Numerical Analysis
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Lecture - 48
Numerical Integration - Rectangle Rule

Hi, in this class we will start a new chapter on numerical integration and differentiation. We will develop some numerical methods to find approximate value of an integral, which is also called quadrature formulas, and we will also develop some methods to approximate derivatives of a given function, which are called finite difference formulas. In this lecture in particular we will develop rectangle rule and trapezoidal rule for approximating an integral of a given function.

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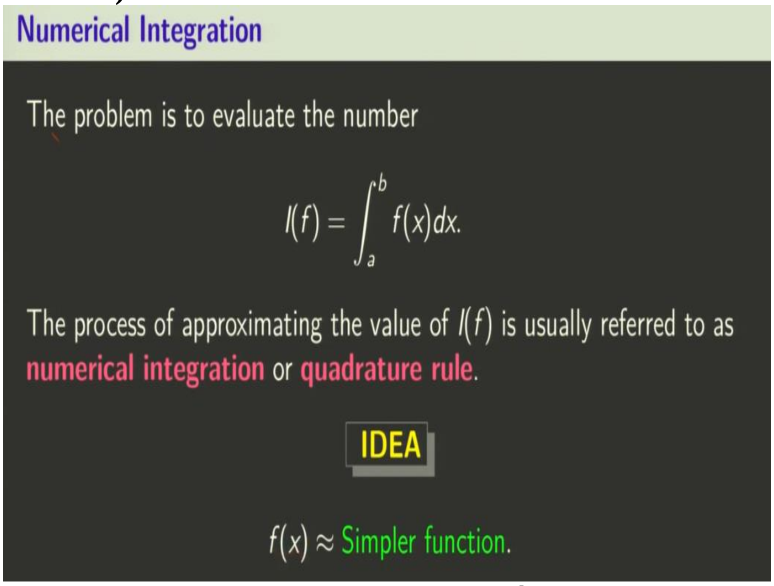

Therefore, our problem in this section is to evaluate $\int_a^b f(x)dx$ for some given function f defined on the interval $[a, b]$. The process of approximating the value of the integral is usually referred to as numerical integration or quadrature formula. We will use the notation $I(f)$ to denote the exact value of the integral $\int_a^b f(x)dx$. The idea is to first look for a simpler function that can approximately represent the given function f.

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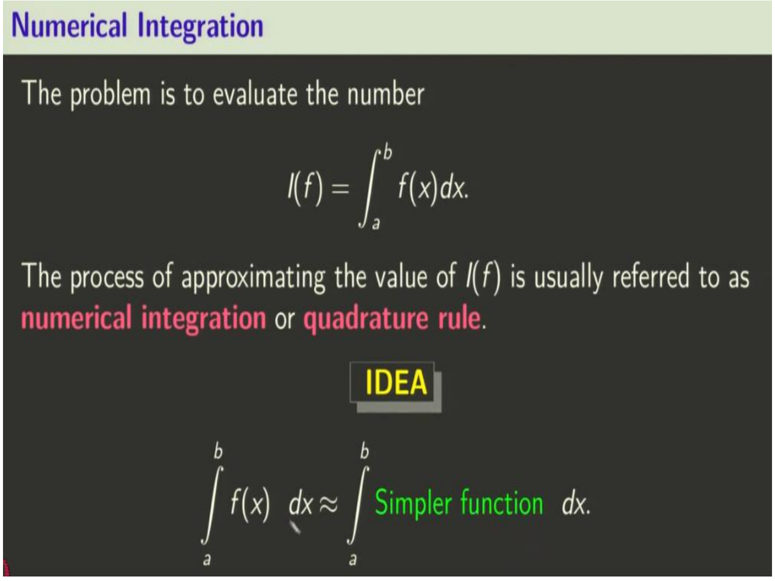

And then we want to find the integral of that simple function and consider the value of this integral as an approximation to our original integral. So that is the broad idea of developing numerical methods for integration or quadrature formulas. Now, the question is what kind of functions that we will consider as an approximation to our given function?

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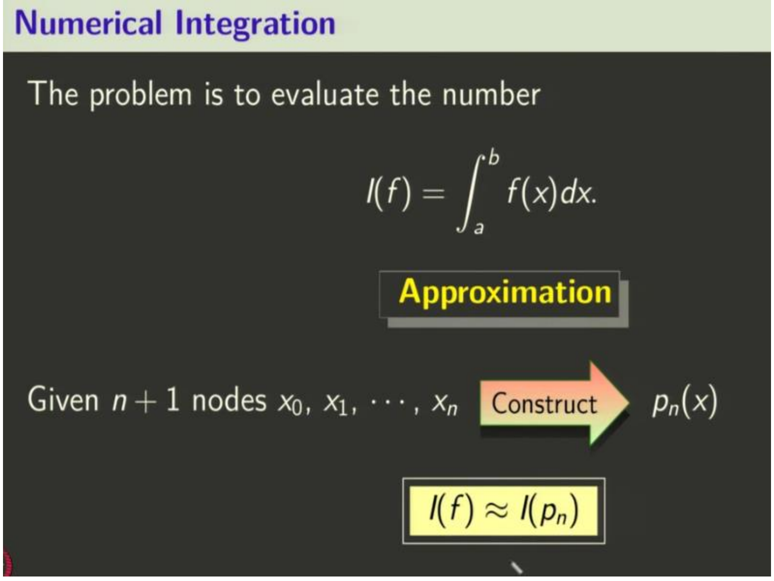

Of course, we already studied such approximations through polynomial interpolations. If you recall, we are given $n + 1$ nodes, $x_0, x_1, \cdots, x_n$, with that we will first construct a polynomial that interpolates the function f at these nodes. Once you have this interpolating polynomial, then you can go for finding the integral of this polynomial and consider that as the approximate value to your original integral.

So that is the idea, once you have this idea then you have scope to develop many methods. For instance, you can give a value of n and then you give nodes, you get a polynomial. Different value of n gives different polynomials and also for a fixed n, if you choose different nodes that also can lead to different interpolating polynomial, for your given function. Each such choices can lead to a quadrature formula. In this way you can find many different quadrature formulas. Here, we will try to develop few numerical integration formulas. Let us see how this integral will look like.

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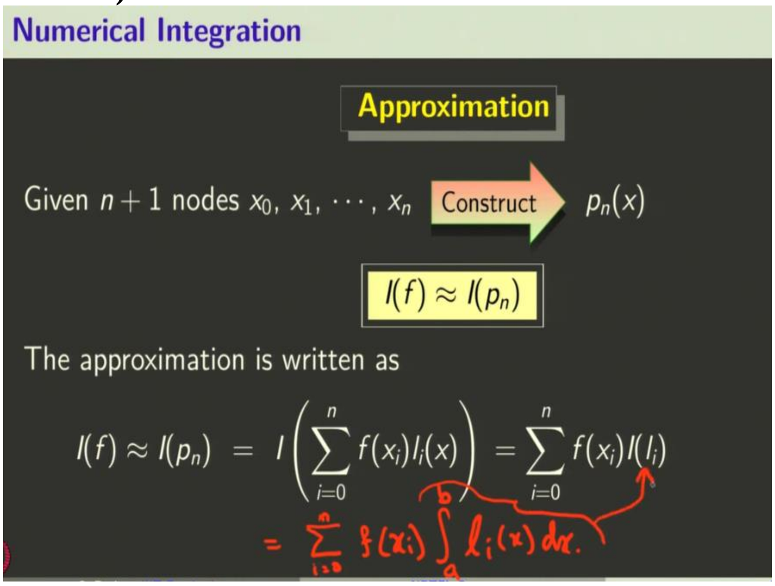

When you go to integrate the polynomial, if you recall $p_n(x)$ can be written in this form. If you recall, I am writing it in the Lagrange form and now I will integrate this polynomial on the interval $[a, b]$ and that can be written as $\sum_{i=0}^{n} f(x)$ into integral of the Lagrange polynomials. Why it is so? We are just taking the integral inside the summation $\sum_{i=0}^{n} f(x_i) \int_a^b l_i(x) dx$. So, this is what we are denoting by $I(l_i)$ here.

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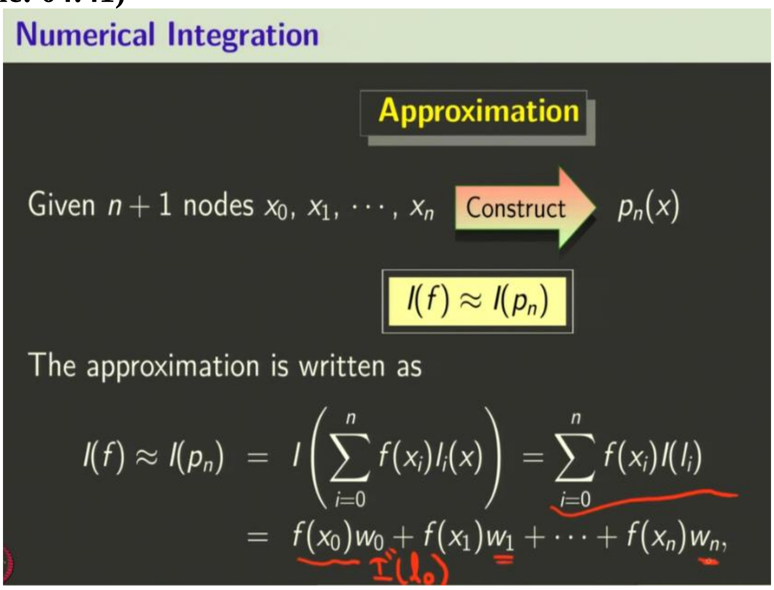

Now, this can be written as $f(x_0)$ and then integral of the Lagrange polynomial at 0 is what we call as w_0, and similarly w_1, and so on up to w_n can be defined.

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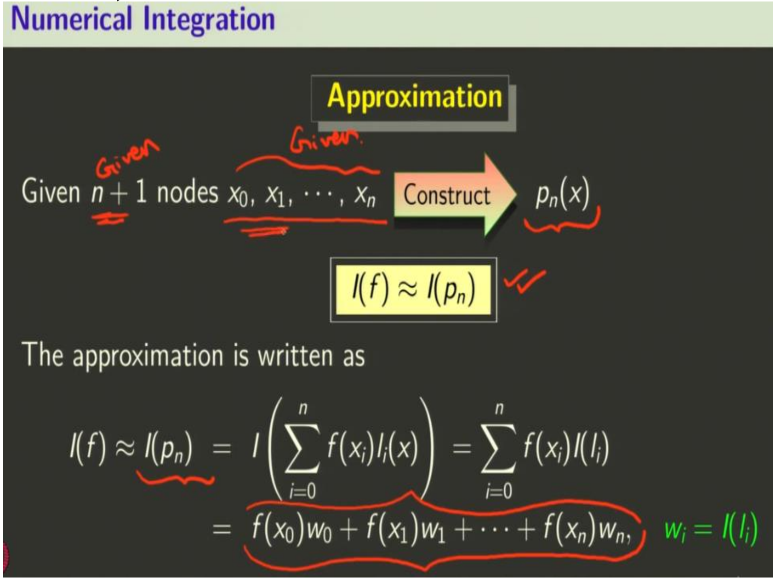

As w_i is equal to the integral of the corresponding Lagrange polynomials. So this is how a general formula for the numerical integration will look like. If you go with this idea, that is you first fix a n and also fix $n + 1$ nodes, so these are given to us. So, this is given to us and all these are given to us. With this, you will generate the interpolating polynomial for the function f and then you will integrate that polynomial and the formula finally will look like this. Let us take some specific values of n and some specific choice of the nodes and see how these formulas look like? Let us start with the simplest formula called rectangle rule.

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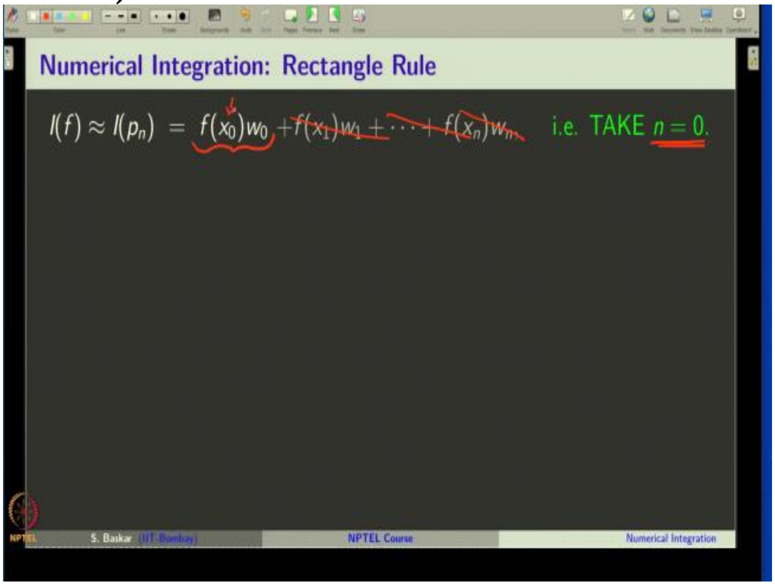

We call our general integration formula or the quadrature rule will look like this. In that we will take $n = 0$ and therefore our integration formula will be simply $f(x_0)w_0$. So, these are not there for us and now you can get different formulas for different choice of x_0.

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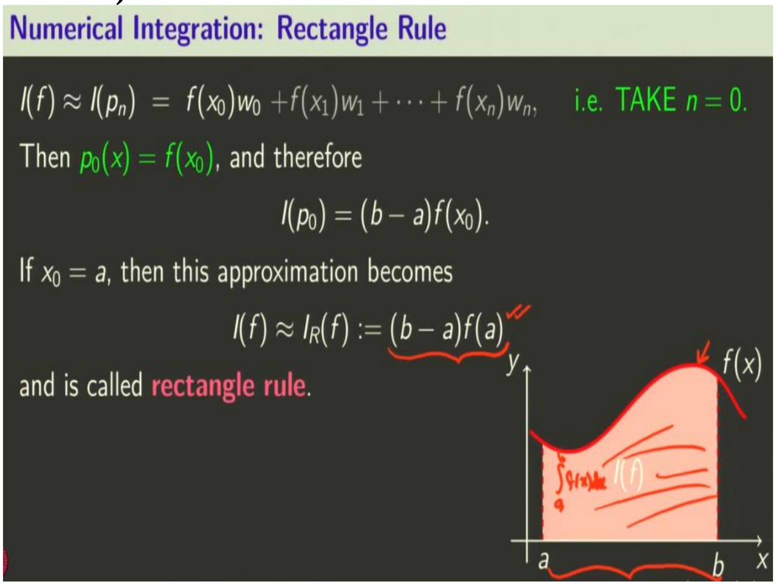

Now, let us go to integrate this formula. You can see that the integral of the polynomial p_0, that is the interpolating polynomial of degree 0 will be simply $(b - a)f(x_0)$. Here you can see the different choice of x_0 will lead to different methods with the choice of $n = 0$. Let us take $x_0 = a$, you can see that the corresponding formula will be $(b - a)f(a)$, and this is called the rectangle rule.

Let us see how this formula will look like geometrically. Let the red solid line indicates the graph of the function $f(x)$ and our interest is to find the integral $\int_a^b f(x)dx$. Geometrically, it is nothing but the area under the graph of the function $f(x)$ in the interval $[a, b]$. So that is what is shown in this shaded region. Therefore, $\int_a^b f(x)dx$ is what is given here.

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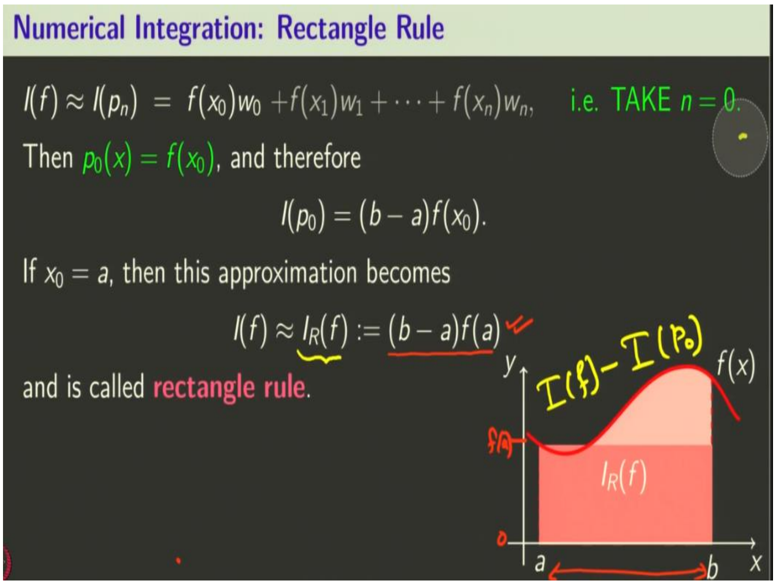

And now, let us see what this rectangle rule gives? The rectangle rule is nothing but the area of the rectangle with sides as a, b and 0 to $f(a)$. Therefore, the rectangle rule gives the area of this rectangle that is precisely given by $(b-a)f(a)$. So, our aim is to get this entire area but the rectangle rule is only giving us this much of area. Now, what is the error involved in the rectangle rule? You can see that there are some regions which are newly included in our area and this region is excluded in our area.

Therefore, the error involved in the approximation of the integral by the rectangle rule is going to take care of this region and then a small region which is included here. Mathematically this is $I(f) - I(p_0)$ and that is what is going to be the error involved in the rectangle rule. We use a special notation for the rectangle rule and that is $I_R(f)$.

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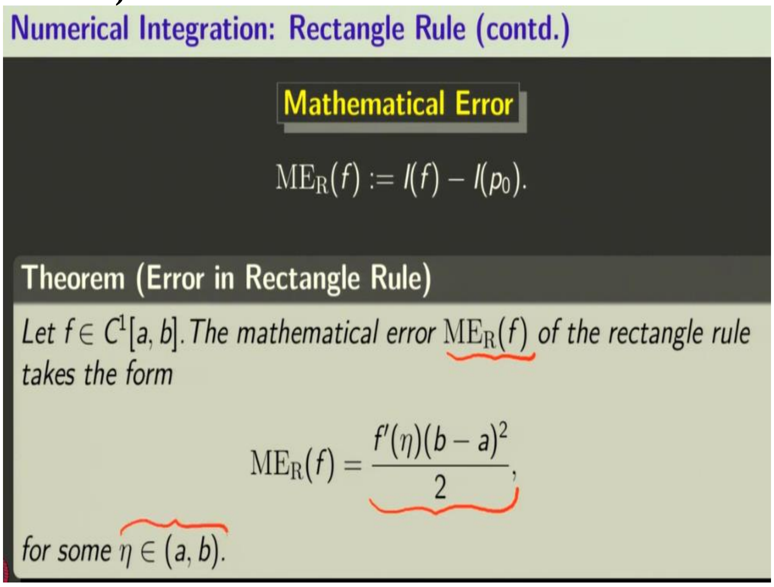

Therefore, the error involved in the rectangle rule is the difference between the exact value and the approximate value and we call this as mathematical error. We can derive an expression for the mathematical error involved in the rectangle rule. Let us state it as a theorem. For this, we need our function f to be continuously differentiable function defined on the interval $[a, b]$. Then the mathematical error involved in the rectangle rule which is denoted by $ME_R(f)$ can be written in the form $\frac{f'(\eta)(b-a)^2}{2}$, for some η in the interval $[a, b]$. Let us see how to prove this theorem.

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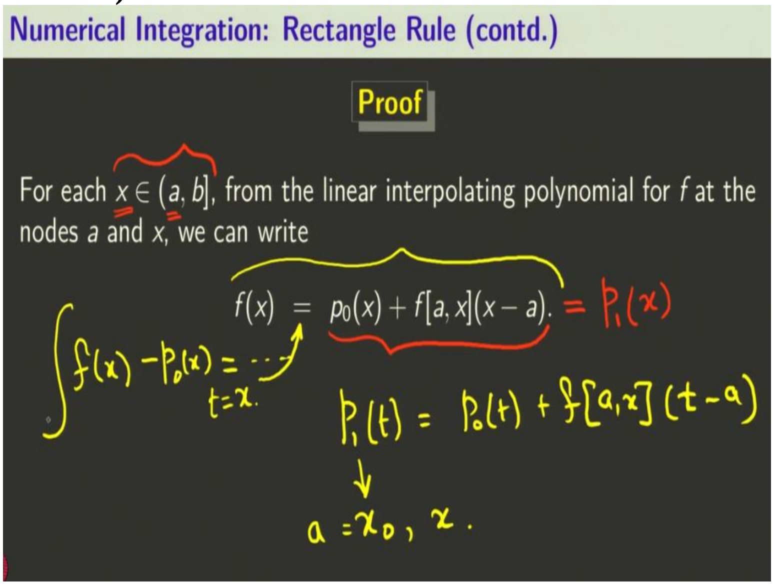

For this, you first take a point x which is not equal to a. Why we are specifically excluding a? Because, a is already included in the formula as the node of the interpolating polynomial. Precisely, we have to take x to b, different from the node that is used in the quadrature rule. Here, the node is a, therefore we are excluding that and taking any x in the interval $(a, b]$ and consider the polynomial $p_0(x) + f[a, x](x - a)$. If you recall, this is nothing but $p_1(x)$.

And if you recall, $p_1(t)$ can be written as $p_0(t)f[a, x](t - a)$, where the polynomial $p_1(t)$ is the interpolating polynomial of f at the node points x_0, which is actually a in our case, and x. Now you put $t = x$ by the interpolating condition. You can see that $p_1(x) = f(x)$. So that is the idea, why we have written $f(x)$ in this form. Why we are interested in this form, because from here you can write $f(x) - p_0(x)$ equal to something and then you take the integral you will precisely get your mathematical error.

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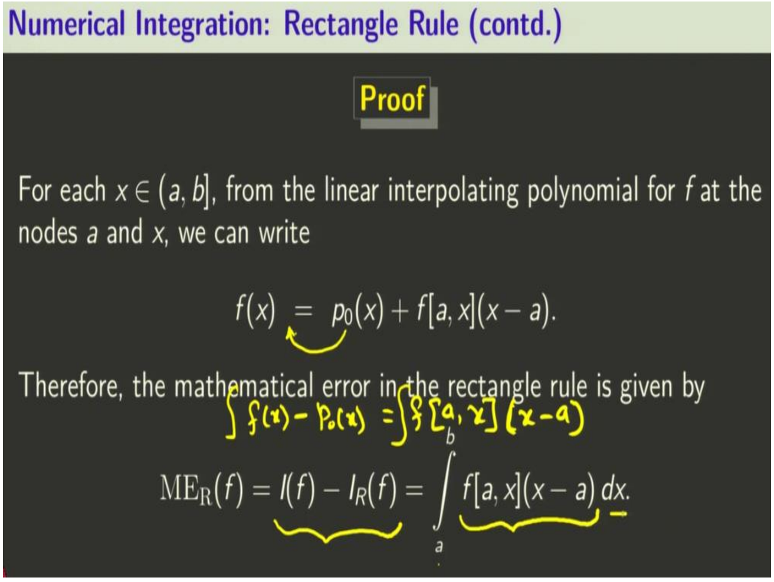

So that is the idea, so now what you do is, you bring this p_0 to the left hand side and $f(x) - p_0(x) = f[a,x](x-a)$ and then you take the integration on both sides you will get the mathematical error on the left hand side equal to $\int_a^b f[a,x](x-a)dx$. Now, let us see how to rewrite this integral in order to get our formula for the mathematical error.

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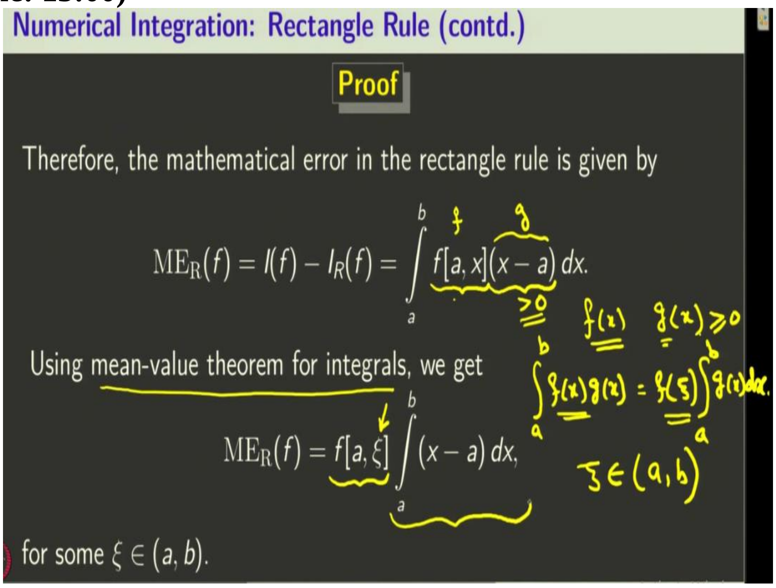

For that, we will use the mean value theorem for integration. If you recall, you have 2 functions $f(x)$ and $g(x)$ and if you know that $g \geq 0$ or equivalently you can also take $g \leq 0$, then you can write $\int_a^b f(x)g(x)\,dx$ as $f(\xi)\int_a^b g(x)dx$. So, you can bring the function f outside the integral by appropriately choosing the ξ, where ξ is some number lying between a and b.

So, that is what the mean value theorem for integration says. Now you can see that $(x-a) > 0$, because we have chosen x in the interval $(a,b]$ and a is excluded in the interval. Therefore, this is greater than or equal to 0 so you can just put the divided difference of f at the nodes a and x in the place of f in the theorem, and put $(x-a)$ in the place of g in the theorem, you can see that the first term of the integrand will come out of the integral with the unknown ξ and then you are left out with only this integral which can be easily integrated.

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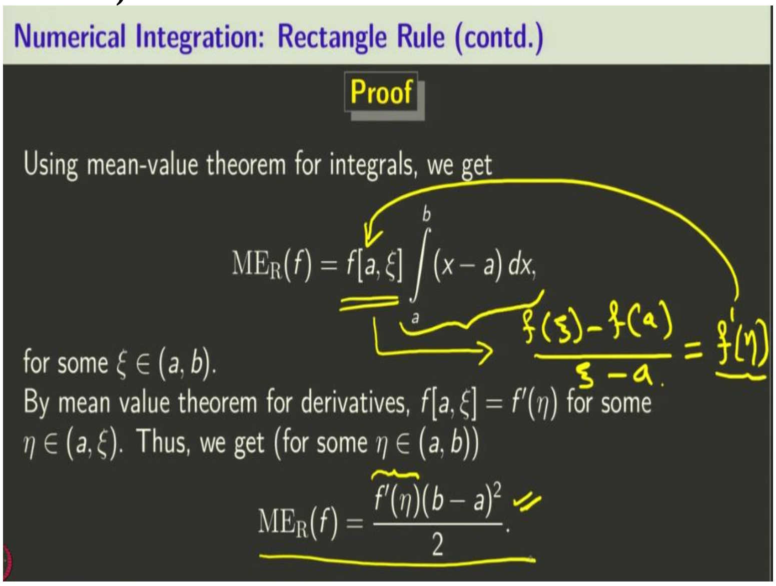

Now again, you can see by the definition of the divided difference this is nothing but $\frac{f(\xi)-f(a)}{\xi-a}$. Again, you can put the mean value theorem for differentiation and you can say that there exists an η such that this is equal to $f'(\eta)$. So that is precisely the mean value theorem for derivatives and I am just putting this in this place to get my mathematical error equal to $f'(\eta)$ and then I am evaluating this integral and then writing it here as $\frac{(b-a)^2}{2}$.

So, this is what the mathematical error expression that we wanted to prove and we have achieved it, and this completes the proof of the theorem. So this is all about the rectangle rule. Now it is very simple for you to evaluate an approximate value of the integral through rectangle rule. You simply have to use this formula. Let us try to change the position of the node. If you recall, in the rectangle rule we have taken the node x_0 as a.

Now you can choose that x_0 as any point in the interval $[a,b]$. In particular, you can take b as the node, then you will get another method and here we will take x_0 equal to the midpoint of the interval $[a,b]$ and we will get an important formula called midpoint rule.

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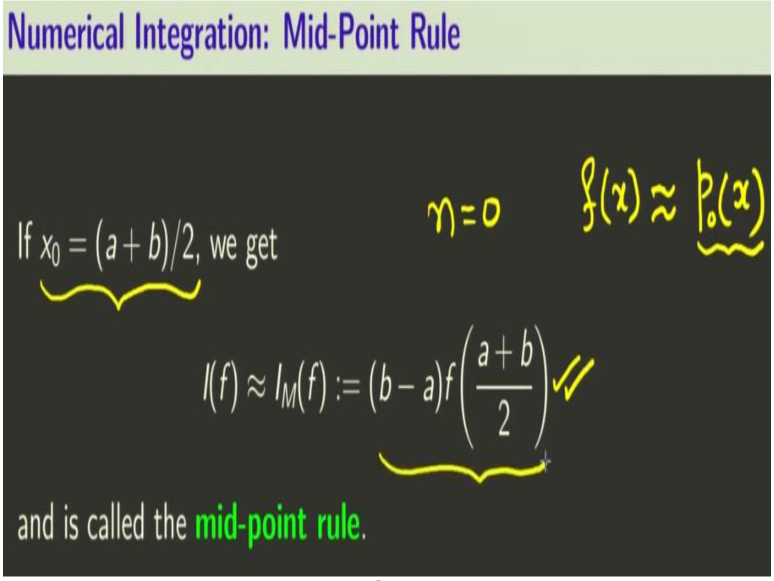

So in midpoint rule, we will only take $x_0 = \frac{a+b}{2}$, that is the only difference in the midpoint rule. Otherwise we are taking $n = 0$ and thereby we are approximating the function f by the interpolating polynomial of degree 0. But to construct p_0 in the rectangle rule we have taken the node as a, now we are taking the node as $\frac{a+b}{2}$. You can clearly see that choice of x_0 will give us this formula and this is called the midpoint rule.

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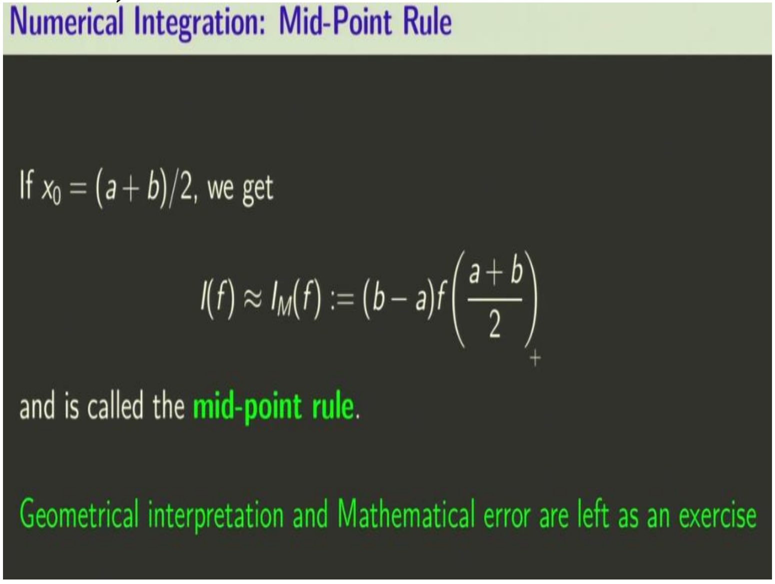

I leave it to you to see the geometrical interpretation of the midpoint rule and also, I leave it to you to derive the mathematical error involved in the midpoint rule.

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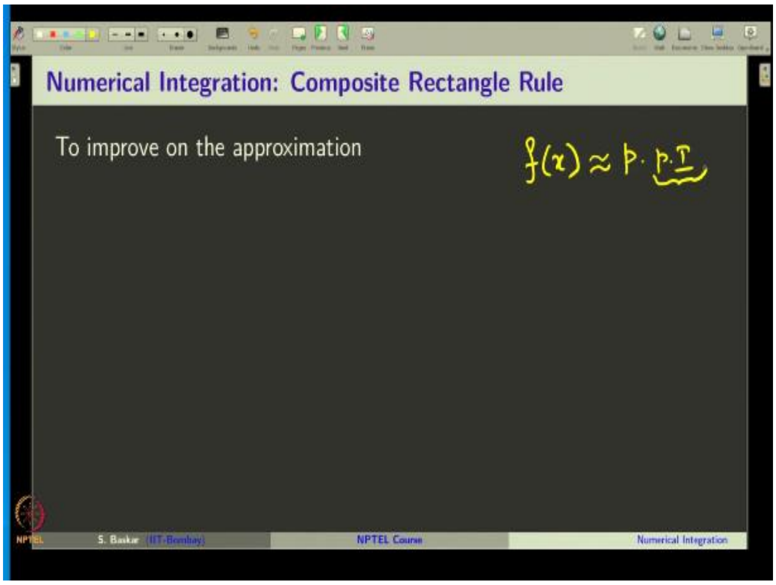

The next is a composite rectangle rule. What is the idea of composite rectangle rule? If you recall, when we were discussing polynomial interpolations, we have introduced piecewise polynomial interpolations. Piecewise polynomial interpolations have their own advantages. The same idea can be adopted in the numerical integration also. Instead of approximating the function $f(x)$ by one single polynomial in the interval $[a, b]$, you can approximate it by piecewise polynomials. Then you will get a composite rule of whatever interpolating polynomial that you use.

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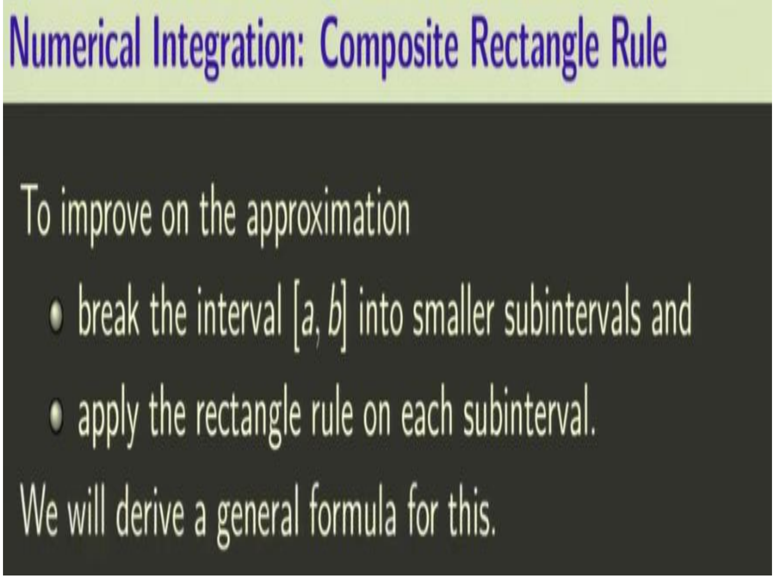

Let us see in the case of rectangle rule, how to derive the composite version of the rectangle rule. For that, you have to break the interval $[a, b]$ into smaller subintervals. Remember this is what we do in the piecewise linear or any piecewise interpolating polynomial. First we will subdivide the given interval into some n number of points and then in each piece, we will put an interpolating polynomial of certain degrees, so that is the idea.

Similarly, here also we will break the interval into smaller subintervals and then apply the rectangle rule on each subinterval. I hope the idea is clear, it is a very simple idea. Let us derive the formula in general.

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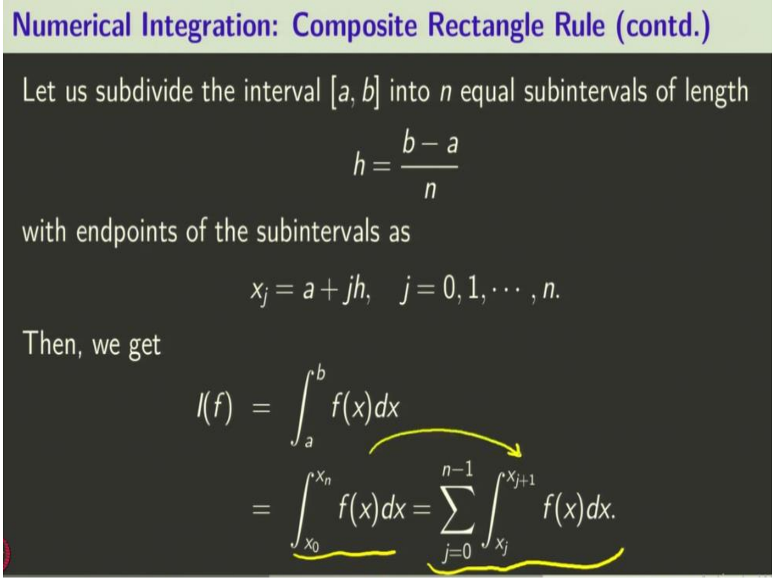

First, you have to break this interval into some *n* number of subintervals. For that, we will define our h, that is the length of each subinterval. For the sake of simplicity, we will take the subintervals with equal length as $h = \frac{b-a}{n}$ and then we will define the partition points that is the nodes $x_j = a + jh$, where $j = 0, 1, 2, \cdots, n$. These are the nodes, but now we will not put one interpolating polynomial in the entire interval.

But we will put polynomial of degree 0 in each subinterval and then we will integrate them. Now, what we will do is, we have this integral. This is what we want to evaluate now. We will break that interval into *n* pieces. For that, we will first write a, which is equal to x_0 and *b* is x_n., just for the convenience and then we will go to write this integral as the sum of the integrals over the subinterval. So, this is a very elementary property of the integration.

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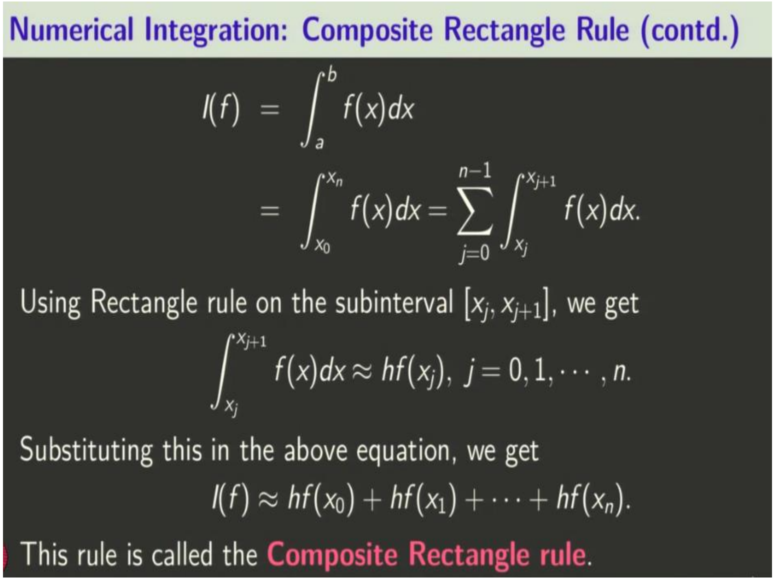

Now, what you do is, you go to put the rectangle rule on each of the subintervals that is you take each of this integral and put the rectangle rule. Remember, rectangle rule is $(b - a)$ here b is x_{j+1} and a is x_j, and the difference is precisely $hf(x_0)$ is what we have defined in the rectangle rule. Here, x_0 means the lower limit in the integral and you have to evaluate the function f at this lower limit and that is what is sitting here.

Therefore, each integral in this sum is approximated by this quantity and thereby, we have the formula like this and this is the composite rectangle rule. Next, we will see how to derive a quadrature formula using linear interpolating polynomial and we will also see how to derive quadrature formula with a quadratic interpolating polynomial. We will continue our discussion in the next class. Thank you for your attention.