Numerical Analysis
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Lecture - 52
Numerical Integration: Tutorial Session

Hi, in the last week we have learned some quadrature formulas to approximate a given integral. In this class, we will do some problems on quadrature formulas.

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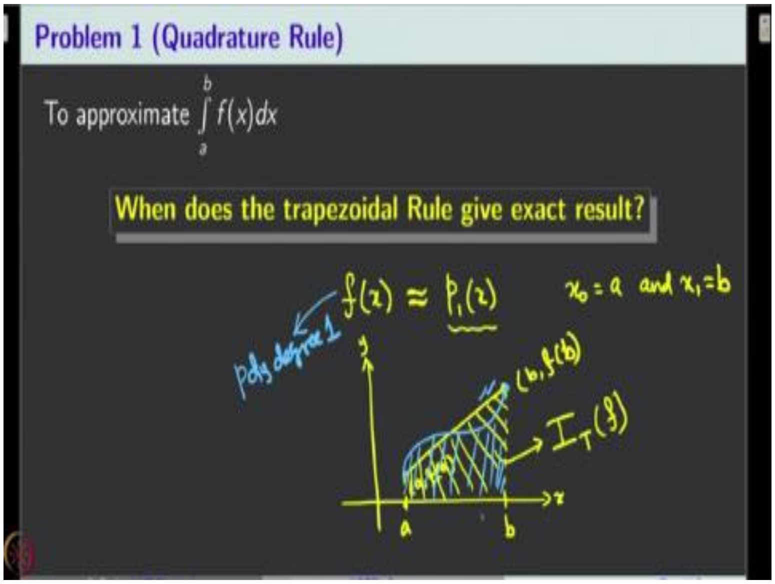

Recall, that in quadrature rules our aim is to approximate the integral $\int_a^b f(x)dx$. We first approximate the given function $f(x)$ by an appropriate interpolating polynomial $p_n(x)$ and then we will take the approximation of the required $\int_a^b f(x)dx$ as $\int_a^b p_{n(x)}dx$. With this idea we have derived few quadrature formulas. One is rectangle rule, another one is trapezoidal rule and then we have also learned Simpson's rule.

Let us first consider the trapezoidal rule and ask this question. When does the trapezoidal rule give exact result? Recall in trapezoidal rule, first we will take the linear polynomial interpolation for the given integrand $f(x)$ and this linear polynomial interpolation is obtained with nodes as $x_0 = a$ and $x_1 = b$. Therefore, we are actually taking the straight line joining two points say $(a, f(a))$ and $(b, f(b))$ and then we are finding the area under the graph of this line.

And this is the trapezoidal rule. Generally, we denote it by $I_T(F)$. Now you are given any function, say the function like this then you are supposed to get the area under this curve. But what we obtain finally is, the area under this line. So, that is the trapezoidal rule idea and therefore we do commit some error in this approximation. Now our question is, when does this trapezoidal rule give exact result.

Obviously, if this integrand is a polynomial of degree one then we will get exact result, from their trapezoidal rule.

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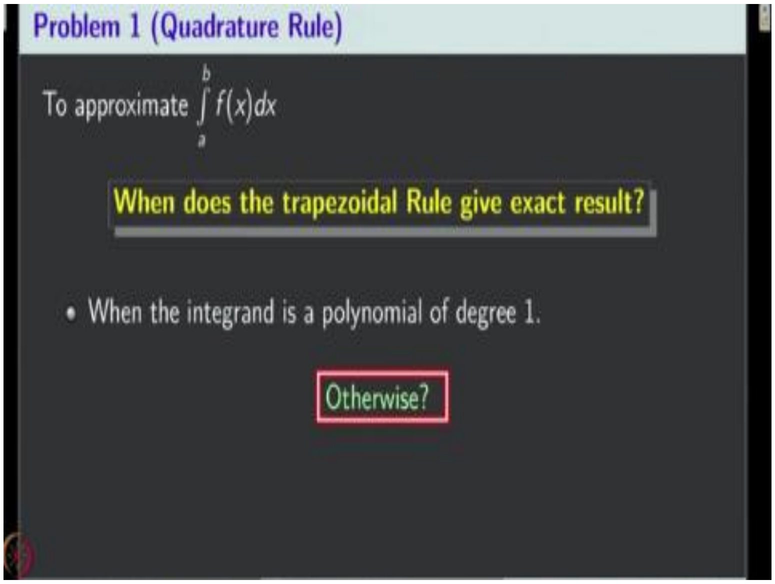

Now the question is, there any other function which is not a polynomial of degree 1 for which the trapezoidal rule will give the exact result, the answer is yes.

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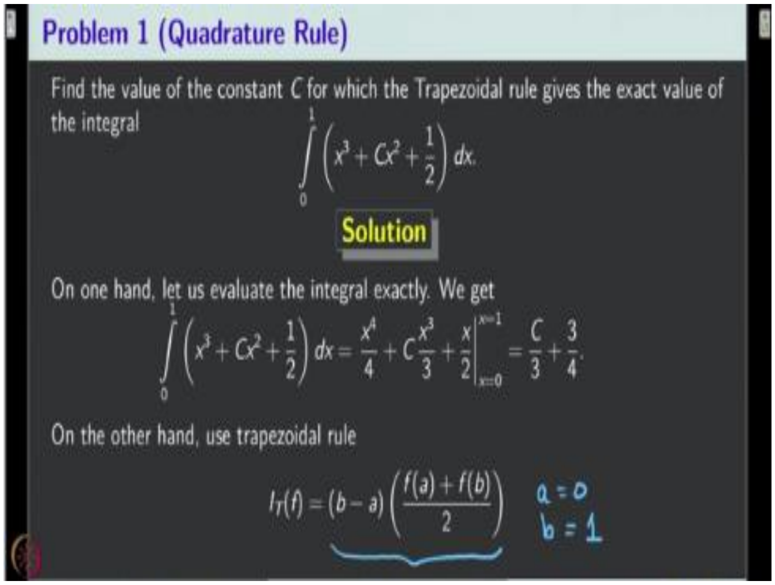

So, let us try to have an example for this. In this problem we have to find a constant C such that when you try to approximate $\int_0^1 \left(x^3 + Cx^2 + \frac{1}{2}\right) dx$, then the trapezoidal rule should give the exact value of this integral. So, how to find such a C, let us see. It is not very difficult what you do is first on one hand evaluate this integral exactly. Remember it is just a polynomial therefore you can obtain the integral explicitly and it will be like $\frac{C}{3} + \frac{3}{4}$.

On the other hand, you use the trapezoidal rule and find an approximation to this integral. Recall the trapezoidal rule formula is given like this where $a = 0$, $b = 1$. Therefore, you can plug in these values in the formula with f given by this expression.

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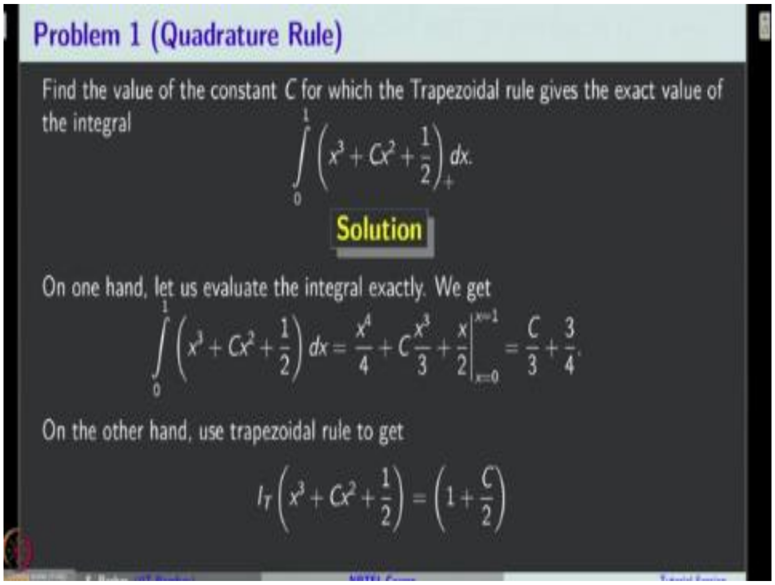

Then you can see that the trapezoidal rule applied to this integral will take the value $1 + \frac{C}{2}$.

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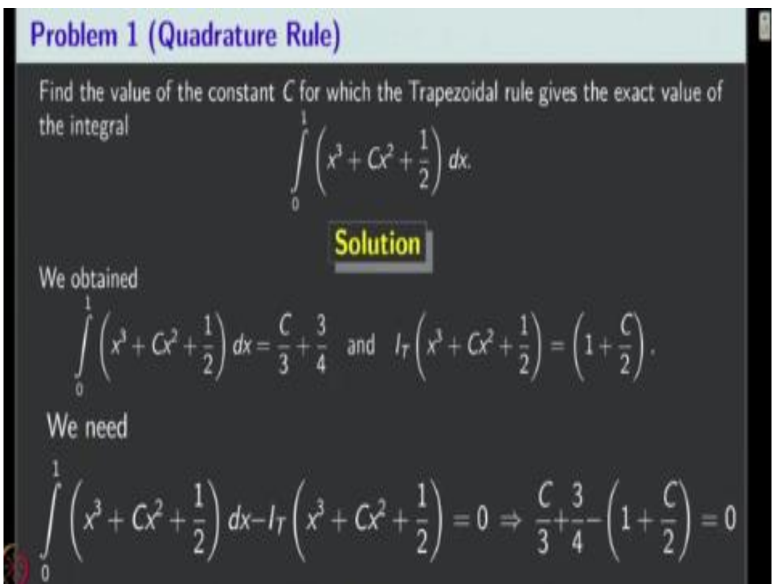

Therefore, we obtained the exact integral on one hand and we also obtained the value from the trapezoidal rule. Now what we want? We want to find this constant C such that this trapezoidal rule gives exact result. That is equivalent to saying that we have to find the constant C such that the exact value which is coming from the direct integration is equal to the value coming from the trapezoidal rule. Therefore, you subtract these two quantities we should get 0.

Let us substitute these expressions into this equation and let us see what we get? We get $\frac{C}{3} + \frac{3}{4} - \left(1 + \frac{C}{2}\right) = 0$.

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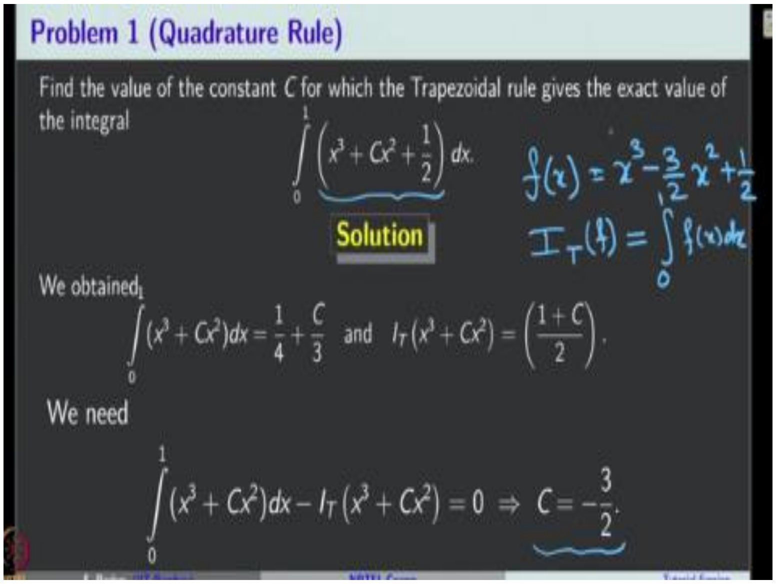

From here, you can directly find the value of C as -$\frac{3}{2}$. It is very simple you can see that the integrand is not a straight line but when you take C = - $\frac{3}{2}$ that is when you take $f(x) = x^3 - \frac{3}{2}x^2 + \frac{1}{2}$, then the trapezoidal rule applied to this function in the interval $[0,1]$ will give you the exact result $\int_0^1 f(x)dx$. You can see that the function *f* is not a straight line but still you get the exact result.

Let us try to view this geometrically, to see what exactly made trapezoidal rule to get exact value of the integral for this particular integrand.

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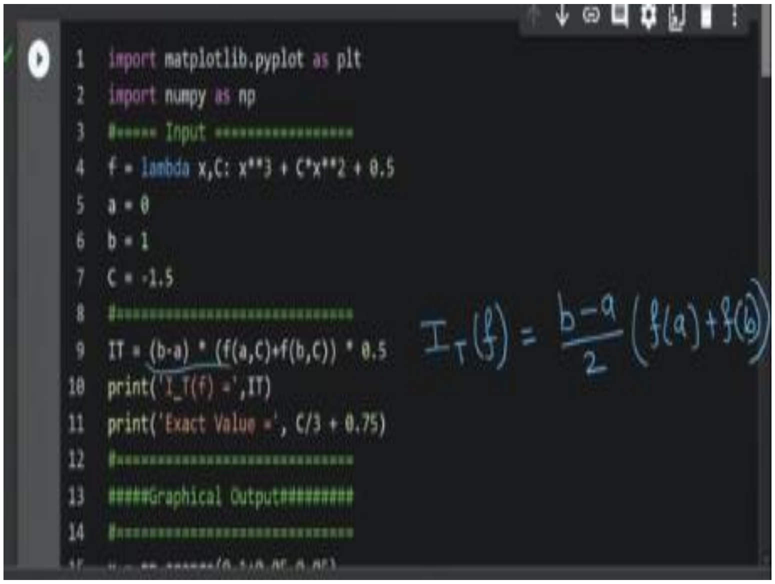

For that, let us try to develop a python code for the trapezoidal rule. First let us define the integrand f as $x^3 + Cx^2$+ 0.5. Recall this is the integrand that we have taken in our problem with a C and we have also found that C = $-\frac{3}{2}$ and our integral is over the interval $[0,1]$. Therefore, these are the inputs that we have to take in our code and if you recall the formula for the trapezoidal rule is given by $I_T(f) = \frac{b-a}{2}\left(f(a) + f(b)\right)$.

So, that is what is written here (b – a) divided by 2 into f(a). Here what I am doing is, I am also sending the value of C as a parameter into my lambda expression. Therefore, I have two arguments in f, one is the point at which I want to find the value of the function f and another one is this constant C. So, therefore as per the formula, I have to evaluate f(a) which I am doing here plus f(b) which I am doing here in order to get this expression.

So, this is precisely the formula for the trapezoidal rule. Then I am asking the python to print the value computed using the trapezoidal rule and I am also asking the python to print the exact value. Recall the exact value is C/3 + 0.75, this is what we have evaluated exactly from the integral. I am just directly taking that expression here and asking the python to print the value of this expression.

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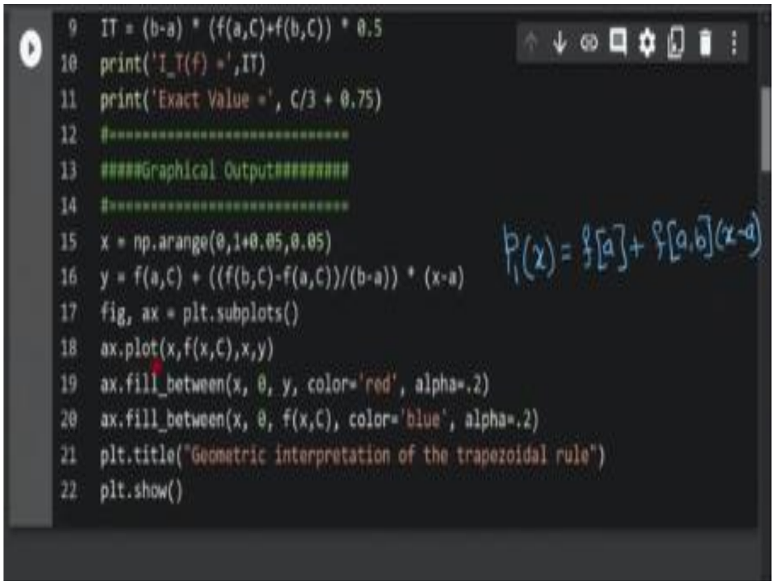

```
IT = (b-a) * (f(a,C)+f(b,C)) * 0.5
print('I_T(f) =',IT)
print('Exact Value =', C/3 + 0.75)
#============================
#####Graphical Output#########
#============================
x = np.arange(0,1+0.05,0.05)
y = f(a,C) + ((f(b,C)-f(a,C))/(b-a)) * (x-a)
fig, ax = plt.subplots()
ax.plot(x,f(x,C),x,y)
ax.fill_between(x, 0, y, color='red', alpha=.2)
ax.fill_between(x, 0, f(x,C), color='blue', alpha=.2)
plt.title("Geometric interpretation of the trapezoidal rule")
plt.show()
```


Now I am also interested in seeing the graph of the function *f* as well as the graph of the polynomial $p_1(x)$ which interprets the function *f* at the node points *a* and *b*. Let us see how to do that. First, let us generate the *x* vector which will be in the *x* coordinate and for the *y* coordinate what we are doing is, we are taking the linear interpolating polynomial at the node points *a* and *b*.

Remember with Newton's form we can write $p_1(x) = f$[a], which is nothing but the value of *f* at a only, + $f[a,b](x-a)$. So, that is the interpolating polynomial for the function *f* at the node points *a* and *b*. That is what I am writing here, $f[a]$ plus this is the first order divided difference $f[a,b]$ which is $\frac{f(b)-f(a)}{b-a}(x-a)$ is written here. Therefore, I am interested in plotting *x* versus p_1.

And also, I am interested in plotting the graph of the given function *f*. So what I am doing here is, I am plotting *x* which is $f(x)$ as the *y* coordinate and again *x*, *y*. Now *y* is the linear interpolating polynomial. So, I am plotting both these graphs in one command and then I want to fill the area under the graph of these two functions. That is what I am doing in these two commands and then I am just giving the title to my graph and finishing my plotting part.

Let us execute this code and see how the output looks like. The code is executed successfully. As we wanted, python printed these two print commands. By that we have the trapezoidal rule value is 0.25 and the exact integral value is also 0.25. How we got that? Because we have taken the constant C as - 1.5. If you recall that is what we obtained here as long as you are taking C = - 1.5 you will have the exact result from the trapezoidal rule that is what we have seen computationally here.

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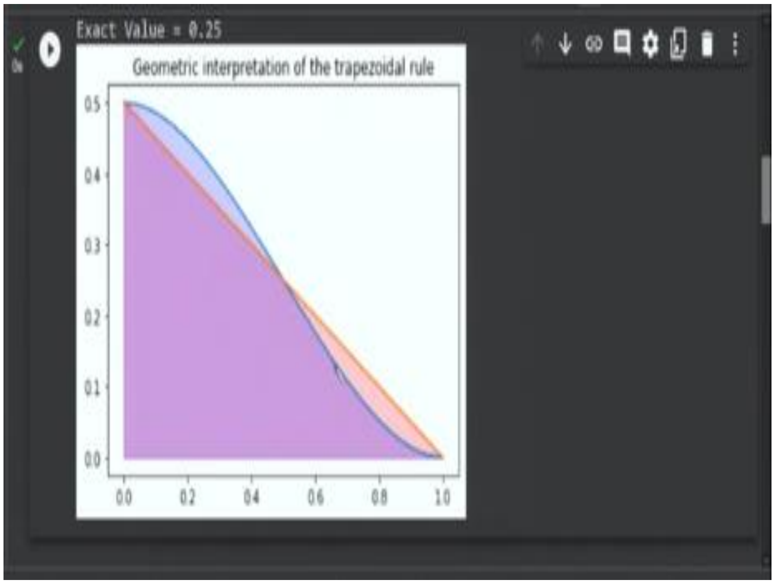

Now let us see how this graph looks like. Well as we have given the title of the graph, geometrical interpretation of the trapezoidal rule, that is printed as the title in the graph. Let us see how the graph looks like. We have plotted the function $f(x)$ and that is shown in the blue colour and we have also plotted the linear interpolating polynomial of this function *f* with nodes as *a* which is 0 and *b* which is 1 in this example and in between that it is just a straight line because it is a linear interpolating polynomial.

And we have also asked the python to fill the area under the graph of these two functions, that is what is written in these two. The first one is to fill the area under the graph *x*, *y*. *y* is nothing but your p_1. So, your *y* is p_1 therefore this command will fill the area under the graph of p_1 with red colour and this command will fill the area under the graph of the function *f* with blue colour. So, that is what we are seeing here you can see that this purple colour is the area which is covered both by the exact integral as well as the trapezoidal rule.

Whereas trapezoidal rule had excluded this area which is actually there in the exact value but it included this area which is actually not there in the exact integral. Now it so happened that the area excluded by the trapezoidal rule is equal to the area included by the trapezoidal rule. That is why even though the function and the polynomial are looking entirely different, the integrals are the same.

So, that is the beautiful geometric interpretation of the trapezoidal rule and it explains geometrically, why trapezoidal rule has captured the exact value of the integral in this particular example. With this let us pass on to the next problem.

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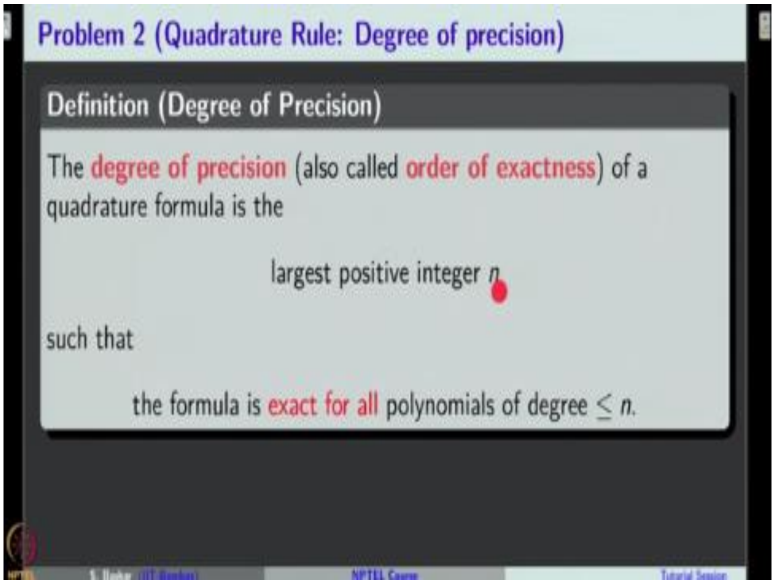

The next problem involves an important concept called degree of precision. We are given a quadrature formula. An important question generally people ask is, what is the degree of precision of our quadrature formula. What does it mean? The degree of precision is nothing but the largest positive integer n such that the quadrature formula is exact for all polynomials of degree less than or equal to n.

So, this is what is called the degree of precision. Let us see how to find the degree of precision of some quadrature formulas.

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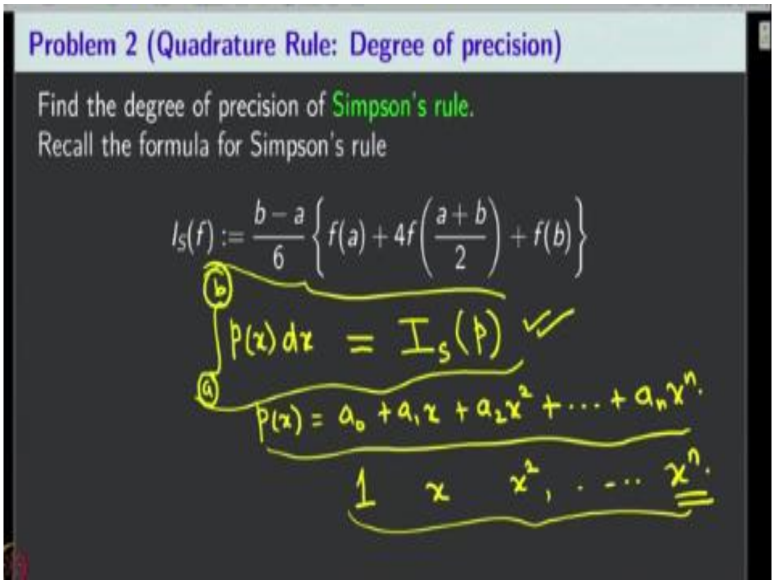

As an example, let us try to find the degree of precision of the Simpson's rule. Recall that the Simpson's rule for the integral $\int_a^b f(x)dx$ is given by this formula. Now our interest is to find the degree of precision of the Simpson's rule, that is we have to find the largest positive integer *n* such that integral a to b that polynomial $p(x)dx$, which is a polynomial of degree less than or equal to *n*, is actually equal to the value computed from the Simpson's rule.

To achieve this, we will do the following steps. We want to show that a to b the integrand is a polynomial of degree *n*, should be equal to the value obtained from the Simpson's rule. Remember any polynomial of degree n can be written as a$a_0 + a_1x + a_2x^2 + \cdots + a_nx^n$. Now you can not verify this condition for all polynomials which looks like this for any given set of constants $a_0, a_1, a_2, \cdots, a_n$.

So what you can do is, we will check this equality only for the basis of this polynomial. Remember the monomial basis of this are precisely $1, x, x^2, \cdots, x^n$. We will simply apply this condition on only the monomial basis and see till what *n* does this equality holds. The largest *n* for which this equality holds will be the degree of precision. For that we have to fix an interval $[a, b]$.

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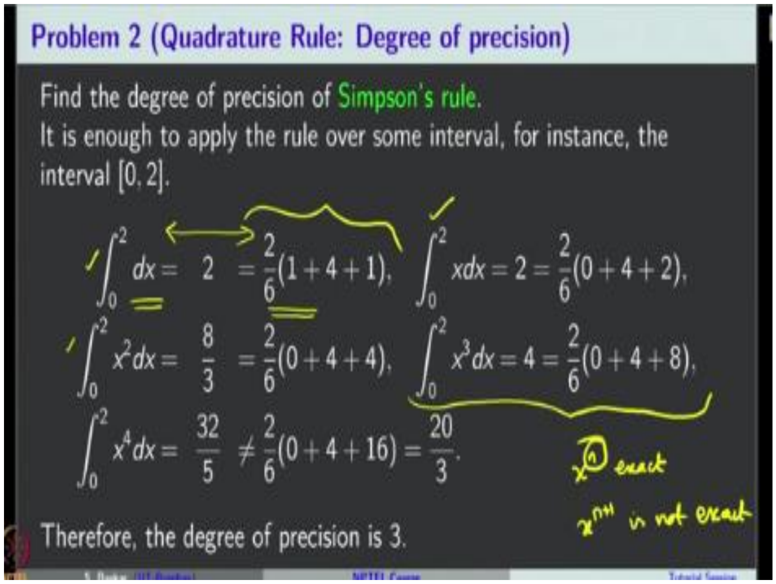

In general, you can fix any interval you want because the process of finding the degree of precision does not depend on the interval that we work with. Therefore, you can fix the interval conveniently so that the calculations are not so difficult. I will choose this interval as $[0,2]$ because I have to evaluate the Simpson's formula at a, b and then the midpoint of a, b. So, if I choose my interval as $[0,2]$ then my midpoint will become 1. All are integers, that will make my computation more easy.

Therefore, I am choosing the interval $[0,2]$, you can also choose the interval $[-1,1]$ that will be little more easy in fact. Let us see, I have chosen the interval like this. With that we have to check the equality of this value obtained from the Simpson's rule with the exact integral for the monomial basis. First let us start with $f(x) = 1$. On one side you have the value 2 from the exact integral on the interval $[0,2]$. On the other hand from the Simpson's method, you also have the same value.

Therefore, $f(x) = 1$, that is the degree 0, is satisfied. Let us go to check the degree 1 for that we will take $f(x) = x$ and now you have $\int_0^2 x\, dx = 2$ on one hand and also the Simpson's rule applied to $f(x) = x$ gives us the value 2. Therefore, the polynomial of degree 1 is also exact with the Simpson's rule. Now let us go to take x^2 the exact integral value is 8/3 and Simpson's method also gives 8/3, when we go to apply it on the interval $[0,2]$.

Therefore, degree two polynomials are also fine with Simpson's rule. Remember in one of the previous lectures, in fact we have derived Simpson's rule by assuming that the rule is exact for all polynomials of degree less than or equal to 2. Therefore, it is not surprising for us that the Simpson's rule is giving exact value for polynomials of degree up to 2. Now let us go to check the condition for the cubic polynomial.

For that let us take $f(x) = x^3$ and perform the exact integral of x^3 in the interval $[0,2]$ and that gives us the value 4. Surprisingly even Simpson's rule gives exact value for cubic polynomial also therefore up to now we have seen that degree 3 polynomials are getting exact value from this Simpson's rule when we go to integrate them. Now let us go to x^4. So x^4 when you integrate exactly in the interval $[0,2]$ it gives 32/5 whereas the Simpson's rule gives 20/3.

So, that is the first degree at which Simpson's rule value is not equal to the exact value. Therefore, you have to take 3 and declare that the degree of precision of Simpson's rule is 3. From the way Simpson's rule is derived we have imposed the condition that the rule is exact for polynomials of degree less than or equal to 2. But in reality, its degree of precision is one greater than how we have derived the Simpson's rule. This is the nice part of the Simpson's rule.

Similarly, if I give you any quadrature rule for instance trapezoidal rule or rectangle rule or any Gaussian rule and ask you what is the degree of precision. What you have to do is, you keep on integrating the monomial basis elements and check whether the equality holds between the exact integral and the quadrature rule that was asked. Here we are doing it for Simpson's rule similarly you can do it for trapezoidal rule in which case you have to find the trapezoidal rule value here and see whether this equality holds.

And if that holds then you will go for $f(x) = x$ if that holds you will go for $f(x) = x^2$ and so on. At some n say x^n gives exact value but x^{n+1} is not giving exact value. Then you have to take this n and declare that as the degree of precision of that quadrature formula. So, this is the method in which you can find the degree of precision of any given quadrature formula.

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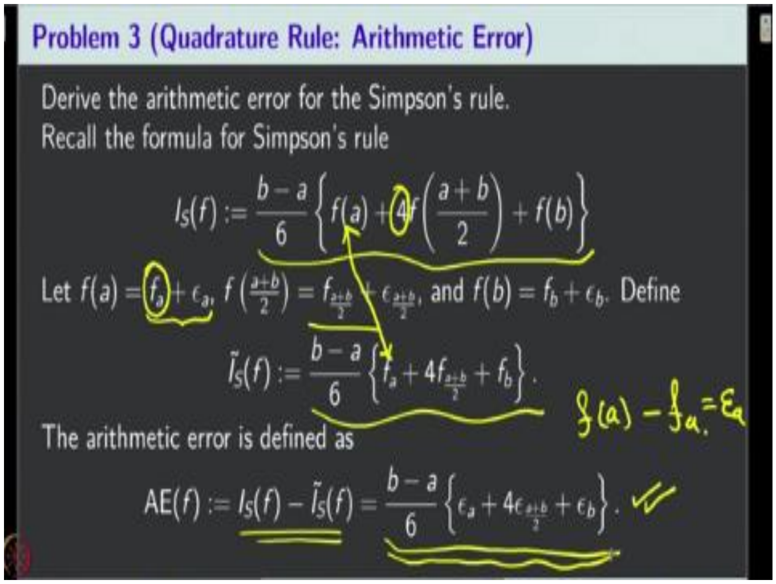

With this let us pass on to the next problem. Here we are interested in deriving the arithmetic error for the Simpson's rule. In fact, the method that we are going to learn here can be used to obtain arithmetic error for any formula, whether it is quadrature formula or finite difference formula, that we will be learning in the next section. The idea is more or less the same. Therefore, let us learn this very carefully. Remember the Simpson's rule is given by this expression.

Now our interest is to derive the formula for the arithmetic error. How the arithmetic error is committed? We want to work with the exact value of the function but due to some reason we only have the approximate value, say just assume that we are only given approximate value of this function value $f(a)$ and the error between these two that is $f(a) - f_a$ is say ϵ_a. So, this error may come into our function value through some rounding process.

Or it may be just a human error, whatever it may, assume that there is an error. That is when we go to apply Simpson's rule, we want these three function values. Due to some reason instead of having exact function values, we are having an approximate value which is denoted by f_a and the error committed in this is just denoted by ϵ_a. Similarly, $f\left(\frac{a+b}{2}\right)$ is given as $f_{\left(\frac{a+b}{2}\right)}$and the error in $f_{\left(\frac{a+b}{2}\right)}$ when compared to the exact value is denoted by $\epsilon_{\left(\frac{a+b}{2}\right)}$.

And similarly, we want to work with $f(b)$ which is the exact value but we are only working with an approximate value f_b and the error between them is denoted by ϵ_b. Now let us take the Simpson's rule and apply the rule with approximate values instead of the exact values given here. Then the Simpson's rule will take this form and let us denote it by $\tilde{I}_s(f)$. Now what is the arithmetic error?

By definition arithmetic error is given by exact value, that is the Simpson's rule obtained using the exact function values minus the approximate value obtained from the Simpson's rule using approximate function values. Now let us directly put the formula and see how this definition looks like finally. You have to take this expression minus this expression that will precisely be written as say $f(a) - f_a$ will come from this first term inside this bracket that is taken as ϵ_a.

So, that is what we are writing here plus 4 times again $f\left(\frac{a+b}{2}\right) - f_{\left(\frac{a+b}{2}\right)}$ and that will be $\epsilon_{\left(\frac{a+b}{2}\right)}$ and finally we will have $f(b) - f_b$ and that is ϵ_b. Therefore, the arithmetic error formula finally looks like this expression. This is the arithmetic error formula for the Simpson's rule. Similarly, any formula is given and you are asked to derive the arithmetic error formula.

What you have to do is, you take whatever values involved in the expression and write them as approximate value plus some error and then take only the approximate value and put it in the required formula and then find the difference between the exact value and the approximate value and that will be the expression for your arithmetic error. I hope you understood it, you try to write the arithmetic error expression for trapezoidal rule in the same way that I have explained here.

That will give you more understanding of how to write the arithmetic error for any given formula. In fact, in the next section, we will also write the arithmetic error for some finite difference formula.

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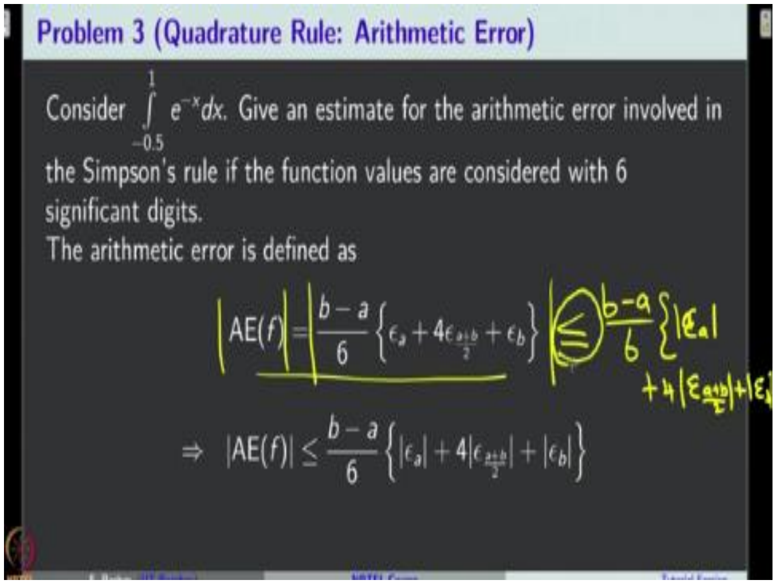

Suppose if you want to find an estimate for the arithmetic error. How to do that? Let us see. Assume that your function values that is $f(a), f\left(\frac{a+b}{2}\right)$ and $f(b)$ are only provided with six significant digits. Then what is the estimate of your arithmetic error that is what the question is asking us now. Let us see how to find the estimate. Remember the estimate means you have to bound your arithmetic error by some constant that is what we meant by estimate.

Here the function f is taken as e^{-x} and a = - 0.5 and b = 1. Now you see, we have to use the Simpson's rule and then get an upper bound for the arithmetic error. We have already found the expression for the arithmetic error in the case of Simpson's rule and that is given by this expression. Now we have to take modulus on both sides and somehow find the values of this ϵ from the given condition that the function values are given with six significant digits.

How to do that? Let us see, first let us take the modulus on both sides of this equation and use the triangle inequality on the right-hand side. That is when you take modulus here, you will have this that is less than or equal to anyway $\frac{b-a}{6}$ is positive. Now when you take the modulus to each of these terms in the bracket, you will have $|\epsilon_a|+4\left|\epsilon_{\left(\frac{a+b}{2}\right)}\right| + |\epsilon_b|$.

So, that by triangle inequality will give us a less than or equal to sign here. That is why I have put less than or equal to into the expression, that I have shown here. So, this is the bound for the arithmetic error. But now we have to get more precise bound from the given condition that the function values are given only with six significant digits.

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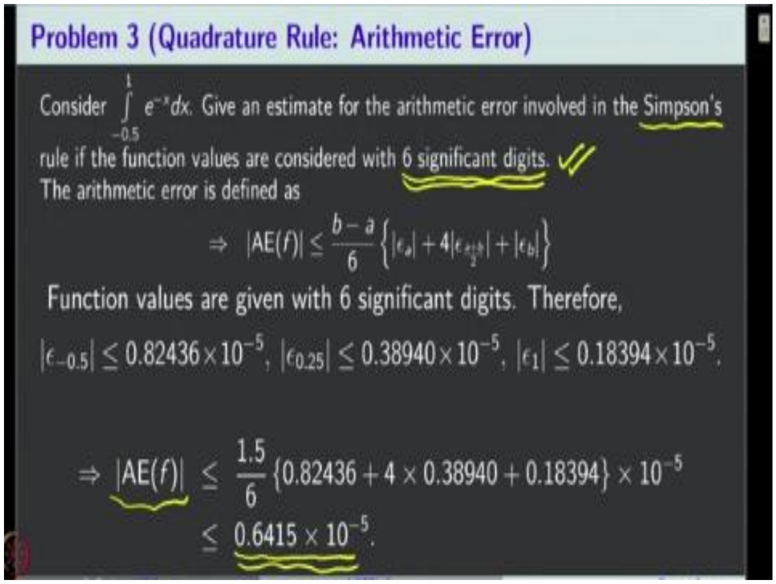

What is mean by function values are given with six significant digits? If you recall, we have seen a definition for significant digits in one of our previous chapters, that is on errors. As per the definition we have, the exact value minus approximate value divided by exact value that is the relative error is less than or equal to $-\frac{1}{2} \times 10^{-6} + 1$. That is what we mean by saying that this approximate value f_a has 6 significant digits when compared to the exact value f of a.

That is why, I have written here $\frac{1}{2} \times 10^{-5}$. Now what you do is, you take this $|f(a)|$ to the other side and you write this inequality as $|f(a) - f_a|$ is less than or equal to this constant which is coming from our significant digit definition into $|f(a)|$. Remember $f(a)$ is e^{-x}, that is why I have put here e^{-a}. Remember number $f(a) - f_a$ is precisely what we have taken as ϵ_a.

Therefore, we have a bound for ϵ_a now. What is *a*? *a* is – 0.5. Therefore, your ϵ_a which is $\epsilon_{-0.5}$ is less than or equal to this constant which is coming from the definition into $e^{-(-0.5)}$ that is $e^{0.5}$ and that will give us $|\epsilon_a|$ that is $\epsilon_{-0.5}$, that is sitting here and we have now an upper bound for this. Similarly, we have to find the upper bound for this ϵ as well as ϵ_b.

How to do that? You do the same idea you apply the formula for $f\left(\frac{a+b}{2}\right)$ as well as for $f(b)$ that will give you $\epsilon_{\frac{a+b}{2}}$, which is 0.25, is less than or equal to this number and similarly $|\epsilon_b|$ where b is in our case 1 therefore ϵ_1 is less than or equal to this quantity. Now you put this number for ϵ_a and this number for $\epsilon_{\frac{a+b}{2}}$ and this number for ϵ_b and let us see how the upper bound for the arithmetic error looks like.

You have to put these three numbers into this expression. So, this is the idea and when you put that you will have $|AE(f)|$ is less than or equal to, remember $b - a$ is $\frac{1.5}{6}$ into the sum of all these numbers where the middle term is multiplied by 4, into 10^{-5} which is sitting in all the upper bounds of the ϵs. Therefore, finally the arithmetic error involved in the Simpson's rule when we use only six significant digits in the function value is bounded by 0.6415×10^{-5}.

Similarly, if I give you any other formula say trapezoidal rule and give you that the function values are considered only up to some n significant digits. Then now you know how to first write the arithmetic error formula and then how to find the upper bound as per this kind of given condition. With this let us conclude this tutorial session, thank you for your attention.