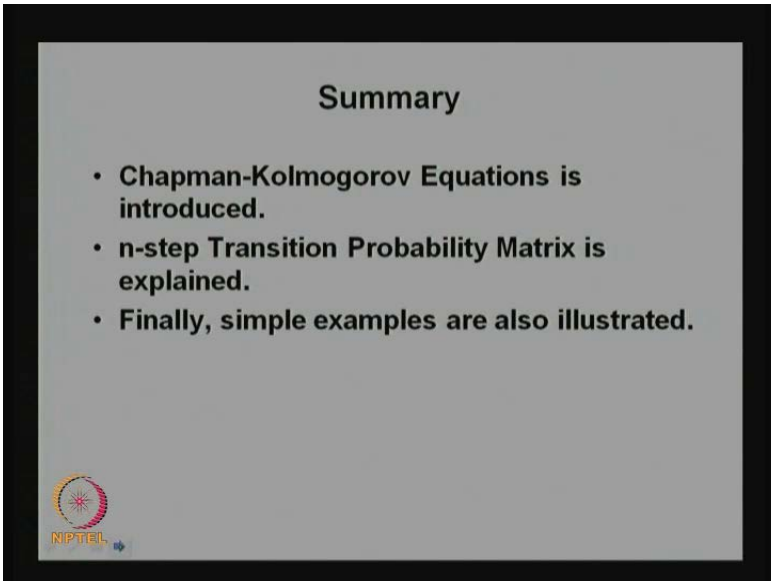

So, in this lecture we have discussed Chapman Kolmogorov equation, and also we discuss the n step transition probability matrix. So, the n step transition probability matrix can be computed from the one step transition probability matrix, with the power of that n. And also we have discussed the four simple examples, for explaining the Chapman Kolmogorov equation, and the n step transition probability matrix.

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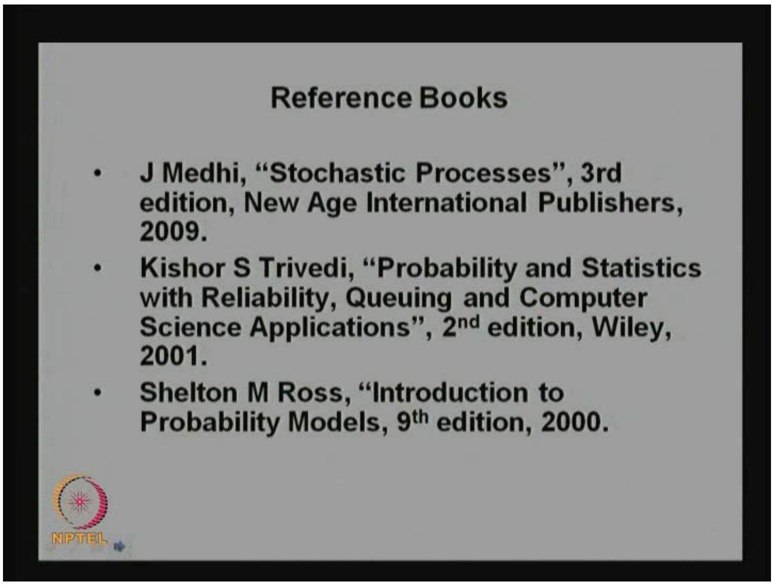

For the lecture 1 and 2, we have used these three books for as the reference. The first one is, J Medhi; stochastic processes book, the second one is the Kishor Trivedi; probability and statistics with reliability, queuing and computer science applications. The third one is, Shelton Ross; introduction to probability models. So, with this, I will complete the lecture 2 of discrete time Markov chain.

Thanks.