**Numerical Analysis**
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**Lecture - 58**
**Numerical ODEs Modified Euler's Methods**

Hi, we are learning numerical methods for first order initial value problems. In this, we have learned Euler method and Runge-Kutta method. In this class, we will introduce 2 methods, one is midpoint method and another one is trapezoidal method. We will use a different idea to derive these 2 methods. We will first write the initial value problem in the form of an equivalent integral equation and then use quadrature formulas to approximate the integrals, appearing in this equation to get these 2 methods.

One is a two-step method and another one is an implicit method. Let us go to derive these 2 methods.

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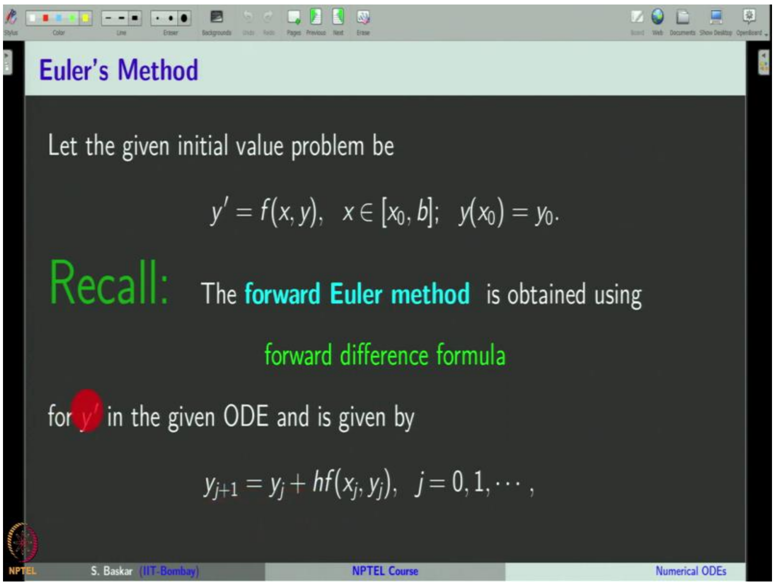


Recall, that the initial value problem that we are interested in, is given by $y' = f(x, y)$ and this equation is posed on a closed and bounded interval $[a, b]$ and we are also provided with an initial condition $y(x_0) = y_0$ where $x_0$ is a point in the interval $[a, b]$. Often, we take $x_0 = a$ just for the sake of simplicity. Recall, we have derived the forward Euler method in one of our previous lectures, where we have approximated the derivative $y'$ in our equation using forward difference formula.

And thereby, we got the formula for the forward Euler method as $y_{j+1} = y_j + hf(x_j, y_j)$. Here, you can see that in order to get $y_{j+1}$, we just need to know the value of $y$ at the previous node, that is at $x_j$. So, in that way this method is a one step method and also you can see that $y_{j+1}$ is obtained fully from $x_j$ and $y_j$, which are known to us. In that way, this formula is an explicit formula for $y_{j+1}$.

And that is why, we say that the forward Euler method is an explicit method. We have obtained this formula by approximating $y'$ by the forward difference formula.

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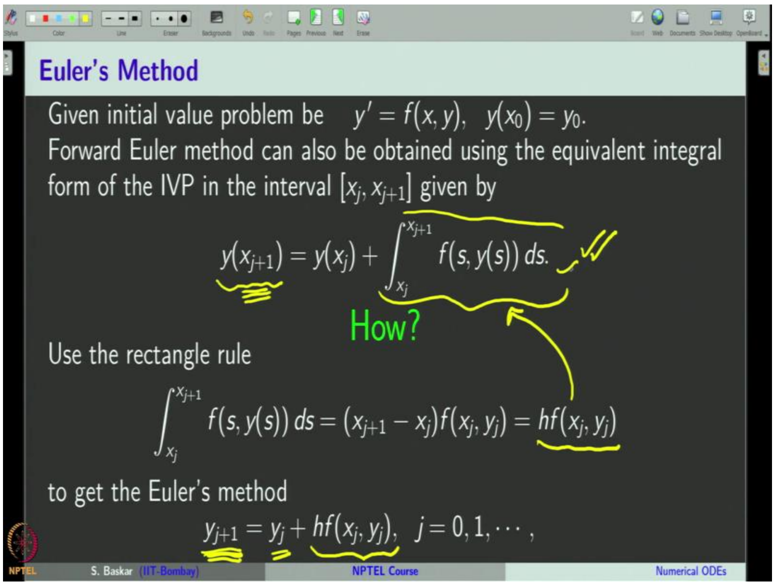


Now, we can also get the same formula using another approach. Here what we will do is, we will write the given equation in the equivalent integral form. How are we getting this? Well, we have $y' = f(x, y)$. Now you integrate it from $x_j$ to $x_{j+1}$, the same has to be done on the right hand side also, $x_j$ to $x_{j+1}$ *ds*. So, let us write it as *s ds* and this is *ds*. Now here you can see that the integral becomes $y(x_{j+1}) - y(x_j)$, which I will take on the other side $y(x_j) + \int_{x_j}^{x_{j+1}} f(s, y(s)) ds$.

So, that is what we are writing here. So, this equation can be equivalently written in this form. We are just using the fundamental theorem of calculus, for that you have to, of course, assume that *f* is a continuous function. With all these nice assumptions, you can see that the given ODE is equivalent to this integral form of the equation. Remember, this is not a formula but it is an equation because you have *y* on the left hand side, which is unknown and that is evaluated in terms of again *y*, which is sitting in the integrand here.

Therefore, it is an integral equation and it has to be solved in order to get $y$. Now we know that there are some integrands $f$ for which we do not know how to perform this integral explicitly. In which case, you can approximate this integral by a quadrature formula. So, that is the idea. Now what we are claiming is that there is a quadrature formula for which we will get the forward Euler method.

The question is, what is that quadrature formula, when I replace this integral by that quadrature formula, leads to forward Euler method. Let us see, if you recall we had studied a rule called rectangle rule. So, what it says, $\int_a^b f(x)dx = (b-a)f(a)$. So, that is what we are going to use here, $\int_{x_j}^{x_{j+1}} f\big(s, y(s)\big)ds = \big(x_{j+1} - x_j\big)f\big(x_j, y_j\big)$, that is the value of the function $f$ evaluated at the lower limit of the integral, and we know that this is nothing but $h$.

Therefore, this can be written as $hf\big(x_j, y_j\big)$. Now you replace this integral by the rectangle rule and thereby you get $y_{j+1}$. Remember, since I put an approximate expression for this, I will not use the notation $y\big(x_j\big)$ because this is used to indicate the exact solution. So, I will use this notation to indicate the corresponding approximation to the exact solution and that is given by $y_j$ which is an approximation to $y\big(x_j\big)$ + this integral is now replaced by the rectangle rule and that is precisely the forward Euler method.

So, the forward Euler method which we derived by replacing $y'$ by the forward difference formula, can also be obtained by replacing the rectangle rule in the equivalent integral equation. Now from here, we get lot of ideas because why only rectangle rule, we can also replace this integral by midpoint rule, trapezoidal rule, even Simpson's rule and Gaussian quadrature rule.

So, we have many formulas from which we have scope to get many methods, this is what is the interest for us in this lecture.

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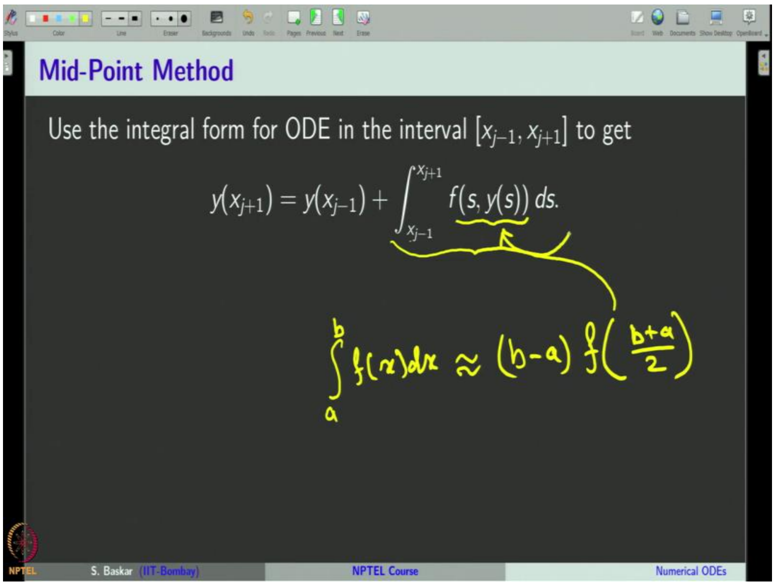


As the first step, let us consider the integral form of the ODE here but there is a slight difference in the way I have taken these limits. I have taken the limits as $x_{j-1}$ to $x_{j+1}$. Remember, how you will write this integral equation is up to you. You just have to integrate $y'(s)ds = \int_{x_{j-1}}^{x_{j+1}} f(s, ys(s))ds$ and then take any limit. So, it is not necessary that you have to take $x_j$ to $x_{j+1}$, you can also take $x_{j-1}$ to $x_{j+1}$ like that you can take the integral over any interval.

But only thing is you have to take the same integral on the right hand side also. So, that is what I am doing here; why am I doing here with a different limit, because I want to. Now apply the midpoint rule. If you recall, midpoint rule is given by $\int_a^b f(x)dx$ is actually $(b-a)f\left(\frac{b+a}{2}\right)$, $f$ evaluated at the midpoint of the interval that is $\frac{b+a}{2}$. Now, when I want to apply this midpoint rule to this integrand, I have to evaluate this integral at the midpoint of the interval on which you are taking this integral.

And that midpoint should coincide with one of your grid points, that you have generated in your problem. That is why, I have taken the limit as $x_{j-1}$ to $x_{j+1}$. So, that $x_j$ is the midpoint of this interval. Remember, we always work with equally spaced nodes. Again, this is only for the convenience but here we are crucially using it. And that will make the evaluation of this quadrature rule exactly at the node point $x_j$, for that reason we have chosen this.

You have to choose similarly, for Simpson's rule also. Because Simpson's rule also evaluates the value of the function at the midpoint of the interval. So, in such cases you have to make sure that all the points where you are evaluating the function should be the grid points of your problem.

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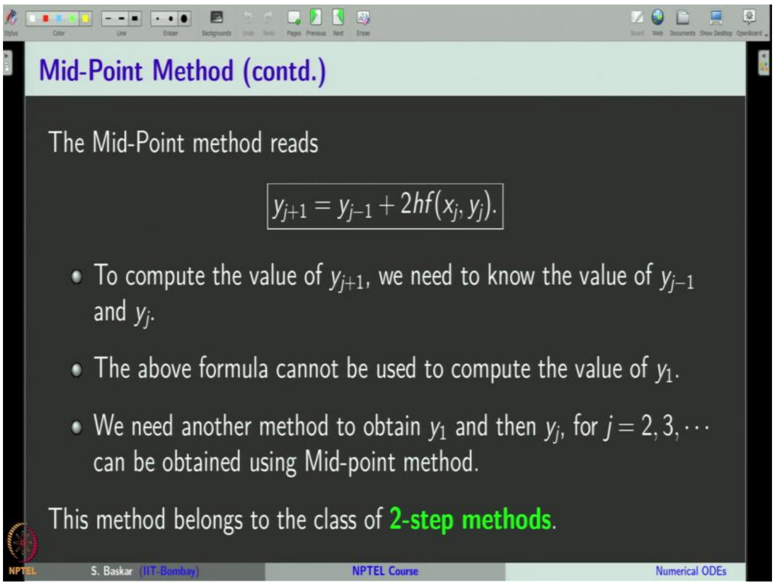


Once you have this, now it is very easy for you to apply the midpoint quadrature rule for the integral and that is given by this, which is nothing but $2h\, f(x_j, y_j)$. So, when you apply this quadrature rule into this equation, you get the formula $y_{j+1} = y_{j-1} + 2hf(x_j, y_j)$, that is why you have $2h$ here, because you have applied the quadrature formula in the interval $x_{j-1}$ to $x_{j+1}$. You have chosen a wider interval in order to fit this function evaluation exactly at a grid point.

So, this is the idea of midpoint rule. We have the midpoint method for our initial value problem and we have the following important observations about this midpoint method. What are they? The first observation is that, to compute the value of $y_{j+1}$, we need to know the value of $y_j$ and not only that, it also depends on the value at $x_{j-1}$. Therefore, $y_{j+1}$ depends on the value of $y$ at 2 grid points one is at $x_j$ and another is $x_{j-1}$, that is the first observation.

Now when you go to find $y_1$, how will I get it? Well, put $j = 0$ to get $y_1$, that will make the first term here as $y_{-1}$, what is that, it is nothing but $y$ evaluated at $x_0 - h$. But if you recall, we have posed the problem only in the interval $[x_0, b]$. We do not care about $x_0 - h$, it is outside our domain. Therefore, we do not know whether the solution exists at this point or not. Even if it exists, we have no interest to calculate the value of $y$ at this point.

So, we cannot go to use the value at this point, that makes the method to be not applicable for $j = 0$. In that case, therefore, you have to compute $y_1$ using some one step method, something like forward Euler method. Once you have $y_1$ from some other method then we can use $y_0$ and $y_1$ to obtain $y_2$ using the midpoint Rule and similarly you can use $y_1$ and $y_2$ to get $y_3$ from the midpoint method and so on.

So, you can apply the midpoint rule only for obtaining $y_j$ for $j = 2,3,\cdots$, whereas $y_1$ has to be obtained from some one step method like forward Euler method. Such a method is called a two-step method because to find $y_{j+1}$, you need to know the value of the solution at 2 previous nodes, here it is $y_j$ and $y_{j-1}$.

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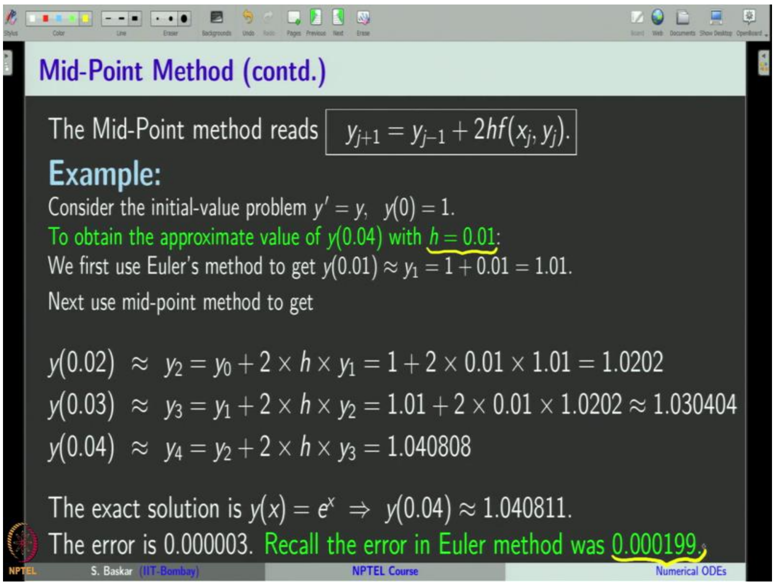


Let us take an example where we have the initial value problem as $y' = y$ and $y(0) = 1$. Our aim is to obtain an approximate value of $y(0.04)$ with the step size $h = 0.01$. As we have already observed, we have to use some one step method to obtain $y_1$ here. We use the Euler forward formula and get the approximate value at the point $y(0.01)$ as 1.01 and is denoted by $y_1$.

So, we got the value of $y_1$ from forward Euler method. Now to get $y_2$, we can go for the midpoint method. $y_2$ is from the midpoint method given by $y_0 + 2h\, y_1$and we know all the values. Therefore, we can plug in all these values and get the approximate value of $y(0.02)$ and it is given by 1.0202 here. Similarly, you can also get $y_3$, $y_3$ is the approximation to $y(0.03)$ and it involves the values of $y_1$, which we have already computed from the Euler method.

And it also involves the value of $y_2$, which we have computed from the midpoint rule from here. So, you can plug in those values and get the value of $y_3$ which is approximately 1.0304. Similarly, you can get $y_4$, which is the approximation of what we want. So, we want $y(0.04)$ and that is now given by approximately 1.04081. Now let us see what is the exact value of the solution $y(0.04)$.

You can clearly see that the exact solution is given by $y(x) = e^x$. From there you can compute the exact value of the solution and that may be taken approximately as 1.040811. So, that is pretty close to what we have obtained. Also, you can see that we have obtained the approximate solution up to six digits rounding. With respect to this exact value, the error is given by this, which is pretty small.

If you recall, we have also obtained an approximate solution for this initial value problem using forward Euler method and for the same value of $h$ that is $h = 0.01$. We got the error from the forward Euler method as this value. From here you can see that at least in this example, the midpoint method performs better than the forward Euler method.

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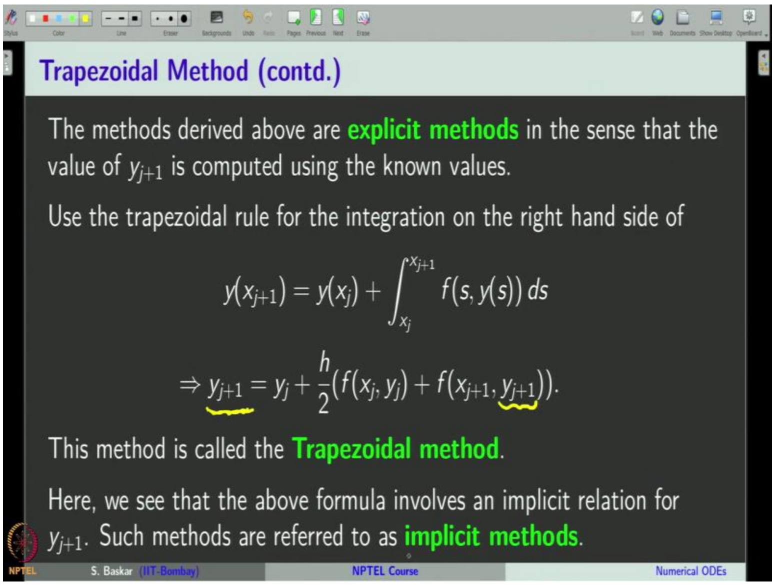


So far, we have only derived explicit methods, where the unknown $y_{j+1}$ is obtained as a solution of $y_j$ and $y_{j-1}$, which are already computed when we go to compute $y_{j+1}$. In that way the methods, that we have derived so far are explicit methods. There are also other cases, where the unknown $y_{j+1}$ is obtained as an implicit relation involving known and unknown quantities, such methods are called implicit methods.

Let us illustrate such a method by putting the trapezoidal rule to the integral equation. Just recall that the trapezoidal rule for this integral is given by $(b-a)$ which is $x_{j+1} - x_j$ by 2 into $f(b)$ that is, $f(x_{j+1}, y_{j+1})$ + $f(a)$ which is $f(x_j, y_j)$. So, we just have to replace this integral by the trapezoidal rule and we get $y_{j+1} = y_j + h(j+1) - hj$ which is precisely $h$ for us, divided by 2 into $f(a + +f(b)$, this relation is called the trapezoidal method.

Here you can see that we obtained a relation for $y_{j+1}$ and it is implicitly represented because to get $y_{j+1}$, we again need to know $y_{j+1}$ on the right hand side also. If you recall, both in the midpoint rule as well as in the forward Euler method, we obtained $y_{j+1}$ purely in terms of the known quantities. whereas in the present case, we only have an implicit relation and for this reason this method is called an implicit method.

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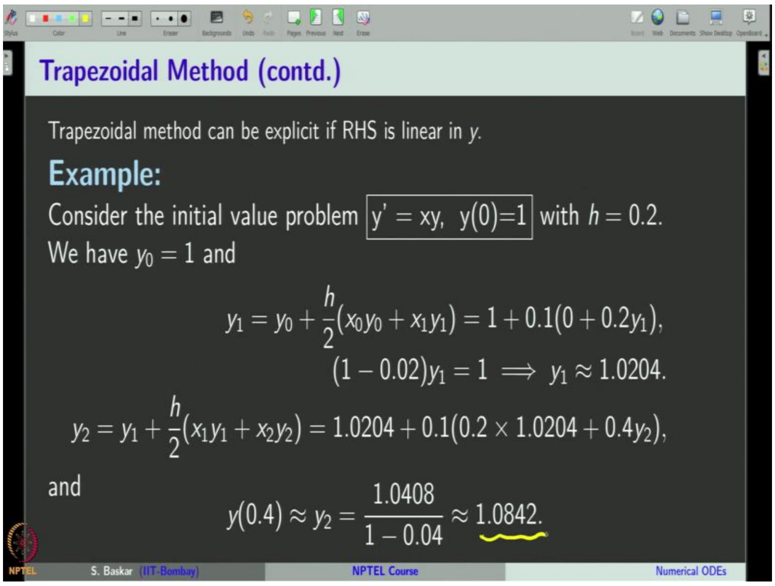


Note that the trapezoidal method can be explicit, if the function $f$ is linear in $y$. Let us see this by an example. Consider the initial value problem $y' = xy$ and the initial condition is given by $y(0) = 1$. Here you can see $f(x, y) = xy$ and you can see that $f$ is linear in $y$. Let us take $h = 0.2$, that is not very important for us now. We will apply the trapezoidal method to this initial value problem and see that we get an explicit relation just because $f$ is a linear function in $y$.

Let us see how it goes. Recall, the trapezoidal rule for this initial value problem is $y_{j+1} = y_j + \frac{h}{2}(x_j y_j + x_{j+1} y_{j+1})$. Of course, we have $y_{j+1}$ appearing on the right hand side but what you can do is, you can take it to the left hand side and then you can easily solve for $y_{j+1}$. So, that is what we do here, for instance, $y_1$ is given like this and when you plug in all the known quantities here, we have this expression where $y_1$ is again not known to us.

But that does not matter. You can take this $y_1$ to the other side and get an explicit relation for $y_1$ like this and that immediately gives you the value of $y_1$ as well. So, therefore as long as the right hand side function $f$ depends linearly on $y$, you can still use the trapezoidal rule just like the explicit method.

In fact, you can also get $y_2$ very easily. Again $y_2$ will be given in terms of $y_2$ again but that is again appearing linearly therefore you can solve it to get $y_2$ and that value is given by 1.0842.

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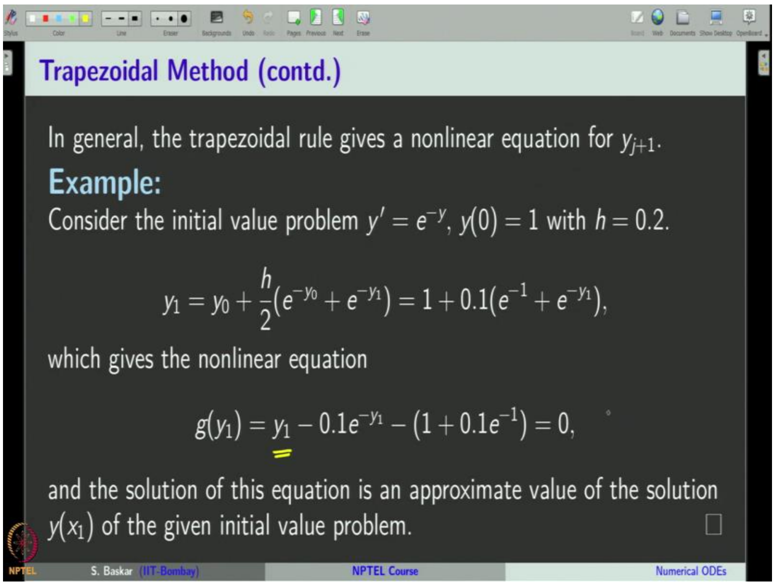


Now the question is, how to solve this in general? Trapezoidal rule gives a non-linear equation for each $y_{j+1}$. Let us see this by another example. Now, let us take the initial value problem $y' = e^{-y}$ with the initial guess as $y(0) = 1$. In this case, $y_1$ is given by this expression, where the right hand side also involves $y_1$ but now it depends on $y_1$ in non-linear way. Now it involves $y_1$ in terms of the exponential function.

Now it is not very easy for us to solve this equation to get $y_1$. Now the question is how to solve this non-linear equation at every grid point. There are 2 ways that we can handle this problem. One is to use some non-linear iterative method. Let us first consider the non-linear equation. The non-linear equation is precisely $y_1$ which is coming from your left hand side - $0.1e^{-y_1}$ - you collect all the constants in one place and that is equal to zero.

Therefore, you have this non-linear equation $g(y_1) = 0$. You have to solve this equation to get $y_1$. If you recall, we have learned some non-linear iteration methods in one of our previous chapters. So, you may use one of those methods to obtain a solution to this non-linear equation which will be the approximation to the solution of your initial value problem at $x = 0.2$. This is one way to handle this non-linear equation.

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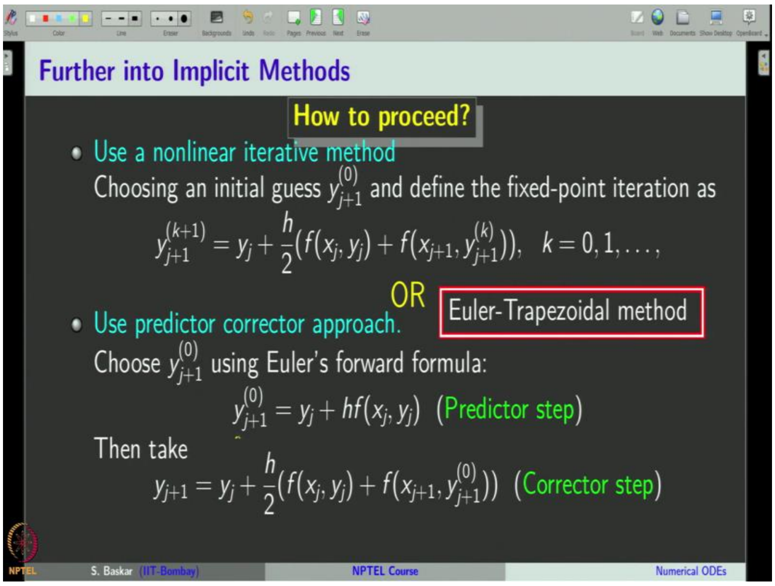


Let us put this idea in a general way. How to proceed when we are working with implicit methods because implicit methods in general gives us a non-linear relation involving the unknown $y_{j+1}$. The idea that we have proposed in the last example, is to go for one of the non-linear iterative methods. Recall, that the trapezoidal method is given like this. This equation can be seen as a fixed-point problem where the iteration function $g(y)$ is given as $y_j + \frac{h}{2} f(x_j, y_j)$.

These are known quantities + $f(x_{j+1}, y)$, so this is unknown to us. So, therefore you can define the iteration function like this, whose fixed point is precisely the point $y = y_{j+1}$. So, that is what we are seeing from the trapezoidal method. So, that is what I have written here. We can view the trapezoidal method as a fixed-point iteration method. Now you can choose any initial guess, which we will denote by $y_{j+1}^0$ and then define the fixed-point iteration method like this, where you will plug in $y_0$ and get $y_{j+1}^1$ and again plug in $y_{j+1}^1$ on the right hand side, get $y_{j+1}^2$ and so on.

So, in that way you generate a sequence of numbers which is expected to converge to the fixed point of the function $g$ and that is precisely the approximation of our solution at $x_{j+1}$. There is another approach which is called the predictor corrector approach, where you take the initial guess from the Euler method. Remember in the first approach, you get the initial guess arbitrarily, whereas in this we are taking the initial guess from the Euler forward formula, which is called the predictor step.

And then use this value here to compute $y_{j+1}$, using the trapezoidal method and that step is called the corrector step. In that way, you have one predictor step, get an initial guess from the Euler forward formula, plug in on the right hand side and thereby the expression now becomes explicit. You can find $y_{j+1}$ from there and that is called the corrector step. And such a method is called the predictor corrector method.

And since we are using Euler forward formula here and then computing the solution from the trapezoidal method, this method is also sometimes called as Euler trapezoidal method.

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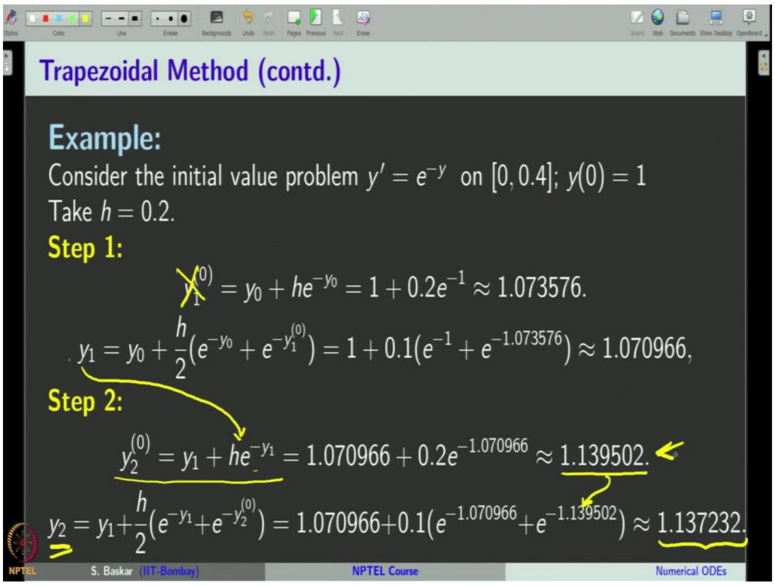


Let us take our previous example $y' = e^{-y}$ on the interval $[0,0.4]$ with $h = 0.2$. We are given the initial condition as $y(0) = 1$. We will now compute the solution using the Euler trapezoidal method, that is the predictor corrector approach, which includes one predictor step, which is coming from the Euler forward formula and that is given by this value. Now you plug in that value into the trapezoidal method and perform the corrector step and get the value as 1.070966.

Similarly, we can do the step 2. For step 2 again, you will take this value $y_1$. Remember, do not take this value for the predictor step of $y_2$. So, you have to take the corrector value from $y_1$ and plug in that to the predictor step of $y_2$ and you get the corresponding value. Once you get that value, you just plug into the corrector step and get $y_2$ from here and that gives us the value for $y_2$.

So, this is not the value for $y_2$,nit is just a prediction and that is further corrected by the trapezoidal method. So, this is a nice illustration for the predictor and corrector method. In this lecture, we have learned 2 important methods, one is midpoint method which is an example for a 2 step method. And another method we learned is, the trapezoidal method, which is an example of an implicit method. We will continue our discussion on multi-step methods with explicit and implicit forms in the next lecture, thank you for your attention.