Numerical Analysis
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Lecture: 50
Numerical Integration: Simpson's rule

Hi, we are learning quadrature formulas to approximate a given integral. In this we have already seen rectangle rule, midpoint rule and trapezoidal rule and also their composite versions. In today's class we will learn Simpson's rule. Simpson's rule is obtained by integrating the corresponding quadratic polynomial, interpolating the integrand at 3 specific nodes.

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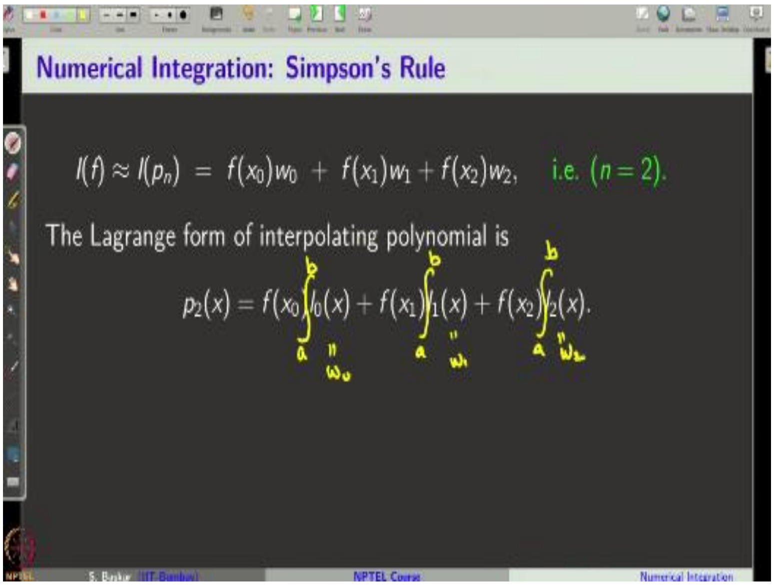

Recall, that we obtain the quadrature rules by first approximating the given integrand by its interpolating polynomial of certain degrees, by supplying the nodes explicitly. That way today, we will take the number of nodes as 3 that is $n = 2$. By this we have to give 3 nodes x_0, x_1 and x_2. Therefore our quadrature formula will be $f(x_0)w_0 + f(x_1)w_1 + f(x_2)w_2$. If you write the interpolating polynomial p_2 in the Lagrange form, you get this expression for the interpolating polynomial with $n = 2$.

The idea is to integrate this polynomial and thereby you will have integral a to b for the Lagrange polynomials and they give you w_0, w_1 and w_2. So, this is the idea that we have been following to derive the quadrature rules.

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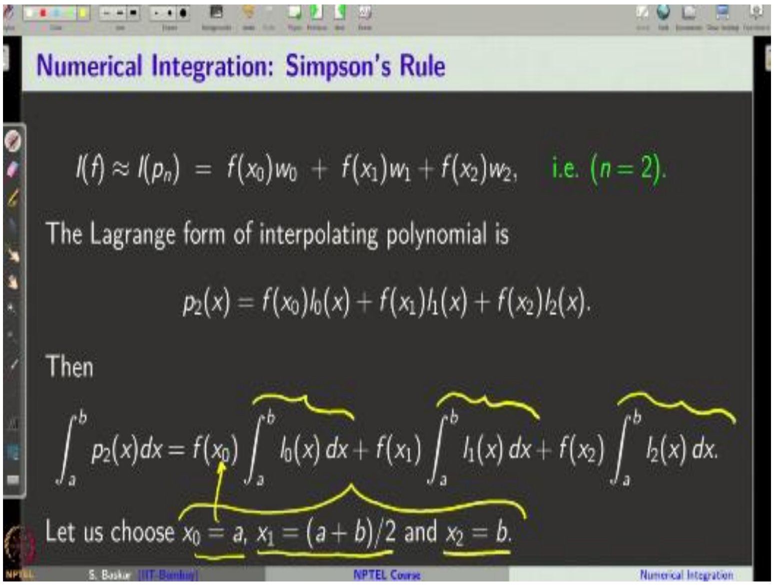

And that leads to this expression. And now let us see how to choose the nodes because we can choose different nodes to get different quadrature formulas. For Simpson's rule we have to choose the nodes like this. They are equally spaced in the interval $[a, b]$ and they are $x_0 = a$, $x_1$is the midpoint of the interval $[a, b]$ and x_2 is b. Once you get this, you plug in this into this expression and you have to perform this integrals, that is what is the difficult job that we have right now.

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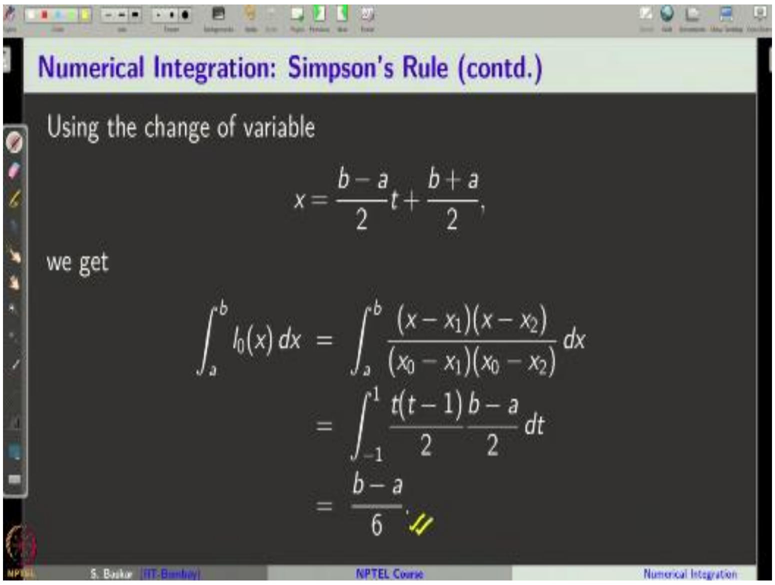

For that, we will use a change of variable in order to make this calculation little simple and calculate these 3 integrals and see how they look like. First let us take $\int_a^b l_0(x)dx$, l_0 is the Lagrange polynomial with $k = 0$. And its expression is given like this and therefore this is the integrand. You can see that it is a quadratic polynomial, therefore you can explicitly integrate it and just to make the calculation simple we will make this change of variable here.

And then integrate it to get the expression for the integral as $\frac{b-a}{6}$. Similarly, we have to evaluate this integral, as well as this integral. Let us see how they come out to be.

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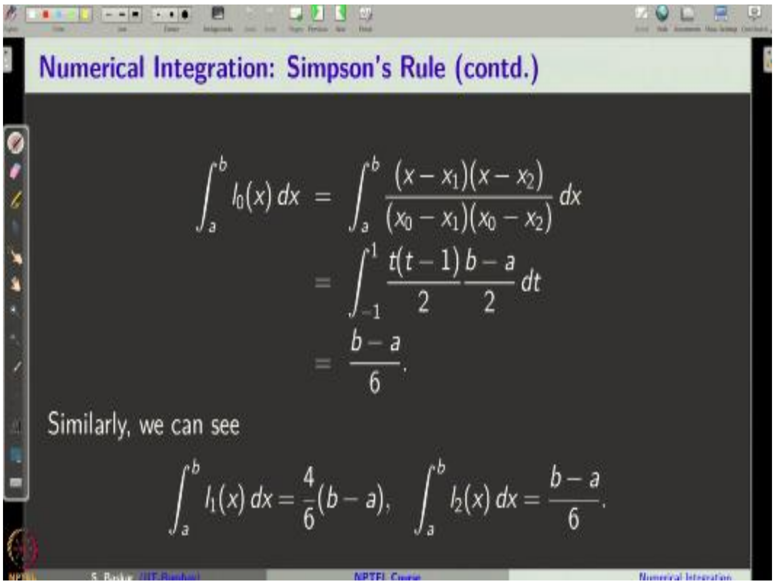

For the second integral, it is $\int_a^b l_1(x)dx$. You can again write the expression for $l_1(x)$ and then use the change of variable, similar to what you did with l_0 and you can see that finally that integral will reduce to $\frac{4}{6}(b-a)$ and a similar calculation will also give us $\int_a^b l_2(x)dx$. Again, it is equal to $\frac{b-a}{6}$. Now you just have to plug in these values into the $\int_a^b p_2(x)dx$.

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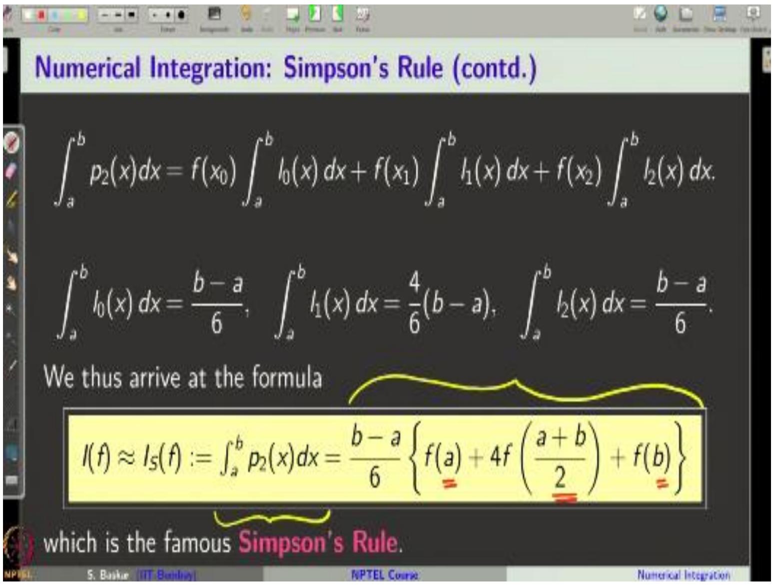

Remember $\int_a^b p_2(x)dx$ is given like this, where you just have to put this value in the first term and this value in the second term and this value in the third term and that gives you the required

Simpson's rule given by this formula and that is obtained by simply integrating $p_2(x)$ with $x_0 = a$ that is what is shown here, x_1 is equal to the midpoint of the interval $[a, b]$ and $x_2 = b$. If you choose the nodes like this, then the corresponding rectangle rule is what is called the Simpson's rule.

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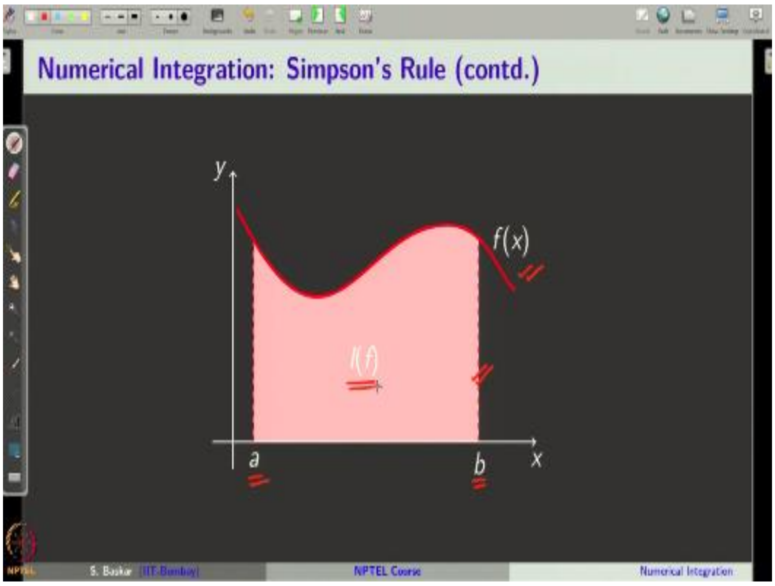

Let us see the geometrical interpretation of the Simpson's rule. Suppose your function $f(x)$ is graphically looking like this, then $\int_a^b f(x)dx$ is the area under the graph of the function $f(x)$ between the interval $[a, b]$. That is the geometrical interpretation of the integral and that is shown in the light red color here and this is the region which you have to find the area and that gives you the $\int_a^b f(x)dx$. This is precisely what we want to find.

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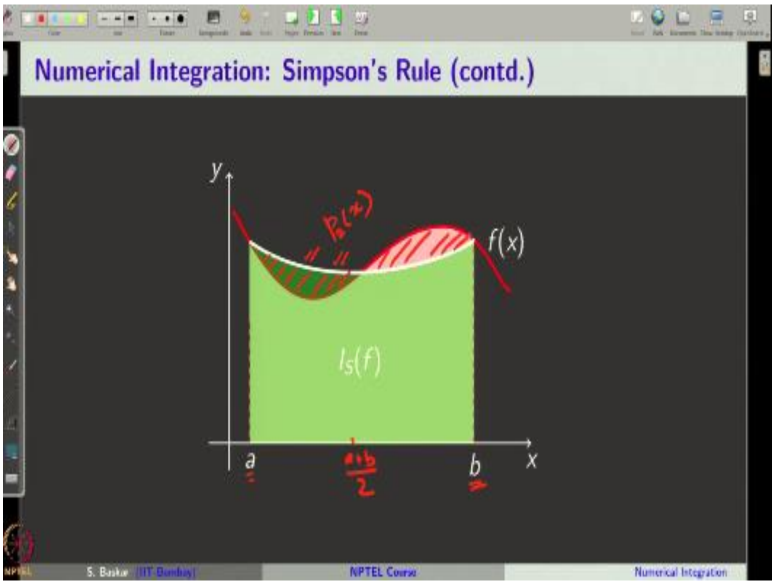

But, Simpson's rule takes the quadratic polynomial interpolating the function f at the nodes a, $(a+b)/2$ somewhere, I am just roughly placing it, It is the midpoint of the interval $[a, b]$ and then b. Say for instance, the graph of the interpolating polynomial is given roughly by this white solid line then the Simpson's rule gives us the area under the graph of $p_2(x)$. This is just a roughly drawn graph for the quadratic polynomial interpolating the function $f(x)$. Just to illustrate the geometry of the Simpson's rule.

Now let us see, how the mathematical error can be derived. Remember, the mathematical error involved in the Simpson's rule is going to include the area covered in this place and the area covered in this place. So, these 2 things are going to be precisely contributing to our mathematical error.

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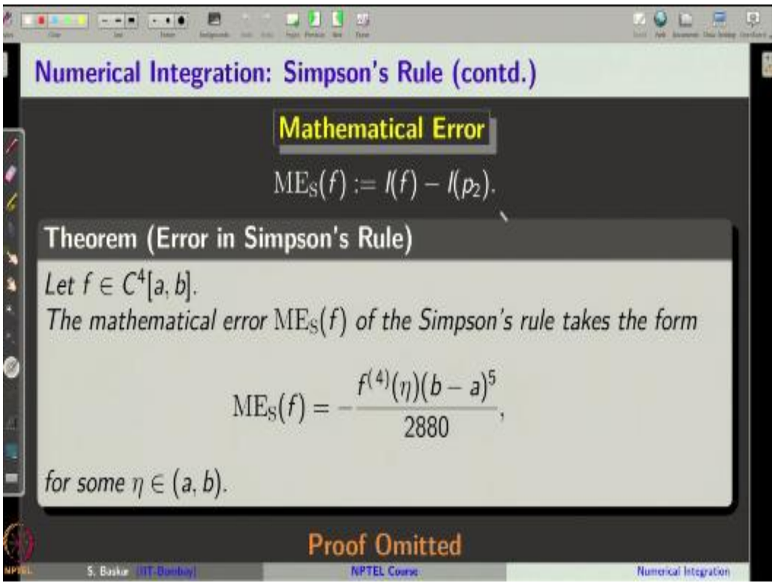

And by definition the mathematical error involved in the Simpson's rule is nothing, but the exact value that is $\int_a^b f(x)dx$ minus the value obtained from the Simpson's rule which is precisely the integral $\int_a^b p_2(x)dx$. So therefore this is the basic definition of the mathematical error involved in the Simpson's rule. As we did in other cases, we can also get an expression for this mathematical error. For that, you have to assume that f is a C^4 function in the interval $[a, b]$.

Once you have that, then the mathematical error can also be represented by this expression. That is what the theorem says it can be written as $-\frac{f^{(4)}(\eta)(b-a)^5}{2880}$, $\eta \in (a, b)$. So that is the expression, that you can obtain for the mathematical error whereas this is the basic definition of the mathematical error.

The proof is not very difficult but it is little bit involved. Therefore, we will omit this proof for our course. However, we have given the proof in our notes. Interested students can go through the proof.

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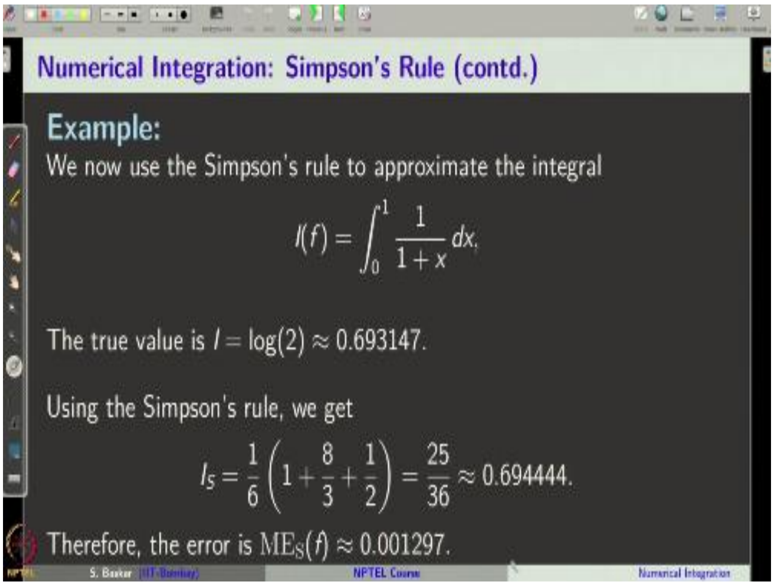

Let us take an example. We will again consider the same integral, which we have been working with the other quadrature formulas also. We will consider evaluating the integral $\int_0^1 \frac{1}{1+x} dx$, here f is taken as $\frac{1}{1+x}$. The exact value of this integral is log 2 and numerically you can approximately take it as 0.693147 and there are more terms but we have just rounded it up to here. Now let us apply the Simpson's rule and see how the value comes out of this Simpson's rule for this integral.

For that, we have to take the formula for the Simpson's rule that is $(b-a)/6$, that is $1/6$ here into $f(a)$, you can check that $f(a)$, a = 0 therefore $f(a)$ is 1, plus 4 times $f(a+b/2)$ that is 0. 5. That turns out to be 8 / 3 + $f(b)$. Now you just calculate this, you will get the value of the integral as 25 / 36, if you use this Simpson's rule. That is only an approximate number and that in the decimal form it gives you 0.694444.

You can see that the error involved in this is roughly 0.001297. Of course this is, strictly speaking, also involving the arithmetic error because we are representing all these numbers after a rounding approximation. Therefore, we should ideally call it as total error but just with a little abuse of notation, we are just calling it as mathematical error only. But you have to bear in mind that strictly speaking this is the total error.

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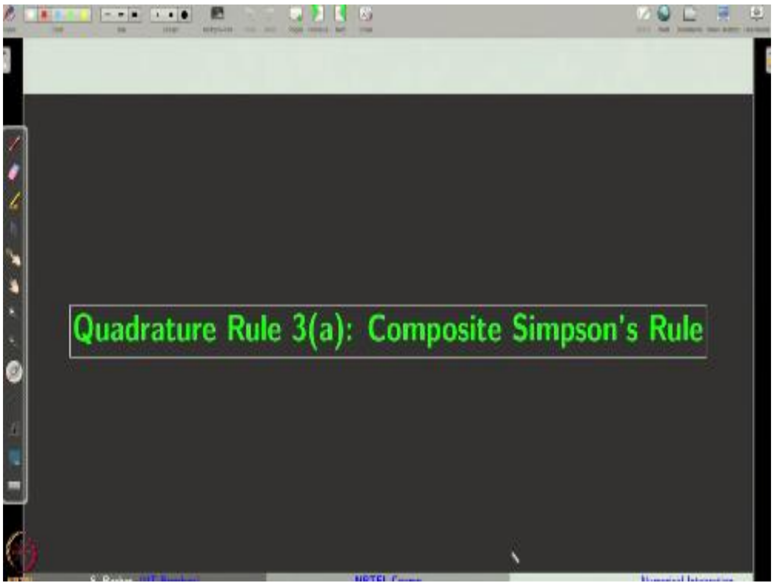

If you recall, we have derived composite quadrature rules for both rectangle and trapezoidal rule. We will also derive the composite Simpson's rule now. What is the basic idea of any composite quadrature rule?

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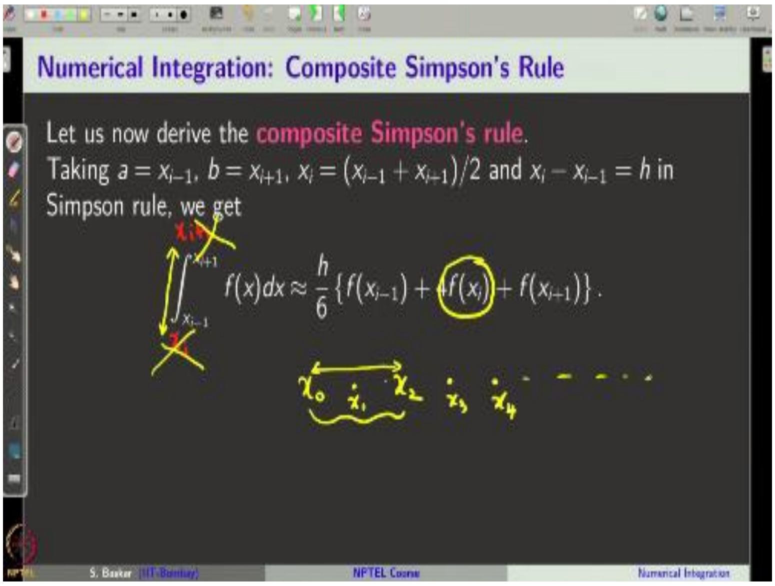

I hope you have seen the previous videos, you know the idea very well. You just have to introduce a partition to your interval $[a, b]$. It can be non uniform partition also. But we have been taking only uniform partitions, just for the sake of simplicity. Here also we will take a uniformly spaced partition with the length size of each partition as h and therefore you can introduce the partition points as $x_i, i = 0, 1, 2, \cdots, n,$ where $x_i - x_{i-1} = h$.

And also, we will make sure that $a = x_0$ and $b = x_n$. Now here is a catchy point, when you are deriving the Simpson's rule, remember you need 3 nodes in order to define the Simpson's rule. Therefore, when you take the partition, you need 3 nodes in that interval. Suppose you take some point as a and some other point as b, there should be 1 point in between these 2 nodes in order to define the Simpson's rule.

Therefore, you have to take the pieces of the Simpson's rule as x_{i-1} to x_{i+1}. So that x_i comes as a node point. Because you are only given the nodes and you are supposed to find the formula based only on those nodes, you cannot generate a new node in order to evaluate the integral. You have to, somehow, manage them with the nodes given to you. Therefore, you cannot take x_i and x_{i+1}.

If you take like that, that is suppose you are taking this as x_i to x_{i+1}, then you can clearly see that there is no node between these 2 but you need one in between node in order to define Simpson's rule. Therefore, this way of taking nodes is not correct. You have to take x_0 and then x_2, this as 1 piece although you have x_1 as a node in your partition you have to take this as 1 interval and similarly x_2 to x_4 you have to choose although x_3 is a node, you cannot take x_2 to x_3.

Then you cannot apply the Simpson's rule in the interval x_2 to x_3, you can only apply Simpson's rule in the interval x_2 to x_4, like that it goes. That clearly says that the number of node points that you take in the interval in order to apply composite Simpson's rule, you need that number to be of even number.

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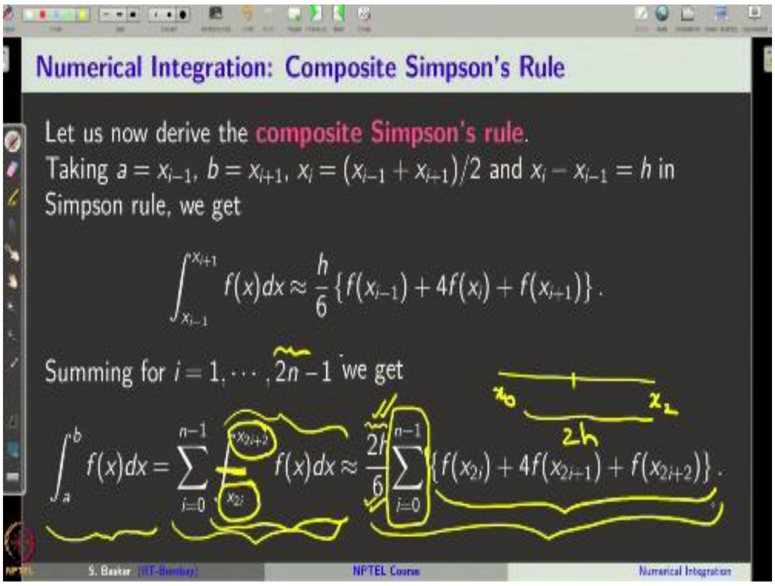

So to make sure that we need even number of nodes, let us take the number of nodes as $2n$. So that is something which you have to remember always, the number of nodes that is needed for us to generate composite Simpson's rule, should be an even number. Now once you make sure then your $\int_a^b f(x)dx$ can be first broken into the smaller integrals taken on the intervals x_{2i} to x_{2i+2}. Therefore, there is 1 node always sitting in between these 2, that is the idea.

Once you split this integral $\int_a^b f(x)dx$ into integrals of this form, remember this is very important, you should not forget it. This is very important for Simpson's rule, you can go back and observe that this complication is not there in trapezoidal rule as well as in the rectangle rule. So therefore, you have to keep this in mind when you are working with composite Simpson's rule.

Then once you make the choice of your intervals properly, you can easily get the formula for the composite Simpson's rule, by applying the Simpson's rule for each of these integrals and that is given by this expression. And now you have to take that with this summation you can see that gives you $2h$ because your interval is always x_0 to x_2 and then x_2 to x_4, like that you are jumping $2h$ length in each sub interval. That is why, you see that here you have instead of $b - a$ you have $2h$ divided by 6 and then the Simpson's formula is applied here.

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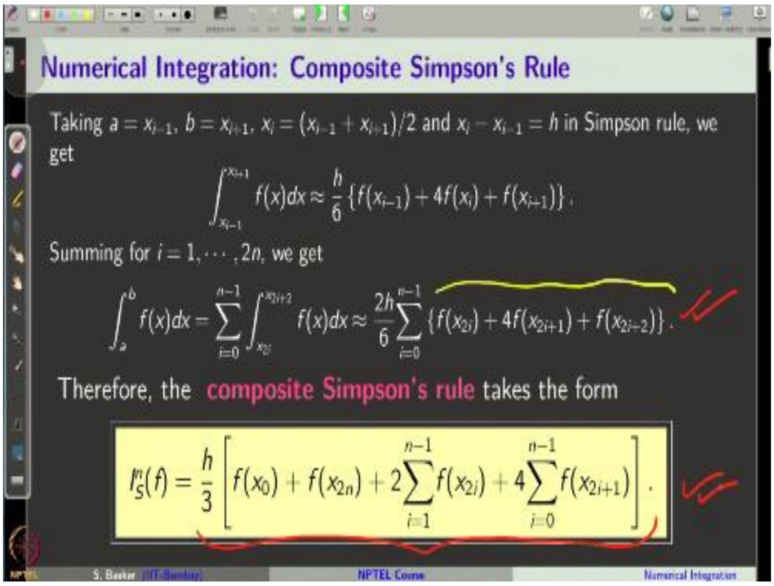

After a suitable rearrangement of these terms, you can finally get the formula for the composite Simpson’s rule. Like this, you can carefully go through and see how I have rearranged these terms in order to get this nice looking formula. And this is the composite Simpson’s rule. Well, like this you can keep on choosing n = 3, 4, 5 and so on and also for every given n, you can also choose nodes at different positions in order to get different quadrature rules.

We have only given 3 cases rectangle rule of course we have also given midpoint rule and then we have given trapezoidal rule for n = 1 and Simpson's rule for n = 2. You can keep on going like this and you can generate many such quadrature rules.

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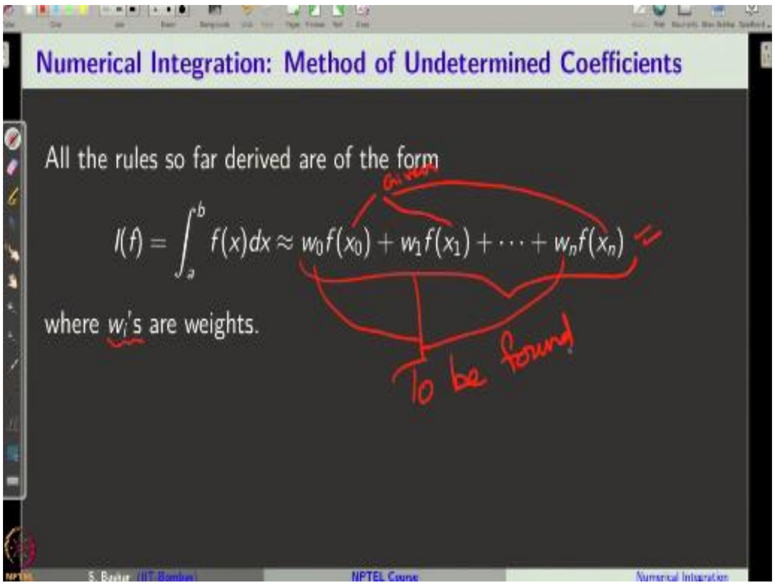

Let us see another way of deriving this quadrature rules, called method of undetermined coefficients. This is particularly important when you go to derive Gaussian quadrature rules.

That is why, we are first introducing this method. If you recall, we have the quadrature rule in this form. You know how this form comes out, I am not going to repeat that all the quadrature rules that we have derived so far. We look finally in this form for a suitably chosen set of nodes for a given n.

And the weights w_i's are computed by integrating the corresponding Lagrange polynomials. Now in this we are given these nodes and the weights are to be found. And these weights are found by just directly integrating the Lagrange polynomials. Now there is another way to find these weights, that is what we will be doing here and this way of finding weights is what we call as method of undetermined coefficients. Let us see how to find this w_i's instead of going for integrating the Lagrange polynomial.

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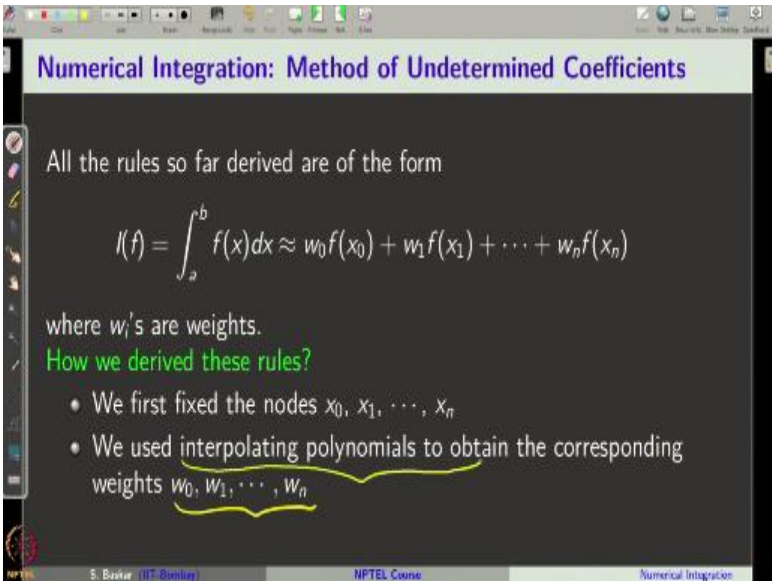

What you do is, first of course we are given the $n + 1$ node points that is given to us and then we have to find these weights and so far we are doing it using the interpolating polynomials.

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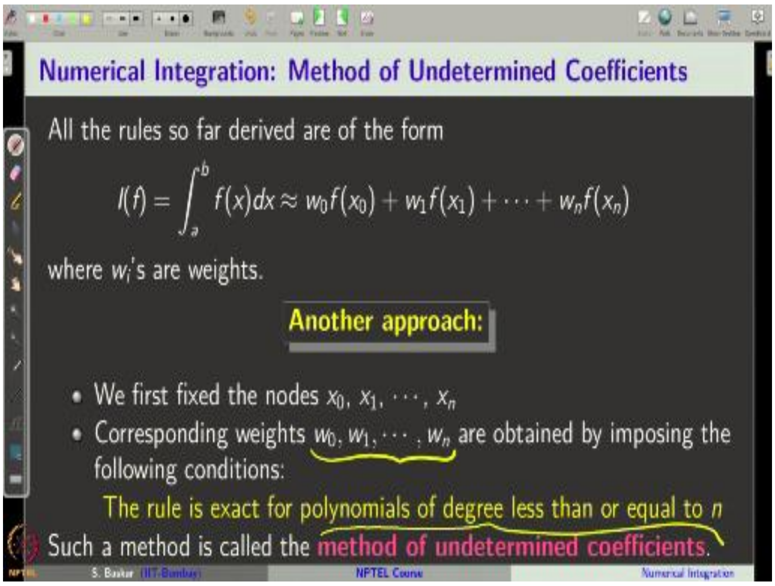

There is another way to do it, let us see how to do that. We first fix this nodes and then to get the corresponding weights we will impose certain conditions. What are those conditions? We will say that this rule is exact for polynomials of degree less than or equal to n, this is the important point. Remember this is the exact integral and this is the approximation for the exact integral, that is why we have put this symbol approximately equal to.

So, this left hand side is not exactly equal to the right hand side but in this condition what we are saying is, this will be exactly equal if this integrand happens to be a polynomial of degree less than or equal to n, that is what we are imposing as a condition. And this will obviously give us $n + 1$ equations involving $x_0, x_1, \cdots, x_n$ and also $w_0, w_1, \cdots, w_n$. In this x_i's are known to us, therefore we can treat this system of equations with unknown as w_i's.

In that way, we will get a linear system of equations with unknown vector as w whose coordinates are $w_0, w_1, \cdots, w_n$. Now you solve this system, if it is possible, and you get the weights. This is the alternate way to get weights and this approach is what we call as the method of undetermined coefficient. This method is well understood through certain examples.

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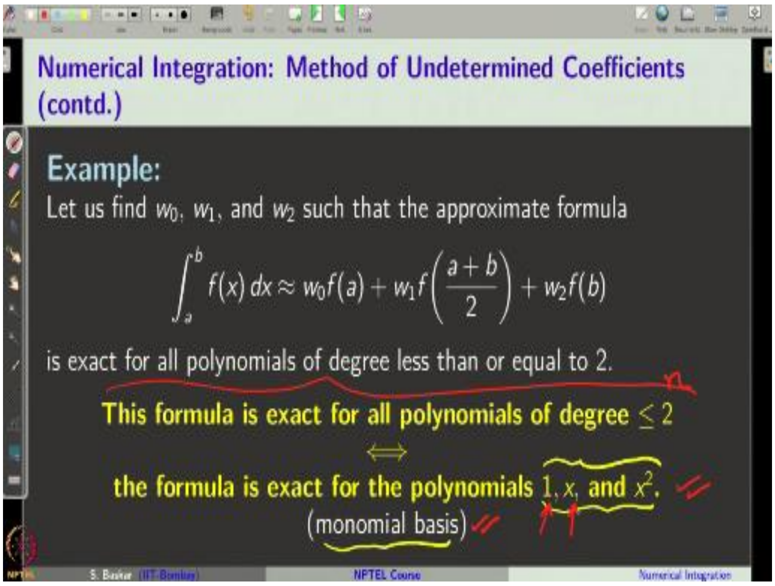

So let us try to understand this method through this simple example. We want to find the quadrature rule for evaluating an approximate value of this integral $\int_a^b f(x)dx$. For that, we want to propose the quadrature rule in this form. In this what we are doing, we are taking x_0 as a, x_1 as the midpoint of the interval a to b and x_2 as b. If you recall, we have already learned how to derive the expressions for w_0, w_1 and w_2 using the corresponding interpolating polynomial.

And we have in fact called that quadrature formula as Simpson's rule in our previous slides. We will take the same situation that is the same set of nodes and we have n = 2 here, because we have 3 nodes. Instead of going for the method of interpolating polynomials, now we will go with the method of undetermined coefficients. And try to get these weights w_0, w_1, w_2 by imposing the condition that this will give you the exact value to this integral.

That is this will be exactly equal to the right hand side, if this function f happens to be a polynomial of degree something less than or equal to 2. So, this is the condition that we will impose and you can clearly see that this condition is equivalent to the condition that the formula is exact for the polynomials which are coming from the monomial basis of the space of all polynomials of degree less than or equal to 2.

Remember the set of all polynomials of degree less than or equal to 2 forms a vector space and the functions 1, x, x^2 will form a basis for this vector space. In general, if this is some n then the set of all polynomials of degree less than or equal to n will form a vector space. And the set of functions 1, x, x^2 up to x^n will form a basis for this vector space and this basis is called the monomial basis.

Now instead of applying this idea, that is this condition on some general polynomial of degree less than equal to 2, you can apply this condition on each member of the monomial basis. That is the idea, remember if you apply this condition to each of this polynomials, you will get a linear equation for each polynomial.

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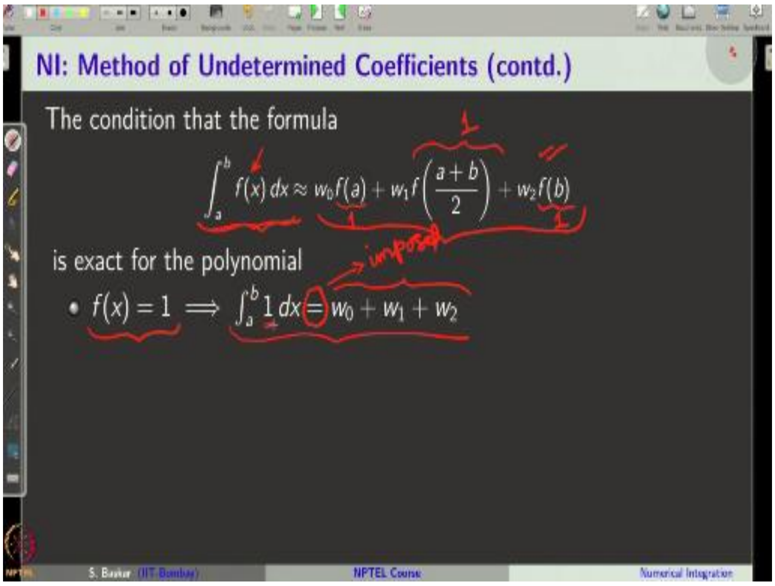

So that is the way to get a system of linear equations. Now, let us see how to do that. Remember, this is the integral that we want to evaluate and this is the quadrature formula that we are proposing to approximate the value of this integral. In this what you have to do is, first take $f(x) = 1$. Remember, just take what happens when this integrand happens to be the constant function $f(x) = 1, \forall x$. In this case, as per our condition this quadrature formula should give exact value.

Therefore, this is the imposed condition under the method called method of undetermined coefficients. So, you are just imposing this condition. On the right hand side, you have to take $f(a)$ = 1 and $f\left(\frac{a+b}{2}\right)$ = 1 and similarly $f(b)$ = 1, because you are now considering your integrand to be the constant function, $f(x)$ = 1 for all *x*. That gives you the right hand side as $w_0 + w_1 + w_2$, and that should be equal to the left hand side integral which is integral *a* to *b*, now your function is just 1. Therefore, we are putting 1 d*x*.

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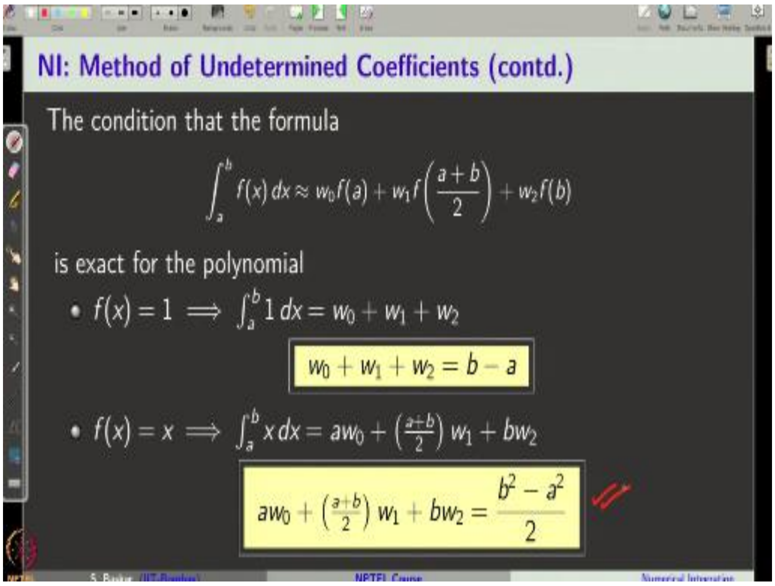

You can explicitly compute the left hand side and that gives you this equation. This is the first equation for our system. Now, you have to impose our second condition. What is the second condition? The second condition is that when the integrand happens to be the identity map that is $f(x) = x$ for all x then again it is a polynomial of degree less than equal to 2. Therefore, your integral value and the value obtained from this formula they should be exactly equal.

That is what you are imposing here the integral is $\int_a^b x\,dx$ and the right hand side formula now gives you this expression. They both should be exactly equal that is what the condition, that we are demanding. And again, that gives us this equation. You can observe that these 2 are equations with unknowns as w_0, w_1 and w_2 and you can also see these are linear in w.

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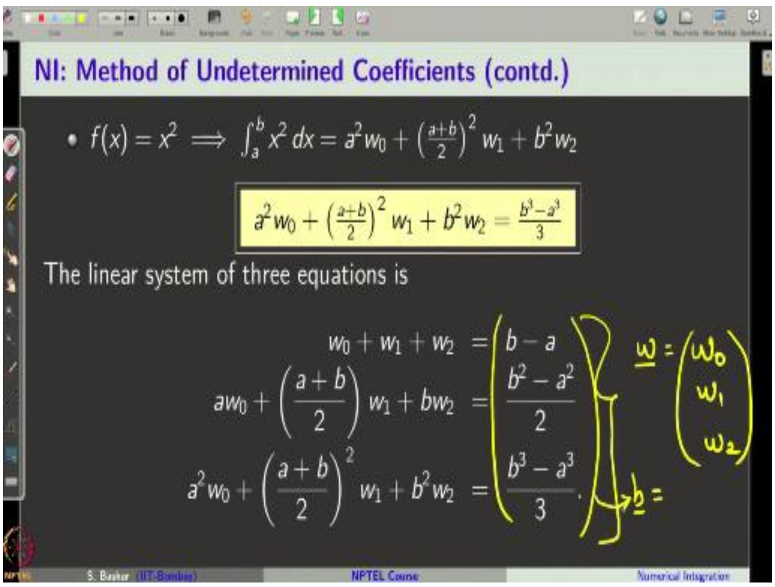

And finally, you have to impose the condition that $f(x) = x^2$, then what happens? Again, you will have $\int_a^f x^2 dx$, is exactly equal to the right hand side expression where you have to put $f(x) = x^2$ there. That is why ,you have a square and this and this. And again, you evaluate this integral explicitly and that gives you another equation. Similarly, if n = 3 then you have to impose this condition with 1, x, x^2 and x^3.

And similarly, for any given n, you have to accordingly take the monomial basis of that vector space, all polynomials of degree less than equal to whatever n that we give. And then you have to impose these conditions one by one, that is, $f(x) = 1$, $f(x) = x$, $f(x) = x^2$ and so on $f(x) = x^3$, you have to put and so on. Whatever n you are given up to that you have to put each case will give you a linear equation. And thereby you will have a linear system of equations. In the present case we have taken n = 2.

Therefore, our linear system will have 3 equations which we have just now derived. And these 3 equations are given like this you see that the unknown vector in this is equal to w_0, w_1 and w_2 and what is the right hand side vector b and that is given like this. So, that is the right hand side vector b. Now you can solve this linear system to get the weights w_0, w_1 and w_2.

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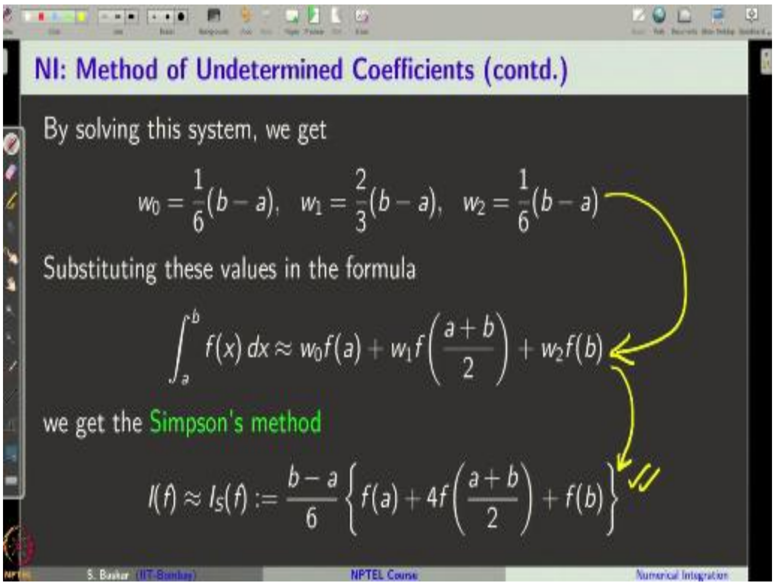

And that is not very difficult, you can explicitly solve it and your weights are given like this and if you recall, that will lead to this formula by imposing these weights into this expression. You get this formula and this is nothing but the Simpson's rule. In fact, method of undetermined coefficient will lead to a quadrature rule which is exactly the same as the corresponding quadrature rule derived by integrating the corresponding interpolating polynomial of the function f.

So, what I am trying to say is, these 2 methods, that is the method of undetermined coefficients and the method by directly integrating the interpolating polynomials, these 2 methods will finally lead to the same quadrature formula, that is all. Only the way we derive these weights are different. In the interpolating polynomial case, we are integrating the Lagrange polynomials and getting these weights.

Whereas, in the method of undetermined coefficients, we are imposing the condition that the quadrature rule is exact for polynomials of degree less than or equal to n. Whatever the n that we choose and with that we are applying this condition on the corresponding monomial basis and we are getting a linear system out of that. And that linear system will give you the weights again that is the method of undetermined coefficients. Whatever may be the method, finally the quadrature rule will be the same, that is the idea.

We will see why we have introduced this method of undetermined coefficients, when we know that the final expression that is the formula is going to be the same as we did with the interpolating polynomials. But there is a major advantage with this method of undetermined coefficients. We will discuss this in the next class thank you for your attention.