Numerical Analysis
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Lecture – 47
Polynomial Interpolation – Tutorial Session

Hi, we have finished our discussion on polynomial interpolation. In this lecture we will solve some tutorial problems, especially these problems are important from the point of view of quadrature rules, which we will be studying in the next chapter.

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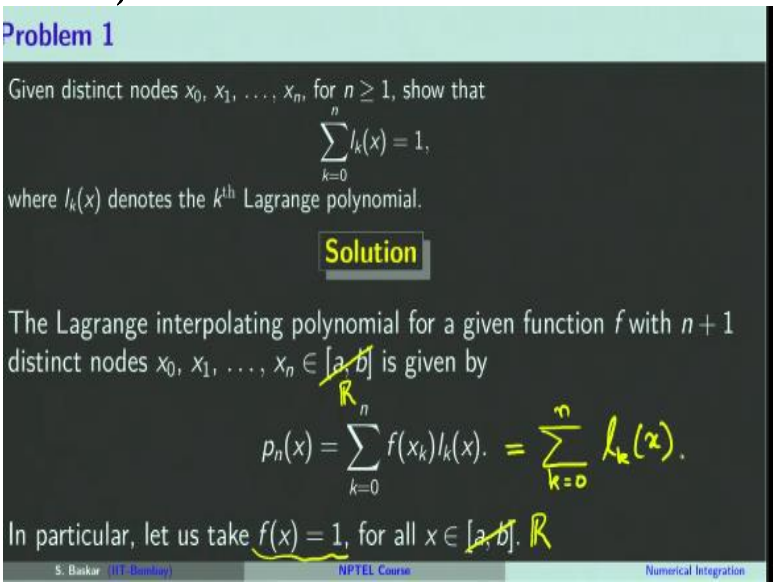

Let us consider this problem as our first problem. We are given n + 1 distinct nodes and we have the Lagrange polynomials, corresponding to each node. We can show that the sum of all this Lagrange polynomials at any point x in ℝ will take the value 1. So that is the problem, this is an interesting and also very important problem. Let us see how to show this, recall that once we are given a set of n + 1 nodes we can write the interpolating polynomial of a given function f at these n + 1 nodes.

And it is given by this formula, it is written in the form of the Lagrange interpolating polynomial. So, in this what you do is, you just want to have this expression. In the polynomial repress,,entation you can see that you have the representation with the coefficients as $f(x_k)$. So, if you can make this term to be equal to 1 then at least the left hand side in our question is precisely what is given here, with $f(x_k) = 1$. For that reason what we will do is, in particular we will take the function $f(x)$ = 1 for all $x \in [a, b]$.

In fact, you do not need to restrict yourself to any interval $[a, b]$. We can in fact generalize it to any $\mathbb{R}$ here, even here you can take any nodes in $\mathbb{R}$, it does not matter, then you can see that for this particular function this expression is written, as $\sum_{k=0}^{n} l_k(x)$. Now what happens to the left hand side?

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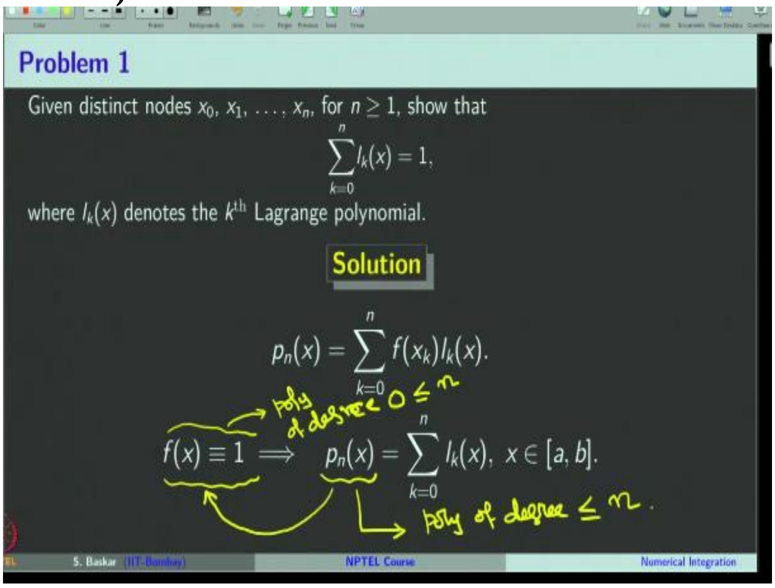

Let us see the left hand side is the polynomial interpolating the function $f(x) = 1$ and what is this function $f(x) = 1$ this is precisely the polynomial of degree 0, which is a particular case of a polynomial of degree less than or equal to n and what is this polynomial? It is a polynomial of degree less than or equal to n again. And it is interpolating the function f which is again a polynomial of degree less than equal to n.

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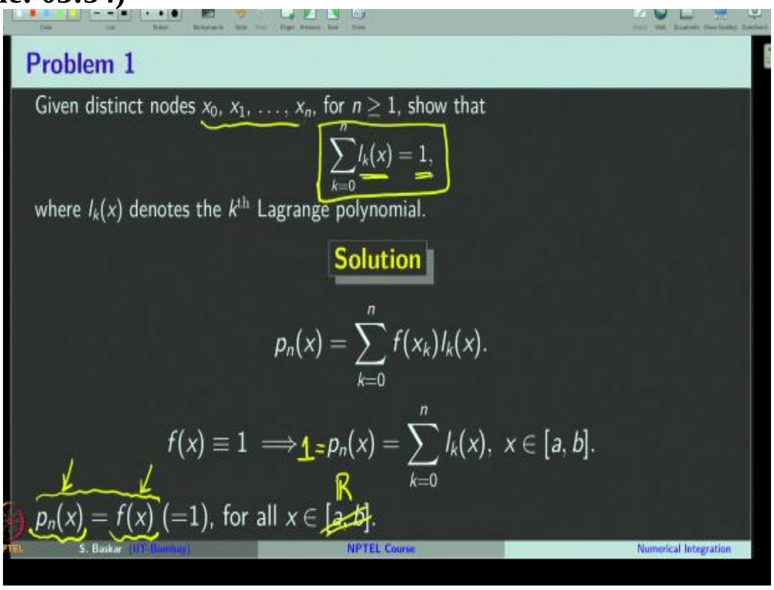

Therefore, by uniqueness we can see that the interpolating polynomial $p_n(x) = f(x)$ because this is also a polynomial of degree less than equal to n and this is also a polynomial of degree less than or equal to n. Therefore, by uniqueness they have to coincide. If you recall in our very first lecture on interpolating polynomials, we have proved a theorem on existence and uniqueness.

From that we can see that $p_n(x)$ is a unique interpolating polynomial at the given nodes for the given function *f*. Now in this particular case, *f* also happens to be a polynomial of degree less than equal to n. Therefore, $p_n(x)$ will coincide with $f(x)$ for all *x*, in fact you can say for all $x \in \mathbb{R}$ and thus you can see that the left hand side is precisely equal to 1 this is what we want to show in our problem. Therefore, you choose any $n + 1$ nodes and if you sum all the corresponding Lagrange polynomials at any real number *x* then you will land up by getting the value 1 only. So that is an interesting and also important result.

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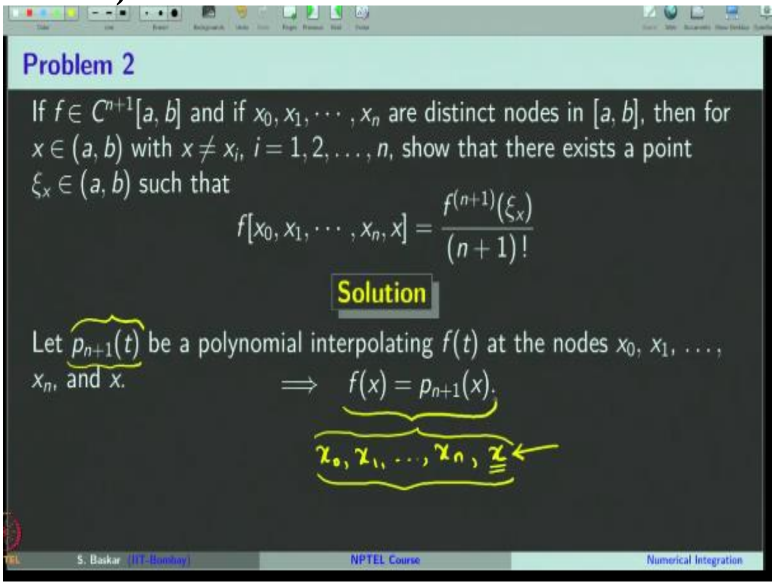

Let us pause on to our next problem. Suppose you have a function *f* which is a C^{n+1} function that is the function *f* is $n + 1$ times continuously differentiable on an interval $[a, b]$ and we are also given $n + 1$ nodes in the interval $[a, b]$ and they are distinct. Now what we are doing is, we are choosing one point *x* in the open interval $[a, b]$ which is different from the nodes that you have already chosen. Therefore, you now have $n + 1$ distinct nodes, plus a real number *x* in the interval $[a, b]$ which is different from all these nodes.

Therefore, you have $n + 2$ distinct points in the interval $[a, b]$. Now we want to show that if you make the divided difference of the function f at these $n + 2$ nodes, then you can find a ξ corresponding to that *x*. That is why we have the notation ξ_x such that $\frac{f^{(n+1)}(\xi_x)}{(n+1)!}$ is precisely the divided difference of f at these n + 2 nodes. So this is what we want to show, let us see how to show this result.

Remember, we have now given n + 1 nodes $x_0, x_1, \cdots, x_n$ and then we have added one more point x into our data set, by that we have n + 2 distinct nodes. And, therefore we can construct a polynomial interpolating the function f at these n + 2 nodes and it will be a polynomial of degree less than or equal to n + 1. Now since this polynomial is interpolating the function f with respect to these nodes, remember the nodes also include the point x. Therefore, the interpolation with respect to x as a node will tell us that $f(x) = p_{n+1}(x)$. Let us keep this in mind and go ahead and see what happens.

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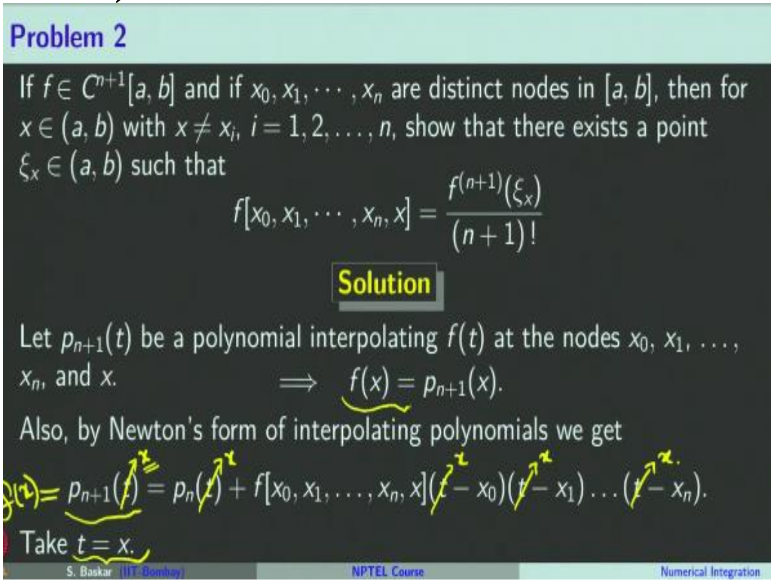

Recall that p_{n+1} can be written in the Newton's form of interpolating polynomial and it is given by the polynomial of degree less than or equal to n plus this extra term. So, this is how Newton's form of interpolating polynomial is written. It is written as the one lower order polynomial plus an extra term and that extra term is precisely given like this and since I am already using the notation x as one of the nodes, so I have used t as the variable in this polynomial. So just keep in mind that t is the variable but x is one of the fixed nodes. So, this is notationally little bit different from what we have used in our chapter throughout this is because we have fixed x as one of the nodes in this interpolating polynomial. Now, what you do is, take $t = x$ and see what happens. If you take $t = x$, you will have x here, x here and all this will be x, but we know that the point x is one of the nodes used in constructing $p_{n+1}(t)$. Therefore, this is in fact is equal to $f(x)$ that is what we have already observed in our first step.

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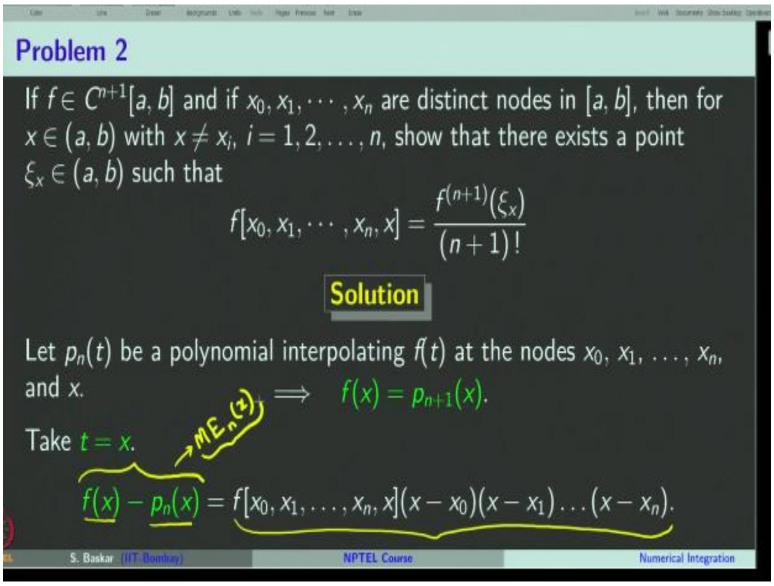

Therefore, you can put this value here and get $f(x) = p_n(x)$ plus this extra term where all this t's in our previous expression now is replaced by x. Now, I will take this to the left hand side and write $f(x) - p_n(x)$ equal to the remaining term on the right hand side. Now you can try to see what this is. You can see that this is nothing but the mathematical error in the interpolating polynomial p_n evaluated at the point x. In one of our previous lectures, we have got an expression for this mathematical error, what is that?

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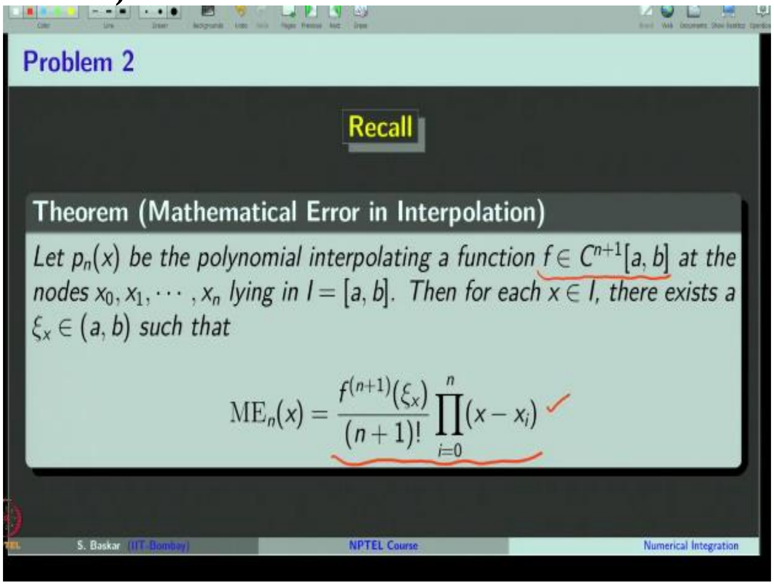

Let us recall that the expression for the mathematical error is given by this, of course we have to assume that f is a C^1 function in order to use this formula for the mathematical error.

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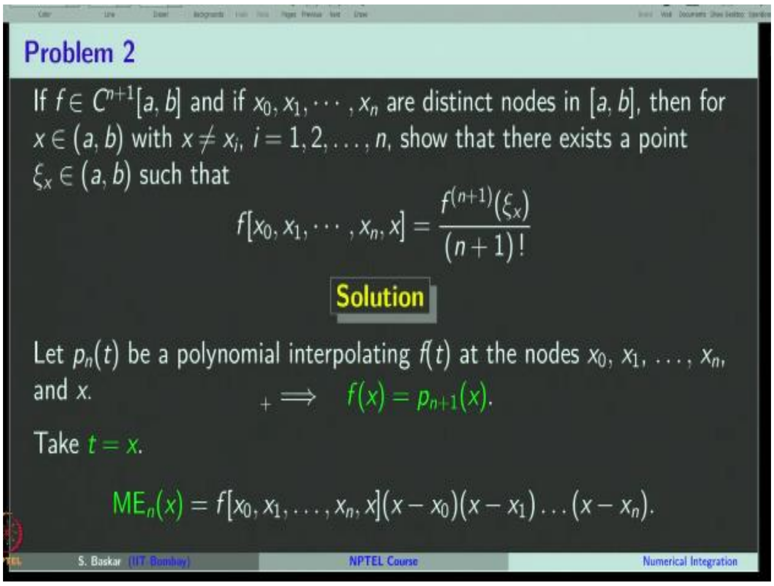

If you recall we have already put the assumption in our problem. Therefore, we can use this theorem and replace the mathematical error on the left hand side by this expression, from our theorem, which we have already proved in our theory class.

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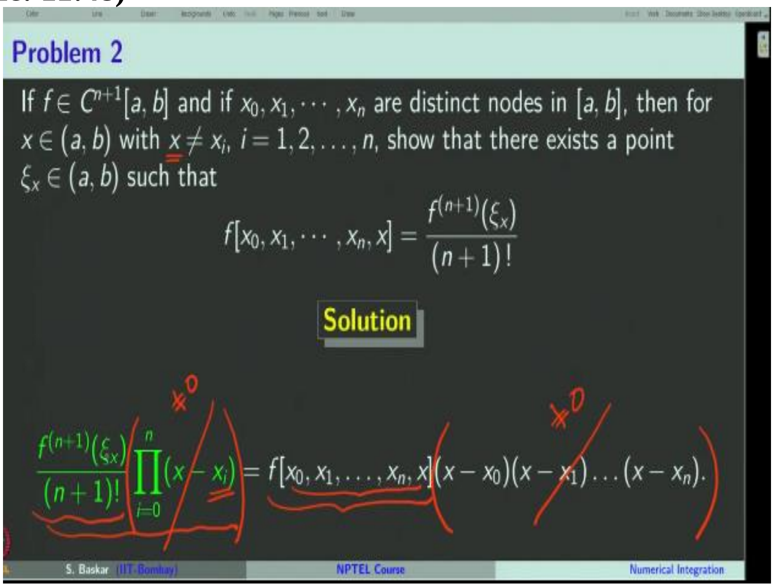

Therefore, I will remove the mathematical error on the left hand side from my previous expression and plug in the expression derived in that theorem and that gives me this equation. Now you can see that this term is, precisely this term on the right hand side therefore you can cancel these two, why? Because we have chosen our x in such a way that x is not equal to any of this node that we have considered.

Therefore, this term is not equal to 0 and similarly this is also not equal to 0. That is why we are canceling and concluding that the divided difference of f evaluated at these $n + 2$ nodes is precisely equal to this expression $\frac{f^{(n+1)}(\xi_x)}{(n+1)!}$. This is also an important result, we often use this in proving the errors for certain quadrature formulas in our next chapter. Therefore, we have to remember this result with this let us pause on to the next problem.

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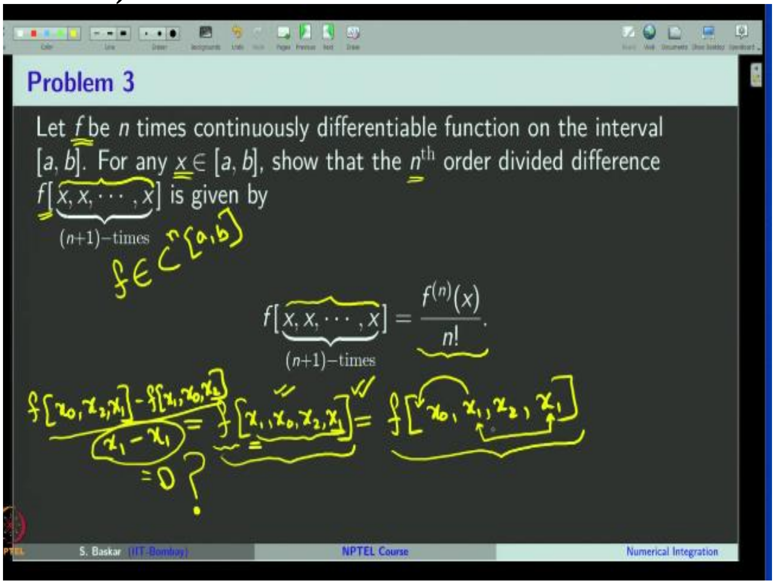

In this problem we again have *f* to be *n* times continuously differentiable function on the interval $[a, b]$ that is to say, that $f \in C^n[a, b]$. Now for any *x* in the interval $[a, b]$, we want to show that the *n*th order divided difference *f*, but here now we have a slightly new situation, that all the arguments in the divided difference are same and it is given by $\frac{f^{(n)}(x)}{n!}$.

So, this is something new to us because so far whatever problem we worked we always assumed that the nodes are distinct, but here we have all the nodes have same value *x*. Suppose you have node say x_0, x_1, x_2 and x_1 again, then you can see that there are 2 nodes which are repeated. Now, can we say that the divided difference for this set of nodes where 2 nodes are repeated is well defined, that is the question.

Because there is a serious problem here. You may write this as $f[x_1, x_0, x_2, x_1]$. If you recall, we have proved that the divided difference formula is symmetric with respect to any permutation of the nodes. Therefore, I can shift x_1 to this position and write the nodes in this order and then apply the divided difference. The value that I obtained from here, will be the same as the value obtained from here also. This is what the symmetric property of the divided difference tells us.

Now let me write the formula for the divided difference with the nodes arranged in this way. How will you write? That it is precisely $\frac{f[x_0,x_2,x_1]-f[x_1,x_0,x_2]}{x-1-x_1}$, that is the last one minus the first one. Now you can see that the denominator = 0 and that gives us a big question whether this divided difference is well defined or not. Therefore, any nodes are repeated then we have a serious question of whether the divided difference corresponding to that set of nodes is well defined or not.

So far, we never encountered this problem because we always assume that the nodes are distinct and in fact while constructing in interpolating polynomials, we actually need distinct nodes. We do not need to consider any 2 nodes or more than 2 nodes repeated that will not make sense as far as the construction of the interpolating polynomials are concerned. But such situations will occur when we were trying to find the errors for quadrature formulas in the next chapter. Therefore, we have to also carefully understand how this divided difference will be defined when 2 or more nodes are repeated.

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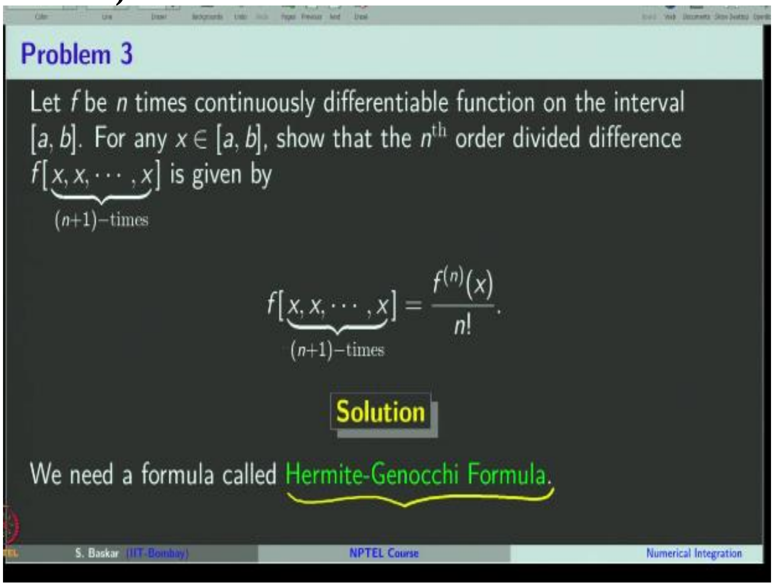

So, this can be done using a formula called Hermite-Genocchi formula. Once we understand the Hermite-Genocchi formula, we can say that the divided differences are in fact defined even if 2 or more nodes are repeated. So let us quickly understand how this Hermite-Genocchi formula is given for divided differences and then we will come back to this problem and try to prove this result.

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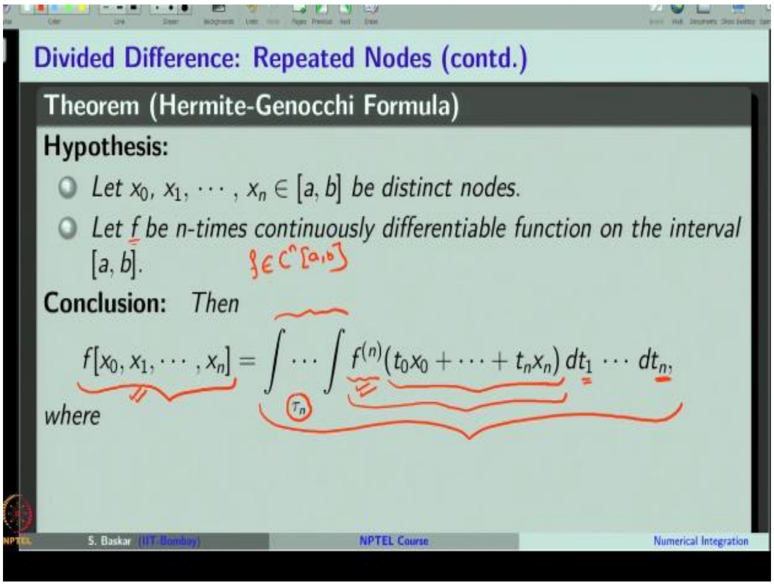

Let us take a small deviation and understand this Hermite-Genocchi formula and then we will come back to our problem. What Hermite-Genocchi formula says, suppose you have $n + 1$ distinct nodes. Remember we will take all these nodes to be distinct in order to prove this formula. Once you prove the formula, you can observe that formula even holds for repeated nodes that is the idea behind this.

Therefore. to derive the formula, you need to have distinct nodes. For that reason, we will assume that the nodes are distinct and f is a C^n function on the interval $[a, b]$. Once you have these 2 conditions, then your divided difference which you know how to obtain using the formula that we have so far used. Now, Genocchi formula says that the same divided difference can also be obtained using this formula which involves this multiple integral.

So here $t_1, t_2, \cdots, t_n$ are the variables used in this integral and they appear in the integrand like this and the integrand is the nth derivative of the function f that is why we have assumed that f is as C^n function in the interval $[a, b]$. Therefore, this makes sense and in fact the integrand is continuous and what is this set on which we are taking this integral.

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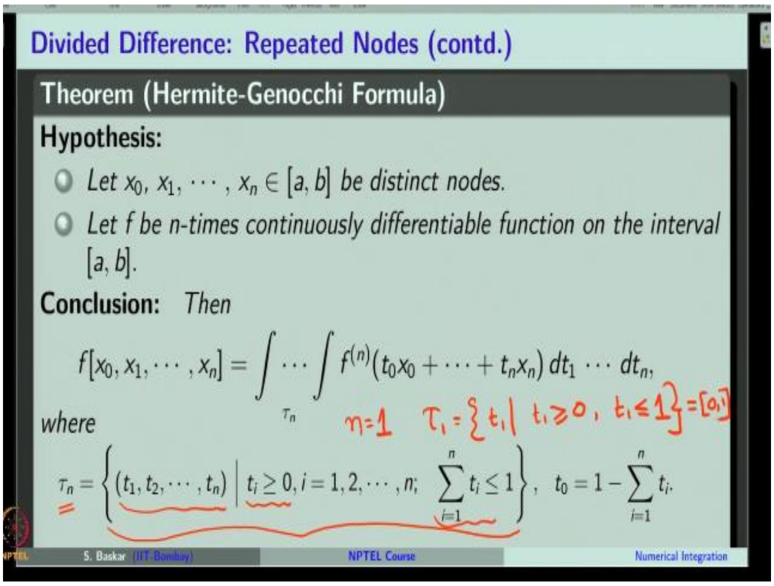

Let us try to understand this set. The set τ_n is given like this, it is the set of all *n* triples such that all t_i's are greater than or equal to 0 and they sum to some number which is less than or equal to 1. Let us try to understand how this set looks like. For instance if *n* = 1, you have τ_1 is equal to set of all, only one number will be there, t_1 such that $t_1 \geq 0$ and from the summation you can see that $t_1 \leq 1$. This is precisely the closed interval $[0,1]$.

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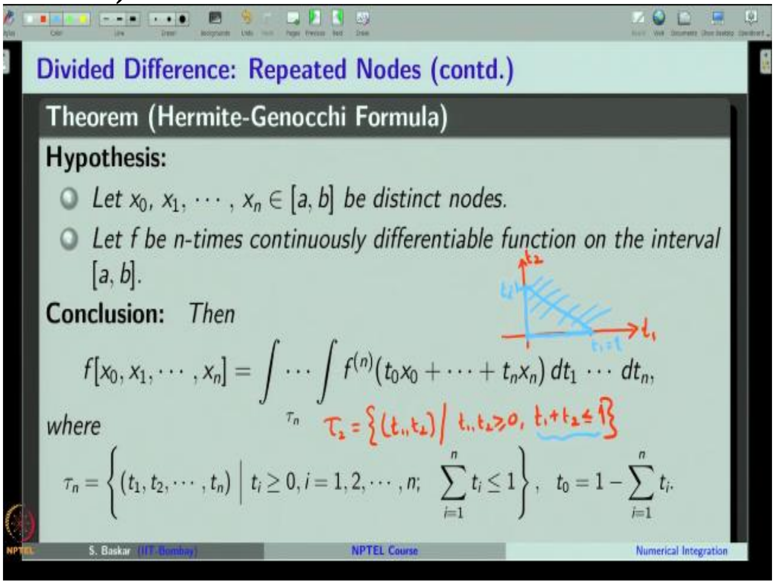

Let us see how this set looks like if *n* = 2. For *n* = 2, τ_2 is equal to the set of all triples (t_1, t_2) such that t_1 and t_2 both are positive and $t_1 + t_2 \leq 1$. How will that look like? Let us try to visualize it in the $t_1 t_2$ plane. So, both t_1 and t_2 are greater than or equal to 0. That tells us that we have to sit in the first quadrant of the plane $t_1 t_2$ and then you also have one more additional information that $t_1 + t_2 \leq 1$ that is, when t_1 = 0 you can have up to t_2 = 1.

Similarly, when t_2 = 0 you can at most have t_1 = 1. So, this set will be the set bounded by these 3 lines. So, this is the set τ_2 which is nothing but the region bounded by these lines. We now understood how $t_1, t_2, \cdots, t_n$ comes and the integral is also with respect to these variables. In addition to these *n* variables, we also have one more variable t_0 and that is given by this expression.

So, now I hope you understood the formula. The theorem says that the divided difference of order *n* which we know how to compute by this time, can also be obtained by this formula, that is what it says. Now look at this formula, you can clearly see that you do not have any problem, that you faced with the previous form of the formula. So, if you recall in the previous form of the formula the divided difference seems to be having problem when 2 or more nodes are repeated.

But in this form, that is in the Genocchi form, you can see that even if 2 or more nodes are repeated you still can have a clear meaning for this integral. So that is what the advantage of Hermite-Genocchi formula.

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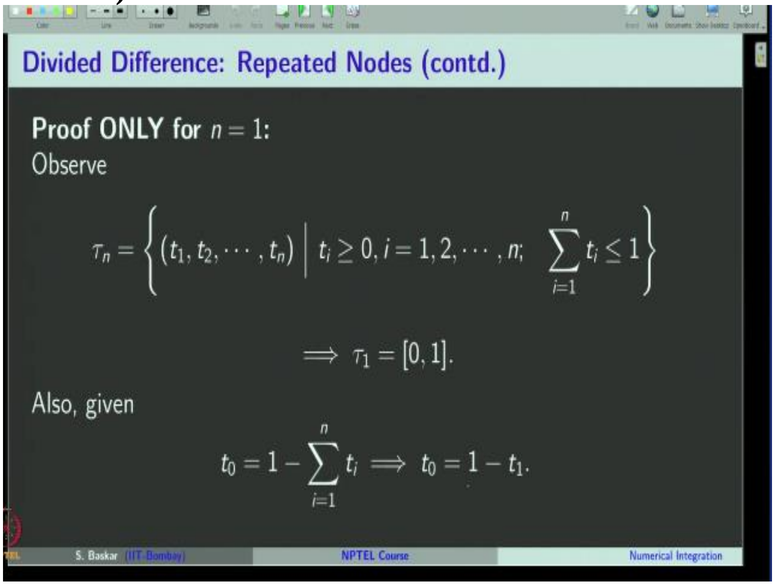

Let us not prove this formula in the general case but let us try to have a feeling of it by seeing how the proof goes when *n* = 1. Remember for *n* = 1, τ_1 is simply the closed interval $[0,1]$ and t_0 is given by $1 - t_1$. For *n* = 1, you can see that you have $f[x_0, x_1]$ and the right hand side is correspondingly written as, $\int_0^1 f'(x_0 t_0 + x_1 t_1) dt_1$. Let us take the right hand side and see how we can obtain the divided difference from this formula.

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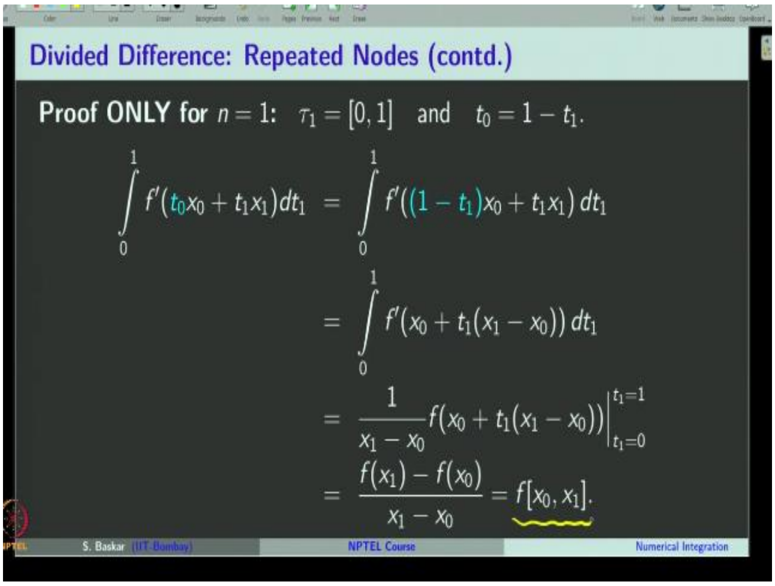

For that, let us take the right hand side with *n* = 1 and that is given by this and remember we have $t_0 = 1 - t_1$. Let us replace t_0 by $1 - t_1$ and now you can slightly adjust this argument and write it as x_0 plus you take t_1 and combine it with the second term and write $t_1(x - x_0)$ and now you are integrating it over the interval 0 to 1 and that can be directly integrated to get this expression. You are integrating with respect to t_1, remember that. Therefore, the integral can be written like this evaluated at these 2 points.

Now how will they come? Well, this limits precisely gives us $f(x_1 - x_0)$ and then we have divided by $x_1 - x_0$ and if you recall this is precisely the first order divided difference of the function *f* at the points x_0 and x_1. Therefore, we can easily prove the Genocchi formula for *n* = 1 and we have seen that this formula is precisely the divided difference when *n* = 1 you can similarly prove it for any *n* using an induction or argument, but we will skip this proof for our course.

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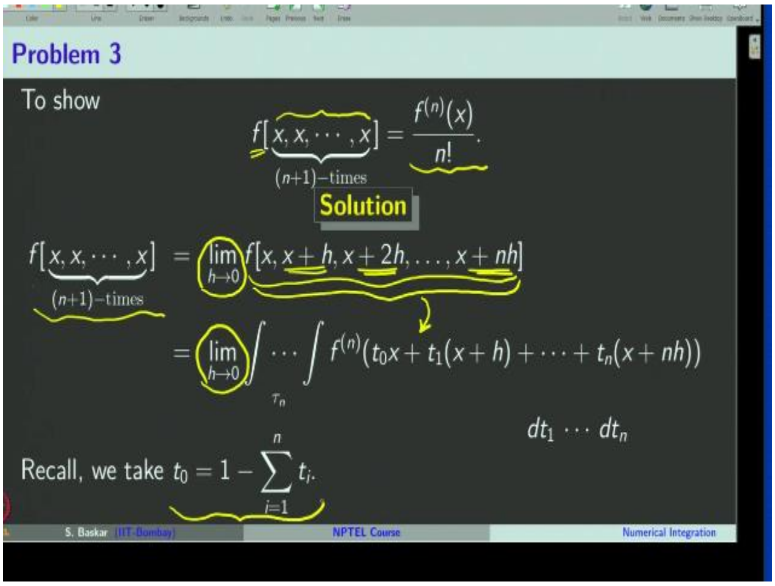

And we will now go back to our problem that we started with. Recall that, we want to prove that the divided difference of f of order n where all the nodes are repeated can be written in this form, so how to prove that. What you do is, you take the left hand side, and you rewrite like this. You add h to the second node, $2h$ to the third node and so on and then take limit h tends to 0, now you see you have all the distinct nodes.

Therefore you can go back to the Hermite-Genocchi formula, this can be written in the Hermite-Genocchi form with the $lim\, h \to 0$. Remember in the Hermite-Genocchi formula, we have $t_0 = 1 - \sum_{i=1}^{n} t_i$. So, we will just plug in this expression into t_0 and we can write it as $lim\, h \to 0$, this multiple integral of nth derivative of f evaluated at this point where t_0 is now replaced by this expression and this argument can be rewritten in this form. I leave it to you to see, it is a simple calculation.

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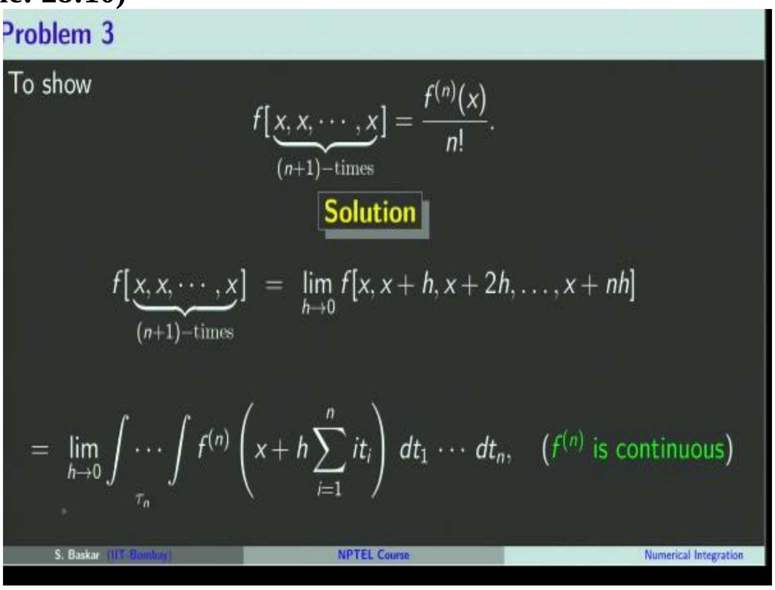

Once you rewrite it in this form, now what you do is, recall that we have assumed that f is a C^n function. It means the nth derivative of f is a continuous function and τ_n is a bounded set. Therefore, you can write this limit as $\int \cdots_{\tau_n} \int f^{(n)}(x) dt_1 \cdots dt_n$. What I am doing is, I am taking this limit inside. You can do that, I leave it to you to see why it is because $f^{(n)}$ is continuous and τ_n is bounded. Therefore, you can take the limit inside and further since $f^{(n)}$ is continuous, you can take the limit further inside this bracket.

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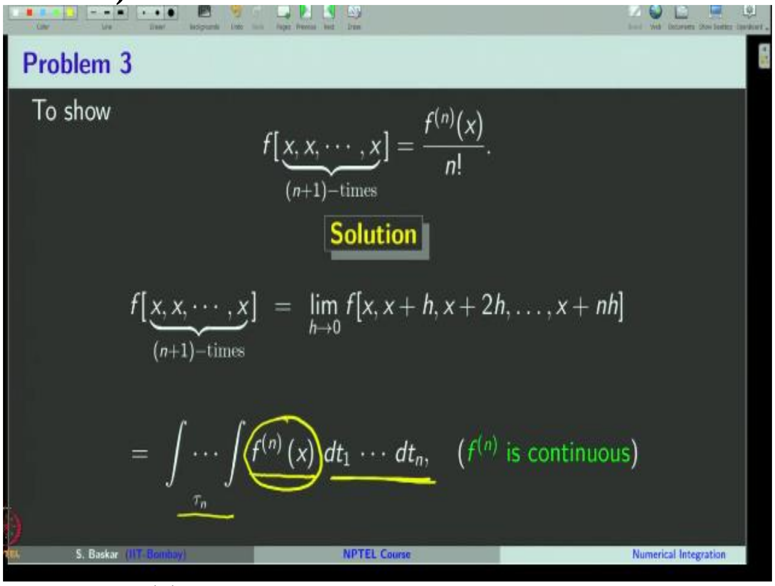

And therefore, you can get $f^{(n)}(x)$ and the integral is taken over this τ_n. Now you see this term is independent of all this t_i therefore you can pull this out.

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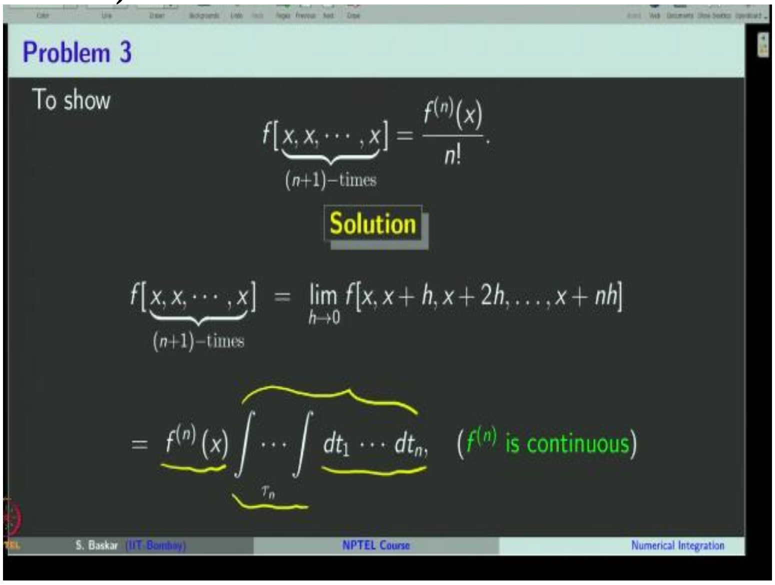

And write $f^{(n)}(x) \int \cdots_{\tau_n} \int dt_1 \cdots dt_n$, now you see this is the volume integral.

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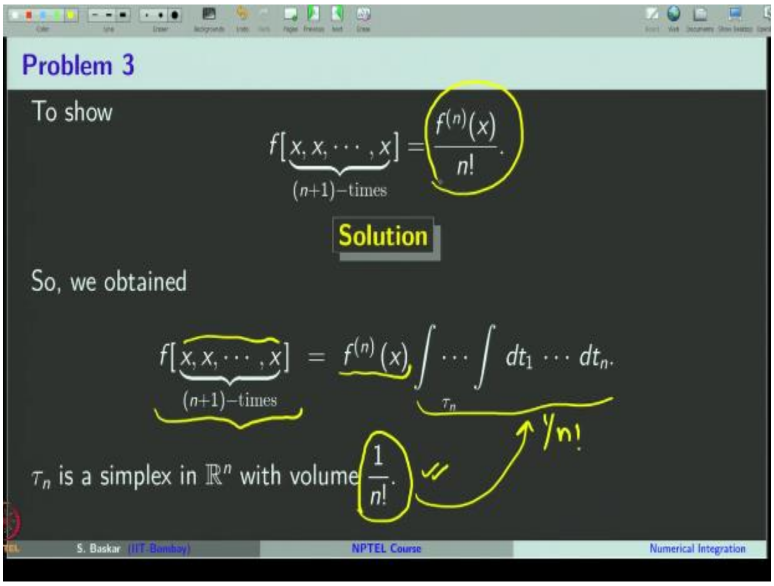

And what is the volume of τ_n? You can see that τ_n is a simplex in $\mathbb{R}^n$ and its volume is given by $\frac{1}{n!}$. So, this result is familiar to us from our multivariable calculus course and therefore you can see that the nth order divided difference with repeated nodes actually is equal to the $f^{(n)}(x)\frac{1}{n!}$. So that is what is given here $\frac{f^{(n)}(x)}{n!}$, so this completes the proof of problem number 3.

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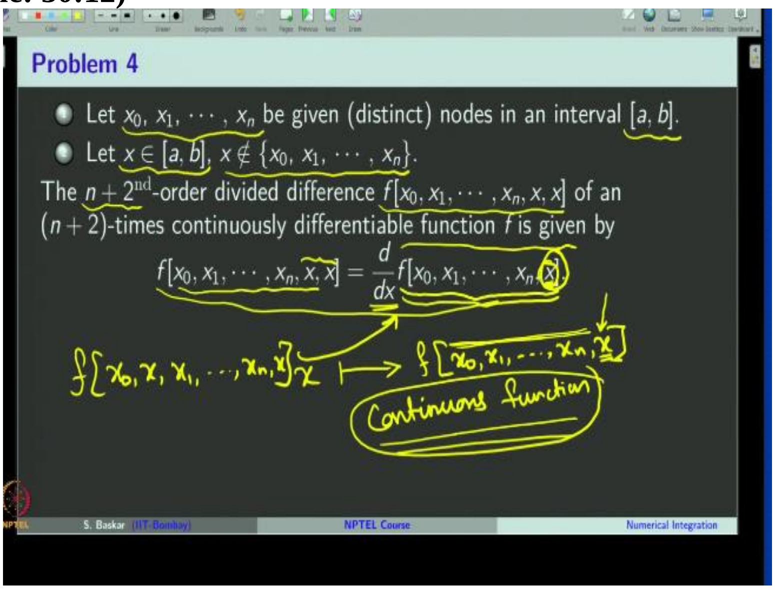

Let us go on with problem number 4. In this problem we have $n + 1$ distinct nodes again given in an interval $[a, b]$ and we are picking up one more point in $[a, b]$ which is different from all these nodes. Then the $n + 2$ order divided difference of the function f with nodes as $x_0, x_1, \cdots, x_n$ and in addition to these $n + 1$ nodes, you also have 2 more nodes which are repeated now. Now here you see you have only 2 nodes repeated and that can be written as $\frac{d}{dx} f[x_0, x_1, \cdots, x_n, x]$.

So, if you have 2 nodes repeated then you can reduce 1 node by introducing 1 derivative of the divided difference of that function. Remember, this is now viewed as a function of x. In fact, from the Genocchi formula you can also show that this function, that is the function $x \mapsto f[x_0, x_1, \cdots, x_n, x]$, this is a continuous function. You can prove this result using Hermite-Genocchi formula but the proof of this is outside the scope of this course.

Therefore, we will not prove this result but it is a very important result. Note that, the position of the node x need not be at the last position. It can stay anywhere in this nodes, because the divided difference is symmetric therefore it does not matter where you place this nodes. Even in this formula, you may have $f[x_0, x, x_1, \cdots, x_n, x]$, this can also be written in the same way. Again this x on the right hand side can be placed anywhere among this nodes.

So, the position need not be like this, that is what I am trying to say. This is clear from the symmetric property of the divided difference. So, you can place these nodes anywhere you want, that is what is very important here.

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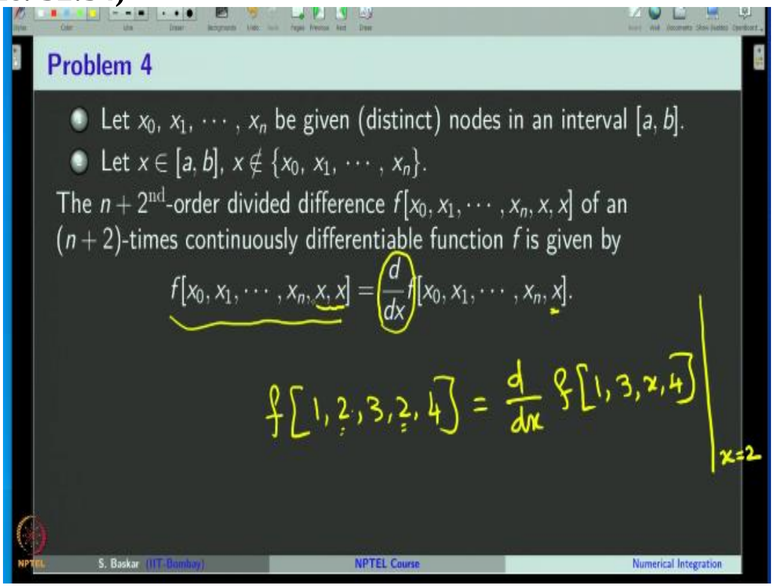

We are just placing the nodes one after the other here and also here we are placing at end. It is not necessary that they have to be placed like this, they can be anywhere. If any 2 nodes are repeated on the left hand side, then you can cut 1 node and introduce this derivative. For instance, suppose if I have $f[1,2,3,2,4]$, Then that can be written as $\frac{d}{dx}$ of f of 1, either you can just remove this or remove this, for instance I will remove this, 3, x, 4 you can do that and then evaluate this derivative at the point $x = 2$.

So, these repeated nodes can be anywhere you want and similarly you can also generalize it to 3 nodes repeated, 4 nodes repeated, and so on.

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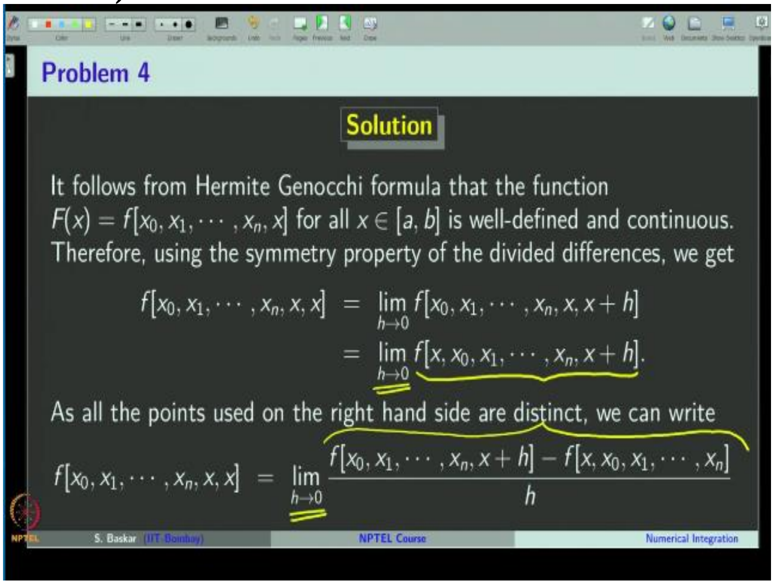

So how to prove this? The proof is not very difficult it just follows from the simple calculus ideas, but for that first you remember that, using Hermite-Genocchi formula you can show that this function that is, x going to the divided difference of f, at these nodes is a well-defined function. Remember from the direct formula, it is not very clear whether it is well defined or not, if 2 nodes are repeated but through Hermite-Genocchi formula, you can see that it is a well-defined function and it is also a continuous function.

Now what you do is, you take the left hand side and just write it as $x, x + h$, something what we did in our last problem. The same idea you do and now you take $h \to 0$. Now what I will do is, by symmetric property of the divided difference, I will just shift this node x to the first position. I am not doing anything, I am just shifting this node to the first position, thanks to the symmetric property. Then what we will do is, remember all these nodes are distinct. Therefore, you can use the classical definition of the divided difference itself and write this formula and then you already have $h \to 0$.

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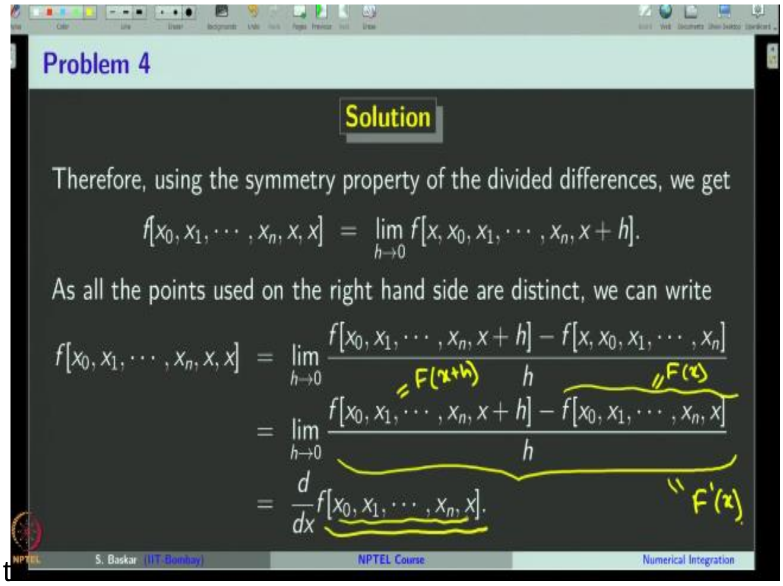

Now, I will again shift this x to the last position and write the same formula in this form. Remember, by shifting 1 node anywhere within this nodes, is not going to change the value of that divided difference. Again, the symmetric property of divided difference is used to write this divided difference in this form. Now let us see what happens, if you recall, this is precisely the derivative of the function F.

Now remember we are viewing this as a function $F(x)$, then this is nothing but $F(x + h)$ and this is the function $F(x)$ divided by h and that is precisely $F'(x)$ and if you recall $F(x)$ is precisely defined as the divided difference of the function f evaluated at these nodes. So that completes the proof of this problem. These are some of the important problems regarding divided difference formula. What we did is, from our theory part we had a formula for divided difference and that formula as far as the nodes are distinct.

But in applications we also come across the situations where certain nodes may be repeated. For that we need to have a different form for the divided difference formula. So, we have introduced Hermite-Genocchi formula for this and we have solved some important problems using Hermite-Genocchi formula with this. Thank you for your attention.