**Numerical Analysis**
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**Lecture - 51**
**Numerical Integration: Gaussian Quadrature Rule**

Hi, we are learning quadrature formulas to obtain an approximation to a given integral on a bounded interval $[a, b]$ in this today we will learn a commonly used method called Gaussian quadrature rule. Gaussian quadrature rule generally gives a better approximation to the $\int_a^b f(x)dx$, when compared to the other quadrature rules, that we have derived so far. Let us see, how to get this better approximation.

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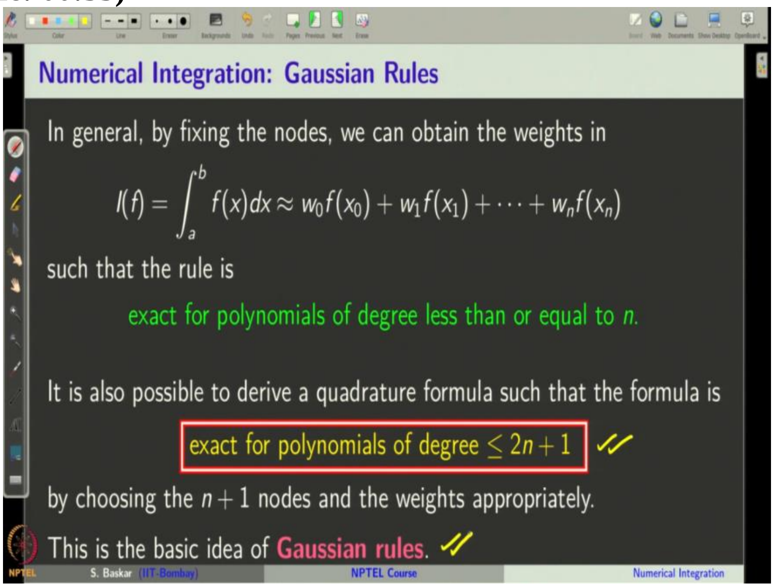


If you recall, the quadrature rules that we have derived so far are of this form where $x_0, x_1, \cdots, x_n$ are given to us. Once we are given the nodes, then the unknowns are only the weights $w_0, w_1, \cdots, w_n$. In the last class, we have seen that we can use a method called method of undetermined coefficients to obtain these weights. of course, we can also obtain these weights by directly integrating the Lagrange polynomials in the interpolating polynomial of the function $f(x)$.

But in the method of undetermined coefficients, we have another approach to find these weights by imposing the condition that this quadrature rule finally gives us the exact value if the integrand $f$ happens to be a polynomial of degree less than or equal to $n$. So, this is what the condition we impose to get the weights. Once you impose this condition this is equivalent to imposing the same condition on the corresponding monomial basis that is what we did in the last class.

In fact, it is possible to derive a quadrature formula in such a way that it gives us the exact value of the integral if the polynomial is of degree less than or equal to $2n + 1$. Can you see how we can achieve this? Just think why we need to impose this condition, that is, why we need to impose that the quadrature formula gives us the exact result for polynomials of less than or equal to $n$? Because in that way you have $n + 1$ elements in the monomial basis and here also you have $n + 1$ unknowns.

So that is how we are matching the number of unknowns with the number of equations in the system and getting a closed system of equations. now if you understand this logic, then you can understand how to get this condition on our quadrature formula, that is, we now want our quadrature formula to be exact for polynomials of degree less than or equal to $2n + 1$. How can we achieve that, why you have to fix the nodes you also consider the nodes to be unknowns.

That is the idea behind getting this condition. So now, we will not fix the nodes but we will also obtain the nodes as well as the weights in that way how many unknowns are there? Just think about that. You have $n + 1$ unknowns coming from the weights. And you have $n + 1$ unknowns coming from the set of nodes. In that way you have $2n + 2$ unknowns. Therefore, you can impose the condition that the quadrature formula will be exact for polynomials of degree less than or equal to $2n + 1$.

In that way, the corresponding monomial basis will have $2n + 2$ elements in it. They are $1, x, x^2, \cdots, x^{2n+1}$, there are $2n + 2$ elements in the basis. And that can lead to a system of equations having $2n + 2$ equations. You have $2n + 2$ unknowns, therefore there is a scope to solve this system to get all these unknowns. So that is the basic idea of this improved version and that is called the Gaussian quadrature rule.

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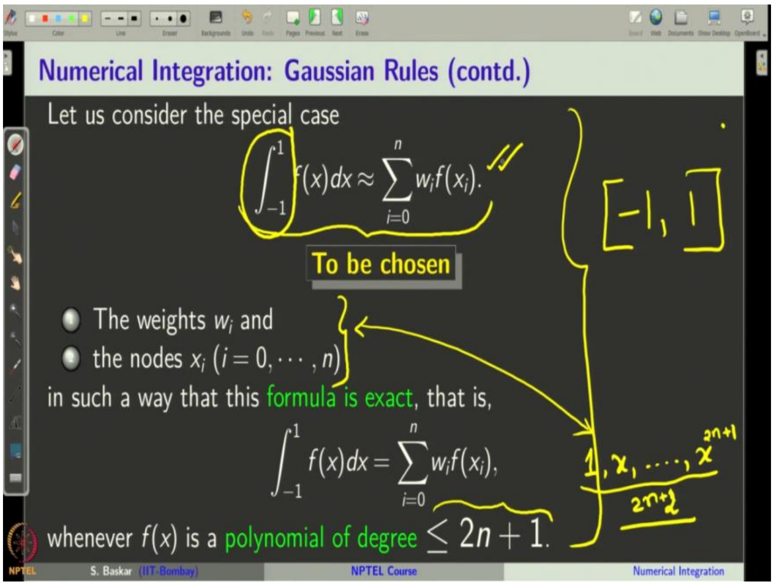


Let us make it more precise. You want to evaluate this integral, for that you are using this quadrature rule which can give us an approximate value to this integral. Now in this process, what are all the unknowns that we have to choose? We have to choose all the weights, they are not given to us but we have to obtain them. And also, now we have to obtain all the nodes, previously nodes are given to us.

But now, we are not going to take the nodes as per our choice but we will also obtain this nodes as the part of the method therefore you have $2n + 2$ unknowns. We have to impose the condition that this quadrature formula will be exact, that is, it gives you the exact value of the integral as long as the integrand $f$ is a polynomial of degree, now less than or equal to $2n + 1$, 1 less because your monomial basis will have 1, $x$ up t,o say if your polynomial degree is $n$, then it goes up to $n$, therefore you have $n + 1$.

So, if you are going up to $2n + 1$, then it has $2n + 2$ members and therefore you will get $2n + 2$ equations. On the other hand, you have $2n + 2$ unknowns therefore you can solve this system to get these unknowns, that is the idea. Remember in order to keep our calculation simple, we will impose this idea on the $[-1,1]$. Remember, our aim is to find a quadrature rule for the integral $\int_a^b f(x)dx$, for any $a < b$.

But in this calculation, we will always restrict ourselves to the interval $[-1, 1]$, keep this in mind. We will first derive the formulas once you have the formula for this integral, that is integral over minus 1 to 1, then we can use certain transformation to get the integral over any given interval $[a, b]$, that is the idea. This restriction is purely because our calculations will become relatively simple in this case, that is why we are doing this.

So let us try to derive the Gaussian rules for the integral $\int_{-1}^{1} f(x)dx$, later we will transform it to any integral $\int_{a}^{b} f(x)dx$.

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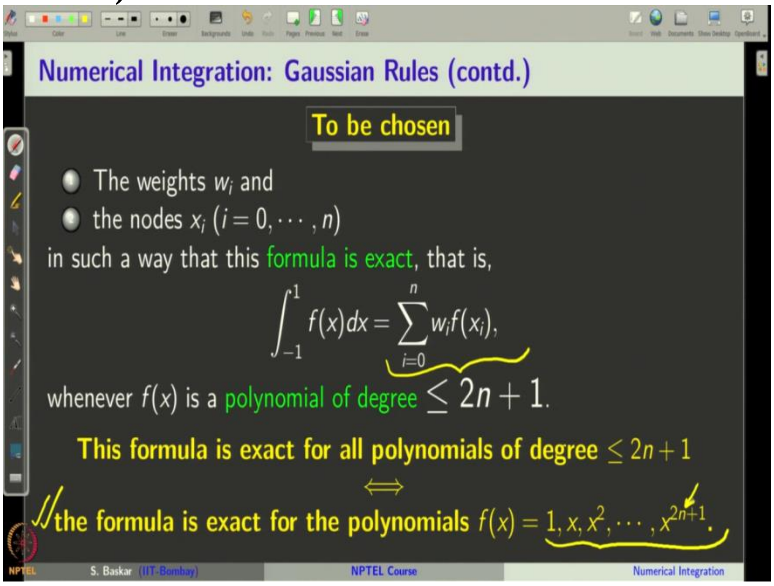


Let us keep this restriction in mind and go ahead. So what we are going to do is, we want to evaluate this integral and we want our quadrature rule in this form. We will assume that this quadrature rule gives exact value if the integrand *f* happens to be a polynomial of degree less than or equal to $2n + 1$. That is equivalent to imposing this condition on the monomial basis only that is, we will impose the condition that this quadrature formula is exact for the integrands $1, x, x^2, \cdots, x^{2n+1}$.

So that is the final condition that we will be imposing. Now you see, you choose n that is all, do not go to choose the nodes. Different $n$ leads to different quadrature rules, $n = 0$ will give you a quadrature rule for $n = 0$. And similarly, $n = 1$ gives a quadrature rule, $n = 2$ gives a quadrature rule. Like that as you go on increasing the values of $n$, you in fact get a better and better approximation to your integral. All these methods are called Gaussian quadrature rules only.

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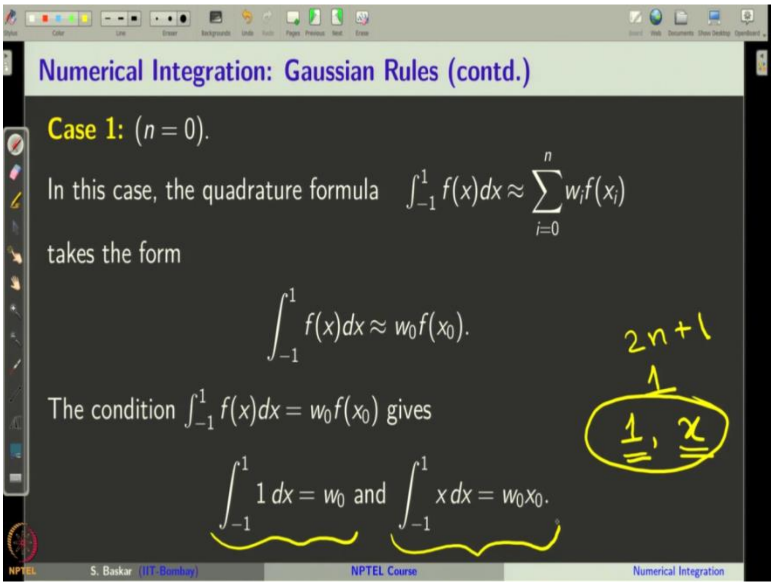


Let us try to derive the quadrature rule for $n$ = 0. Remember this is the general form of the quadrature rule and we want to take $n$ = 0 here. In that way our quadrature rule will look like this and what is the condition that we have to impose now? We have to impose that this is exactly equal to, that is what I am writing here the quadrature rule. If the integrand $f$ happens to be a polynomial of degree $2n + 1$, $n = 0$ therefore it is polynomial of degree less than or equal to 1.

That gives us 2 elements in the corresponding monomial basis that is 1 and $x$ therefore you have to get the weight $w_0$ and the node $x_0$ by imposing the condition that the quadrature formula gives exact value if $f = 1$ that is this, and $f = x$ that gives us this expression.

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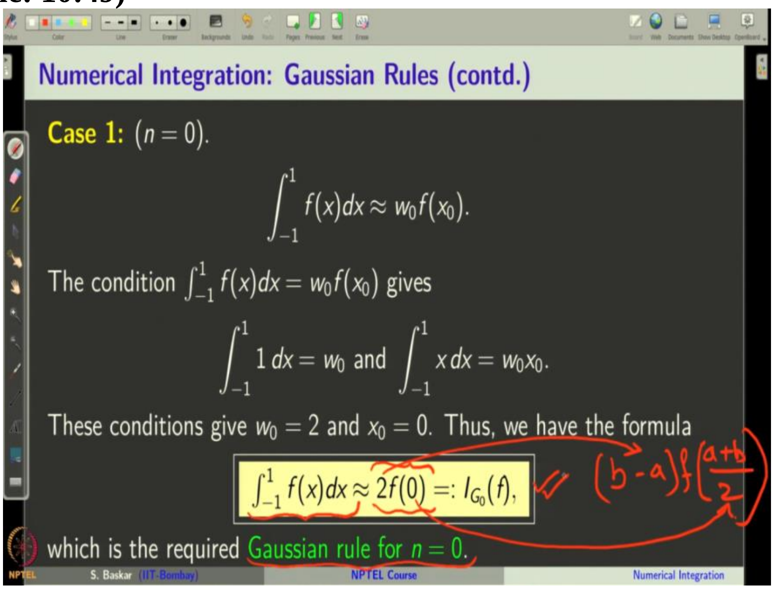


Now from here you can get a pair of equations each coming from these conditions. You see now, we do not have a linear system because the unknowns are $x_0$ and $x_1$. And they are not appearing linearly in this equation therefore in the Gaussian quadrature rule what you get is, finally a non-linear system of equations. That is 1 level difficult in the case of Gaussian rules, when compared to the quadrature rules that we derived in the previous idea.

There we are given the nodes, therefore the unknowns are only $w's$ and in that way it gives us a system of linear equation but that is not the case here. You will get non-linear system of equations but in the present case it is very easy to solve this non-linear system. In fact, you can easily check that it leads to $w_0 = 2$ and $x_0 = 0$. And in that way the quadrature rule finally reduces to this expression.

So, what it says is the Gaussian rule for $n = 0$ for the integral $\int_{-1}^{1} f(x)dx$. Remember this is a particular case only, $\int_{-1}^{1} f(x)dx$ is given like this. If you recall we have come across this method already in one of our previous classes. What is that, well you can go back and see that this is what precisely we called as the midpoint rule. Remember the midpoint rule is $(b-a)$ into $f$ of the midpoint of the interval $\frac{a+b}{2}$.

In the present case the midpoint is 0. So that is what is this and $b - a$ is precisely 2 here in this particular integral minus 1 to 1. Therefore, what you get as the Gaussian rule for $n = 0$ is precisely the midpoint rule, that is what is interesting here.

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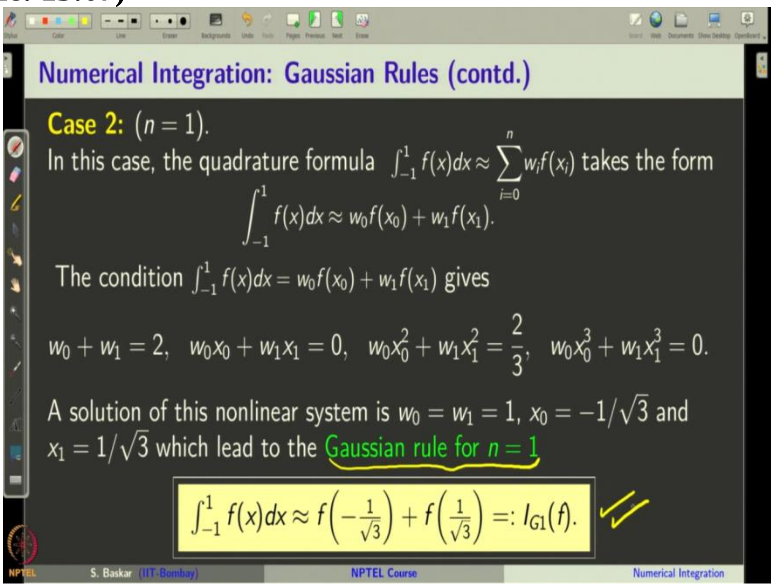


Let us go to the next case, now we will take $n = 1$ and see how the Gaussian rule with $n = 1$ looks like. Again, in this case, we have to take our general quadrature rule and put $n = 1$ in that to get this expression. So, this is the general form of the quadrature rule that we are interested in the present case. Here we have to obtain the weights $w_0, w_1$ and also the nodes $x_0$ and $x_1$. Therefore, you have to impose the condition that your quadrature formula will be exact for all polynomials of degree less than or equal to 3.

Therefore, your monomial basis will now contain the elements $1, x, x^2$ and $x^3$. For each we will get a non-linear equation. Let us see how it comes, when you take $f(x) = 1$, you get this equation. When $f(x) = x$, you get this equation, $f(x) = x^2$, you get this equation and finally $f(x) = x^3$ gives you this equation. You can see that you have 4 equations, it is a system of non-linear equations.

And you somehow have to solve this system you can see that $w_0 = w_1 = 1$ and $x_0 = -\frac{1}{\sqrt{3}}$ and $x_1 = \frac{1}{\sqrt{3}}$ will solve this system of non-linear equations. Therefore, the Gaussian rule with *n* = 1 is given by this formula. Now as you go on increasing *n*, the number of non-linear equations will also increase in your system and also their expressions are quite complicated. And thereby solving the system of non-linear equations will also become more difficult one can go for certain non-linear solvers like Newton's method and so on.

But we will not give any weightage for such problems. We will just stop our derivation of Gaussian rules only up to *n* = 1. However, we will just give an idea of how to go about with *n* = 2, 3 and so on in general for *n*.

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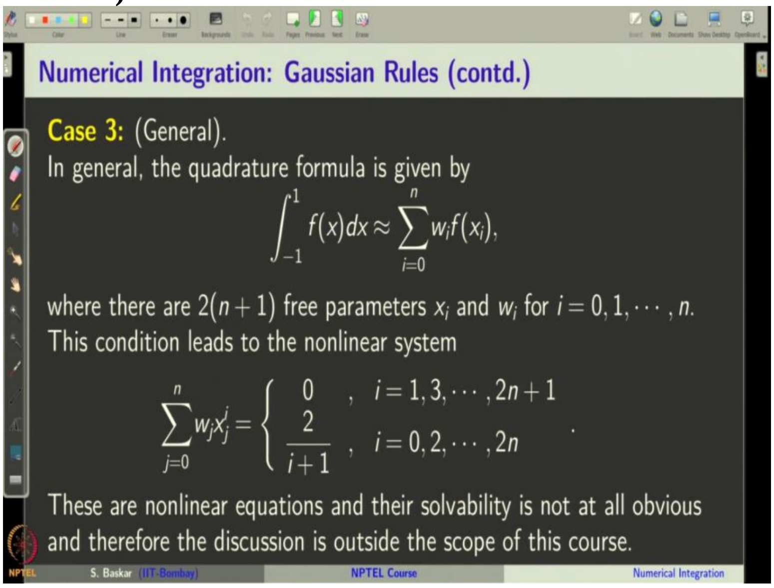


In general, we need to obtain the weights and the nodes such that you can approximate the integral $\int_{-1}^{1} f(x)dx$ by this quadrature formula. For that we have to impose the condition that this quadrature formula will be exact for polynomials of degree less than or equal to $2n + 1$. Because we have $2n + 2$ unknowns in our problem, so those non-linear systems in general are given by this expression and therefore you have to solve this non-linear system. That is a quite difficult task and once you solve this non-linear system and get the weights and the nodes then you have the Gaussian rule for that given $n$.

So, $n = 2$, 3 and so on, one can go on deriving but we will not give any weightage in our course. We will only restrict ourselves to $n = 0$ and $n = 1$ in our course but the idea should be clear how to go for higher values of $n$.

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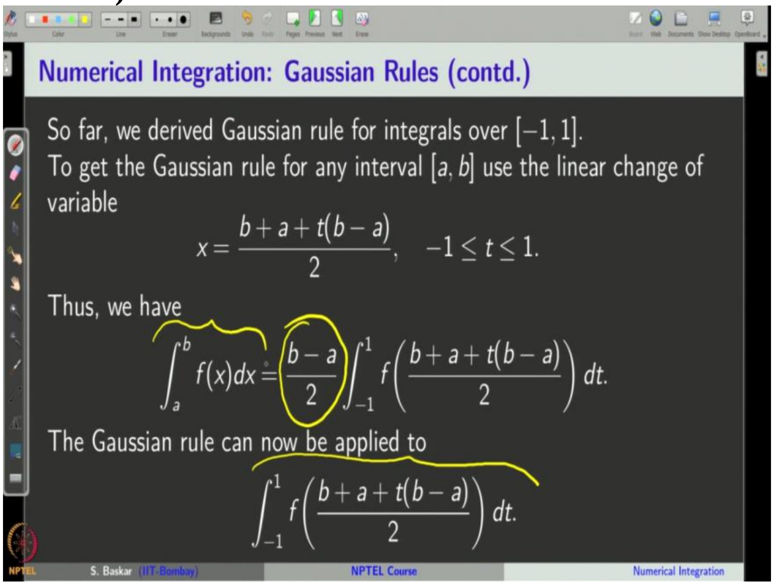


We have derived the Gaussian rule so far only for those integrals over the interval $[-1,1]$. Now, let us see how to generalize it to any given interval $[a, b]$. This can be achieved by this simple change of variable formula that can take the interval $[-1,1]$ to any interval $[a, b]$. So, you just have to impose this change of variable into your integral.

You are interested in finding the integral $\int_a^b f(x)dx$. But, you have the Gaussian rule only for integral $[-1,1]$, that is you have only the Gaussian rule defined for this kind of integrals that is integral over $[-1,1]$. But that is not a serious problem because you can write this $\int_a^b f(x)dx$ as $\frac{b-a}{2}$ into the integral that is comfortable for us to apply the Gaussian rule.

Now remember, if you want to evaluate and approximate value of this integral, you should not put the Gaussian rule for this $f$, but you have to put the Gaussian rule for this function, that is the only extra information that you have to remember. When you are applying Gaussian rule on any integral $\int_a^b f(x)dx$ students make this mistake quite often they just take this *f*. And apply the Gaussian quadrature rule for this *f* only. You should not do that you should apply it to this integrand.

Therefore, the Gaussian quadrature rule should be applied to this integral, and then you multiply it with this number in order to get an approximate value of this integral using Gaussian rule that is one extra work you have to do you should not forget that.

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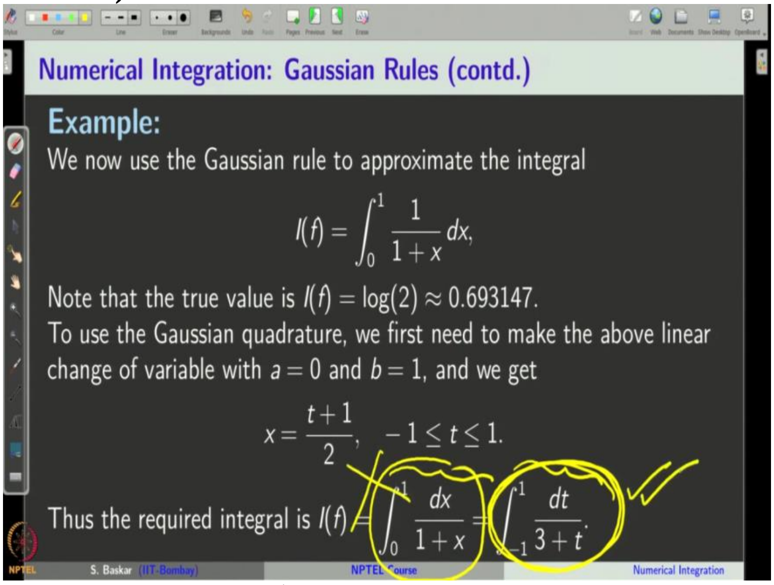


Let us try to evaluate the integral $\int_0^1 \frac{1}{1+x}dx$. Remember, our integral is not over $[-1,1]$. Therefore, you first have to carefully use the change of variable which we have shown in this slide and obtain this function with *a* = 0 and *b* = 1 now, and then apply the Gaussian rule. So, in the present case the change of variable happens to be $x = \frac{t+1}{2}$. Therefore, this is the given integral and that should be now rewritten in this form.

And then apply the Gaussian quadrature rule for this integral. Remember, that is what I am just emphasizing, do not apply the Gaussian quadrature rule for this integral, this is wrong. You apply the Gaussian quadrature rule for this integral.

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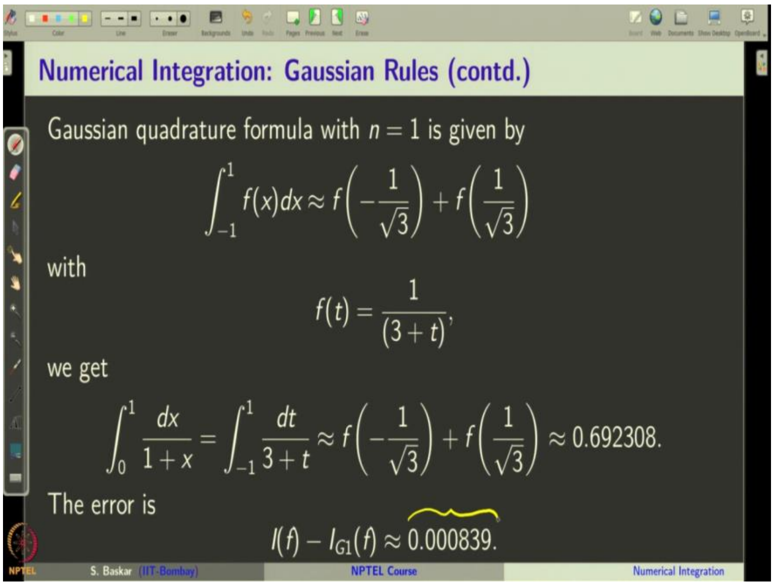


So that is you use this formula, this is the Gaussian quadrature rule for $n$ = 1. Similarly, for $n$ = 0 also you can do. What is $f$ now? $f$ is not the one which is given to us. But $f$ is the one which you obtained after putting the change of variable that is $\frac{1}{t+3}$ and that gives you this value. So, this one transformation that you have to do without forgetting and that is very important. You see, what is the mathematical error involved in this calculation.

It is something given like this, again I am giving you a caution that although I am calling it as mathematical error ideally, it is actually the total error.

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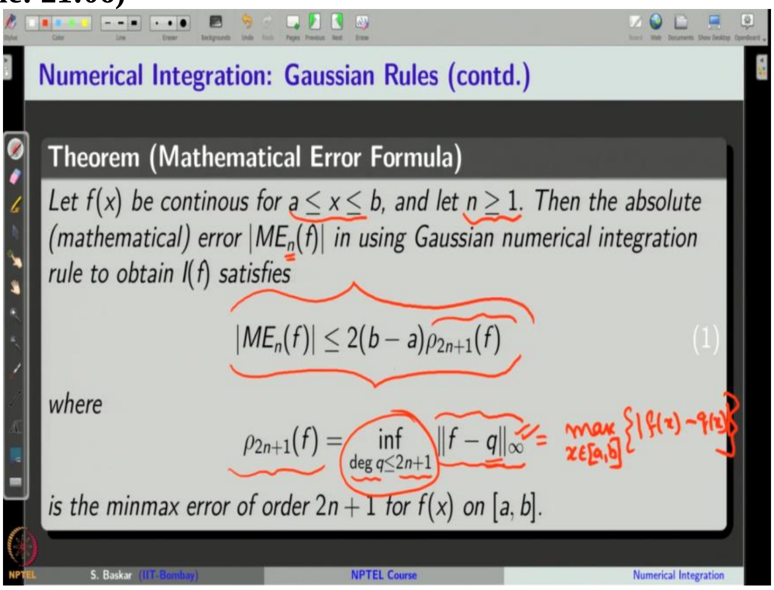


Let us see how the mathematical error estimate looks like. We can obtain an estimate in the case of Gaussian rule. let us assume that $f$ is a continuous function defined on an interval $[a, b]$. And you have some $n$ and you also obtained the Gaussian rule for that given $n$. Then the mathematical error involved in the Gaussian rule with that $n$ is denoted by $ME_n(f)$ and you can estimate the mathematical error by this inequality.

Where this $\rho = \inf_{\deg q \leq 2n+1} ||f - q||_\infty$. Remember, you should go back to our previous classes and see what this infinite norm means. It is nothing but $max\ \{|f(x) - q(x)|\}, x \in [a, b]$, that is what is mean by this notation. And we call it as maximum norm or infinite norm. So what you are doing is, you are taking all the polynomials of degree less than or equal to $2n + 1$.

And obtaining the maximum norm of that minus $f$ and then taking the infimum over all those numbers. And that is what is called as $\rho_{2n+1}(f)$ and the upper bound of the mathematical error involved in the Gaussian quadrature rule is given like this.

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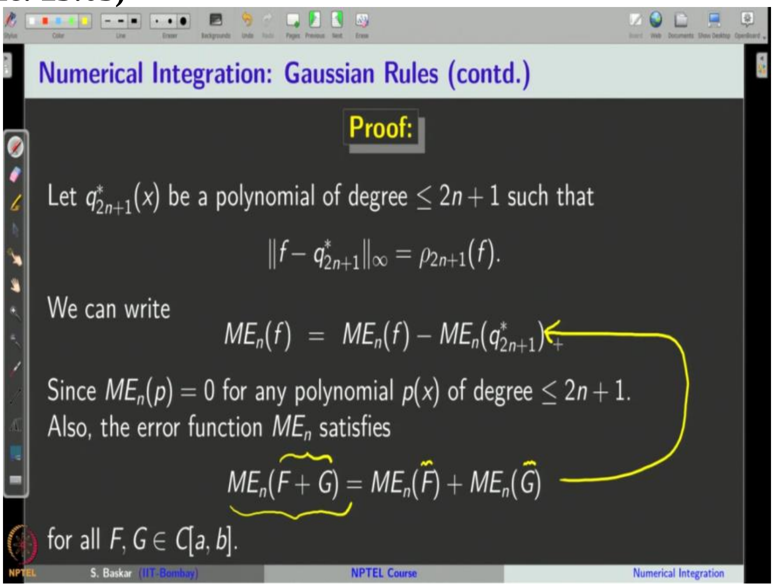


Let us see how to prove this, it is not very difficult. Assume that the infimum is achieved at some polynomial which is denoted by $q^*_{2n+1}$. It is a polynomial of degree less than or equal to $2n + 1$. Then $\rho_{2n+1}(f)$ is precisely equal to this because infimum of this is what is the definition of $\rho$ and we are taking that infimum to be achieved at $q^*$. Therefore, if you take the infinite norm of $f - q^*$ that will be exactly equal to this number that is by definition.

And now look at the mathematical error of $f$. You can see that the mathematical error involved in the Gaussian quadrature rule in evaluating the integral of $f$, is written like this. Why it is so? Because this is actually equal to 0. Because by the derivation, Gaussian quadrature rule gives you the exact value for all polynomials of degree $2n + 1$ and $q^*$ is a polynomial of degree less than or equal to $2n + 1$. Therefore Gaussian quadrature rule gives you the exact integral value.

That means the mathematical error involved in the value, obtained from the Gaussian quadrature rule for $q^*$ is exactly is equal to 0. So, what I am doing is, precisely the mathematical error in $f$ is equal to the mathematical error in $f - 0$. That is all I am putting, I am not putting anything extra here. Therefore, this is always true. Now, you just check that the mathematical error involved in the Gaussian quadrature rule for the integrand $f + g$ is nothing but the mathematical error involved in the Gaussian quadrature rule, with integrand $f$ plus the mathematical error with integrand $g$.

So this is very simple to check, it comes directly from the linear property of the integral, in fact. Now I will use this simple property in this expression.

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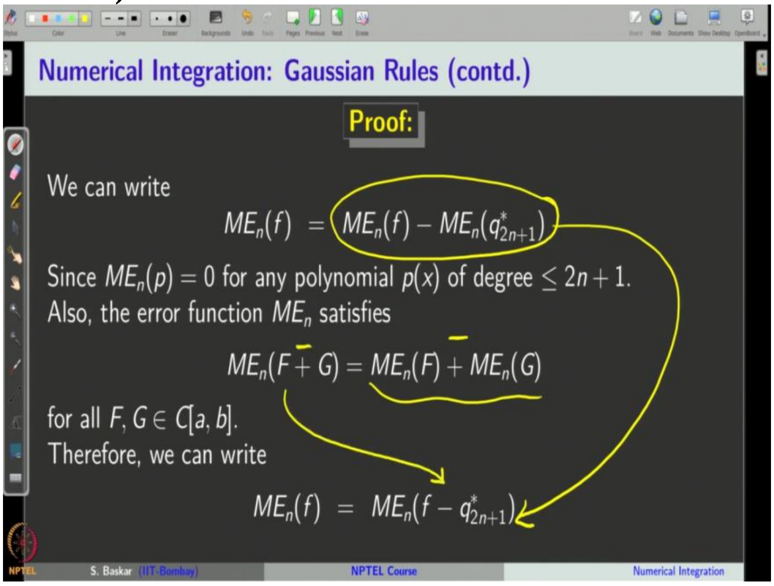


And that tells me that I can write this expression like this so I am just having this, of course with the minus sign here. And that I am writing like this with a minus sign here that is what I am doing.

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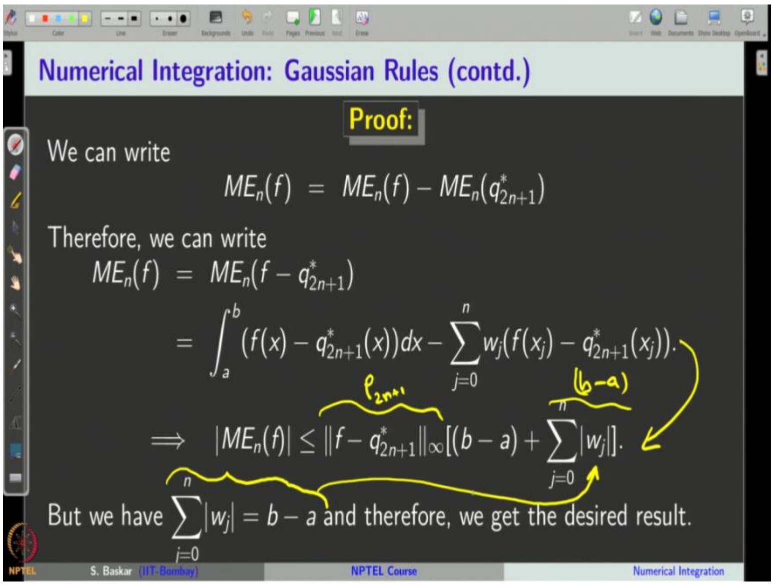


And now we can see that the mathematical error in $f - q^*$ can be written like this. This is precisely the definition of the mathematical error. This is the exact value minus the quadrature rule. That is, the Gaussian quadrature rule is this. So, this is exact value and this is the approximate value. So, that is the mathematical error. Now, let us take the modulus on both sides and use the triangle inequality for the modulus.

And then take the maximum norm on this integrand. I am doing all this in one step, you can see that the right hand side, in fact, can be written like this, after taking a modulus with a less than or equal to sign. So, you are just dominating modulus of this by this quantity. you can easily check this. What I am doing, I am just taking the modulus. And using the triangle inequality for the modulus and I am also using the condition that $|\int_a^b f(x)dx| \leq \int_a^b |f(x)|dx$.

This is also a property, that is well known for the integrals. I am using that also here, I am first taking modulus here and then pushing this modulus inside the integral. And then dominating this term by its maximum. That is how I am having the maximum norm here and then what remains is $\int_a^b dx$ that is nothing but $b - a$. And you can also see that this term can be dominated by this. That is not a difficult thing, of course you take the modulus and then take the modulus inside the summation.

And then you get this. I hope you can do this to this without any problem and now you can see that all these weights with the modulus will sum to the length of the interval, $b - a$. So that is what is very interesting, once you put this into this term you will see that you will get back the inequality that we want to prove. Remember this is what we have taken as $\rho_{2n+1}$. And this will be another $b - a$, therefore $b - a + b - a$ will be $2(b - a)$.

That is what precisely we want to show and this gives us an estimate of the mathematical error involved in the Gaussian quadrature rule. With this we will end our class, thank you for your attention.