Numerical Analysis
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Lecture – 43
Polynomial Interpolation: Cubic Spline Interpolation

Hi, we are discussing piecewise polynomial interpolations. We have seen that piecewise polynomial interpolations give good approximations to our functions. However, they are not differentiable at the node points. In order to rectify this disadvantage, in this lecture we will discuss another type of piecewise polynomial interpolations called spline interpolations. We will first define spline interpolations and then we will learn to construct Cubic Spline Interpolation of a given function.

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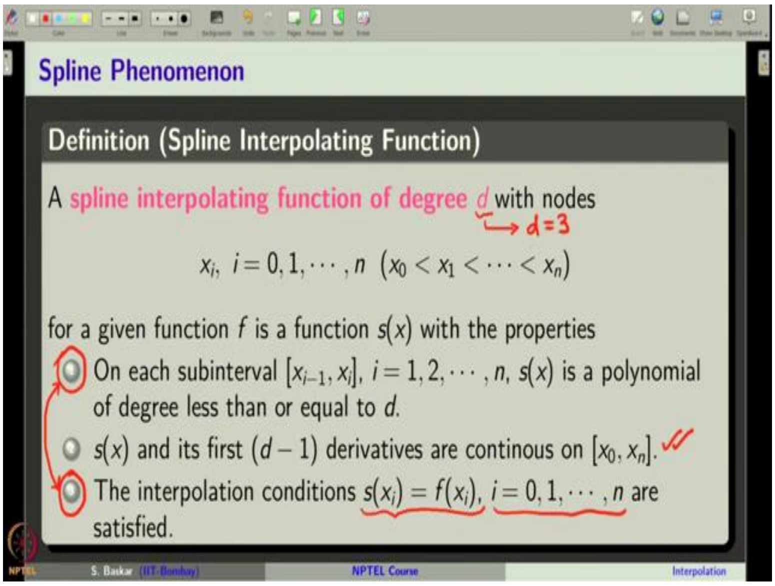

Let us see what is mean by spline interpolating function. A spline interpolation function of degree *d* at the nodes $x_0, x_1, \cdots, x_n$ for a given function *f* is a function which we denote by $s(x)$ with the following properties. Note that the degree of this plane is nothing to do with the number of nodes that are chosen. In polynomial interpolation the degree of the interpolating polynomial depends on the number of nodes that is chosen.

So, we should not get confused with the degree of the spline with the degree of the interpolating polynomials. Let us go to see what are all the properties that this function *s* should satisfy in order to be spline interpolating function of the given function *f* at these node points? The first property is that when you restrict $s(x)$ to each sub interval $[x_{i-1}, x_i]$, then the function $s(x)$ should be a polynomial of degree less than or equal to d.

Up to here, you can see that $s(x)$ is a piecewise polynomial. And of course, if you also impose the interpolation condition then it will become a piecewise polynomial interpolation that is what we have learned in one of our previous lectures. In addition to that, spline also demands certain smoothness of the function s in the entire interval $[a, b]$.

Remember, since s is a polynomial in each of the sub intervals. This smoothness should be mainly achieved on the node points x_i's. However, as a definition, we say that $s(x)$ is continuously differentiable up to order $d - 1$ in the interval $[a, b]$ in which we are interested in approximating the function f by this $spline$. And the last condition is of course, the interpolation condition that the function s should satisfy the interpolation condition with the function f at all the node points.

Therefore, the first condition and the third condition together will say that s is a piecewise polynomial interpolation and the additional condition that it should be $d - 1$ times continuously differentiable on the interval $[x_0, x_1]$ is what is demanded in addition to the piecewise polynomial interpolation. In our course we will restrict ourselves to $d = 3$ in which case we call this plane as cubic spline interpolation.

We will use a rather direct method to construct cubic splines. There are also other methods with which we can also construct higher degree spline interpolations. But we will only restrict to this simple case.

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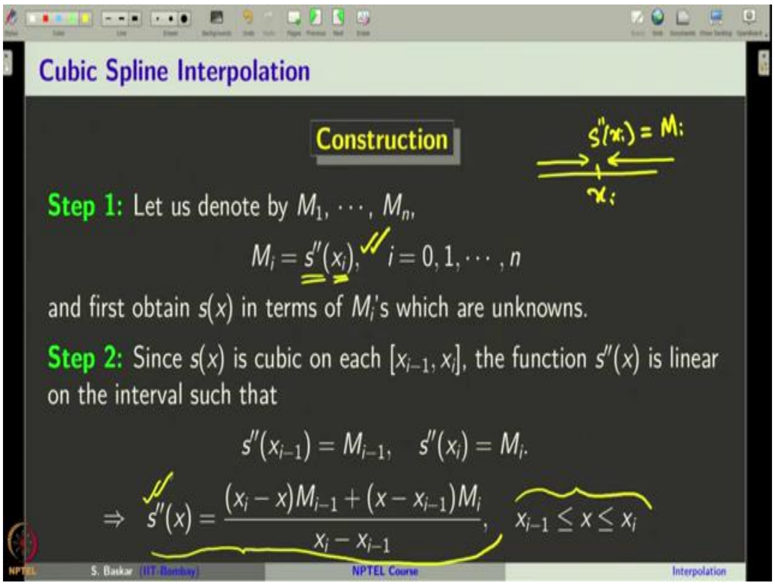

Let us see how to construct cubic spline interpolation. Remember we are given $n + 1$ node points $x_0, x_1, \cdots, x_n$. Let us have a notation M_i to denote the second derivative of s at the ith node point x_i. And the idea is to first obtain an expression for $s(x)$ in terms of M_i's and then we will go to find precisely the values of these M_i's. Now, once you take this notation, you can see that since s is a cubic polynomial in the interval $[x_{i-1}, x_i]$ you can see that $s''(x)$ is a linear polynomial on this interval.

Therefore, you can easily find the equation for s'' in the interval $[x_{i-1}, x_i]$ because we know that it is a straight line joining the points $[x_{i-1}, M_{i-1}]$ and $[x_i, M_i]$. With, this you can immediately write the expression for $s''(x)$ like this for all x in the interval $[x_{i-1}, x_i]$. Now, once you have this, you can get $s(x)$ by integrating s'' twice.

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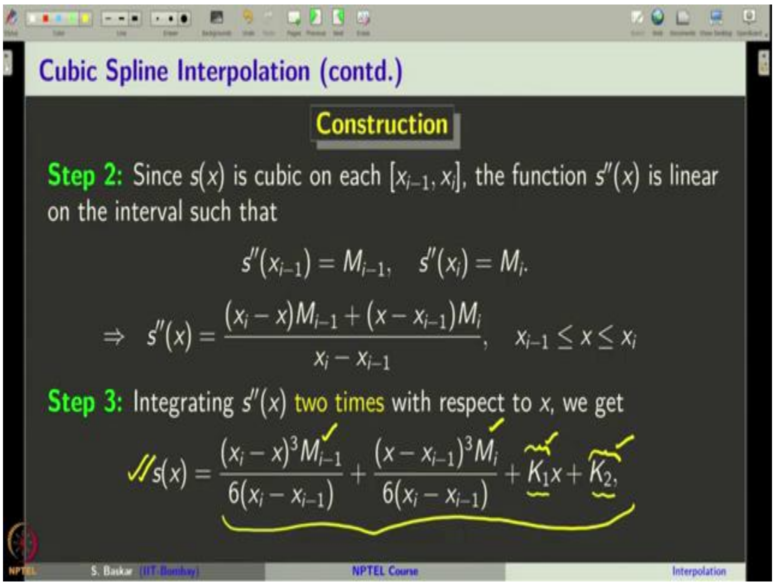

So that is what we will do we will integrate s'' two times with respect to *x* and that gives us this expression. Note that when you integrate for the first time, you will get an integrating constant because it is an indefinite integral. We will denote it by K_1 and then, when you go to integrate it for the second time, we will get another integrating constant which we will denote by K_2.

Therefore, in the interval $[x_{i-1}, x_i]$ you need to now find K_1, K_2, M_{i-1} and M_i there are four unknowns to be determined in the interval $[x_{i-1}, x_i]$. And we have to do this in each of the sub intervals. Let us first try to find K_1 and K_2. Remember, how are we going to find all these unknowns. We, of course, have certain conditions imposed in the definition of spline interpolation. One is it is a polynomial of degree *d* in each of the sub intervals.

In our case *d* = 3 so, we have already used the condition that is *s* is a cubic polynomial in each of the sub intervals. We have already used that condition in order to arrive at this expression for *s* in each sub interval. Now, what are all the other conditions we have? We have to use the interpolation conditions at the node points in this interval. The node points are x_{i-1} and x_i and we have certain smoothness conditions also. Let us see how to use these conditions in order to obtain all these unknowns.

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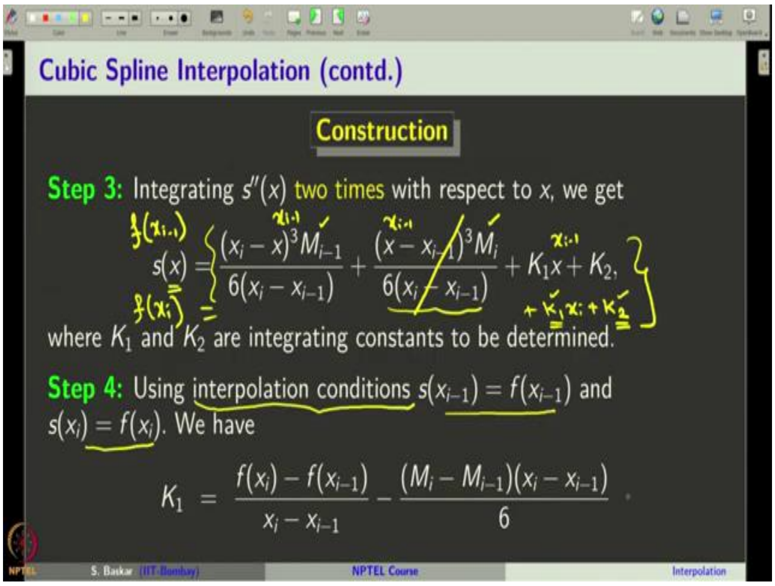

First, let us use the interpolation condition at the node points x_i and x_{i-1}. So, if you put $x = x_{i-1}$ then, on the left hand side you will have $f(x_{i-1})$ which is assumed to be known to us. And now, you have to put x_{i-1} here and x_{i-1} here that will make the second term 0 and you will have x_{i-1} here. Similarly, you have another equation. When you put $x = x_i$ and that will make this first term to become 0 but you will have the second term + $K_1 x_i + K_2$.

So, you have two equations with of course, four unknowns. But now, our aim is to only find K_1 and K_2 in terms of M_i and M_{i-1}. Therefore, we will not touch upon this M_i's while evaluating K_1 and K_2. So, we will just treat K_1 and K_2 as unknowns and we will solve this linear system of two equations with unknowns as K_1 and K_2.

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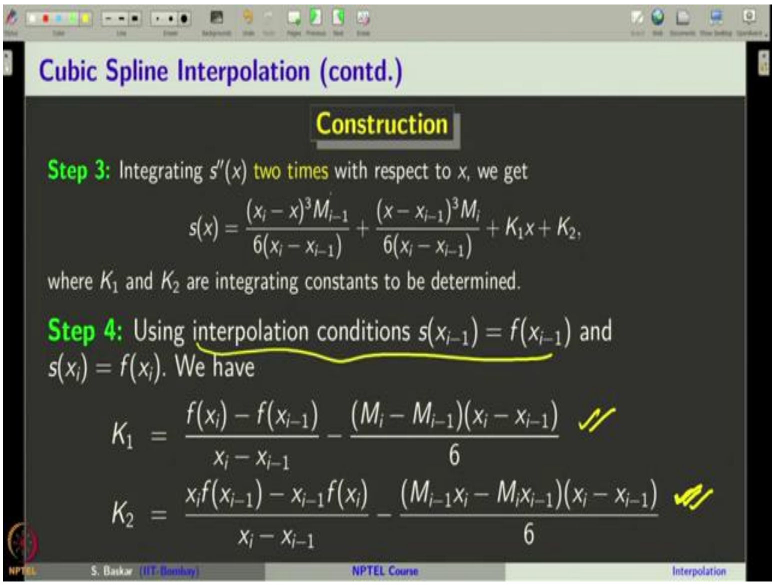

And you can get K_1 as this expression and K_2 as this expression. So, you can easily check these conditions. Therefore, we got two unknowns K_1 and K_2, by imposing the interpolation conditions. Now, we have to get M_i's that is M_i and M_{i-1} in the interval $[x_{i-1}, x_i]$. How are we going to do that?

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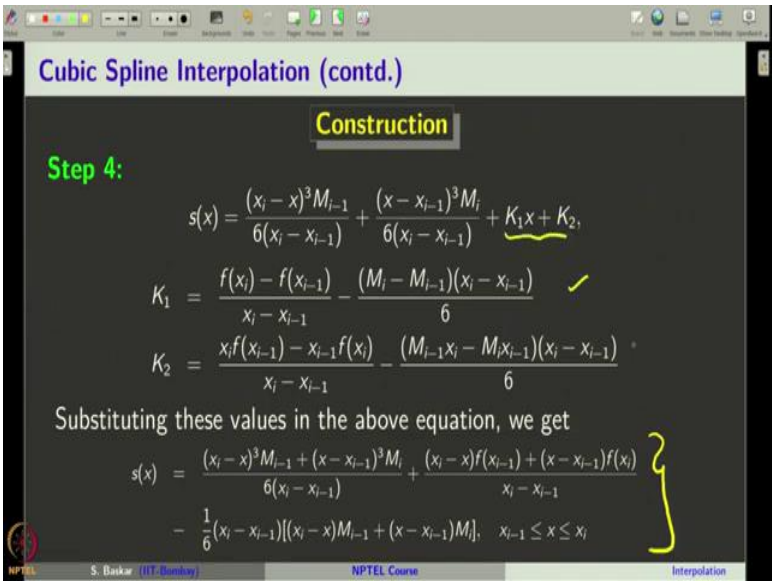

Well, for that let us consider the expression for $s(x)$. Now, we have eliminated K_1 and K_2 in the expression of *s* by substituting these two expressions into it and we got this expression. In this we are yet to find M_{i-1} and M_i.

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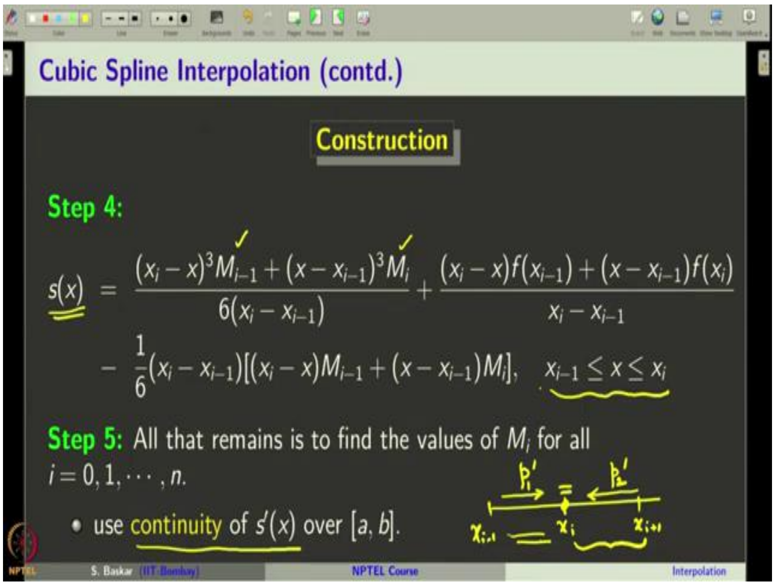

And we will use the continuity of $s'(x)$. Remember we have used the fact that *s* is a cubic polynomial. We also use the fact that s'' is continuous. How we did that? Because that is the way we have constructed s''. You can see that s'' coincides at the boundary points because at x_i whether you approach from this side or this side $s''(x_i)$ has to be M_i.

So, in that way the continuity of s'' is already used in this step and we have used *s* is a cubic polynomial. That is how we have taken the expression for s'' as a straight line joining two points. And we also used the condition that it is an interpolating polynomial. So that is how we got K_1 and K_2. Now, we are left out with only one condition that s' is continuous.

We will use this condition in order to get the values of M_i and M_{i-1}. Let us see how to do that. What you do is at every node x_i you have a polynomial from this side and you have another polynomial from this side. Because s is a piecewise polynomial interpolation. Therefore, in each sub interval $[x_{i-1}, x_i]$ and $[x_i, x_{i+1}]$ you have different polynomials. And now, you take this polynomial, say p_1 and p_2.

And you find the derivative of these polynomials and then you have to equate them in order to achieve the continuity of p' at this node. So, what you do is you have s in this interval that is given by this expression in the interval $[x_{i-1}, x_i]$. Similarly, you write $s(x)$ in this interval then take the derivative of s in these two intervals and then equate them. So, you just have to equate them and this has to be done for all the sub intervals.

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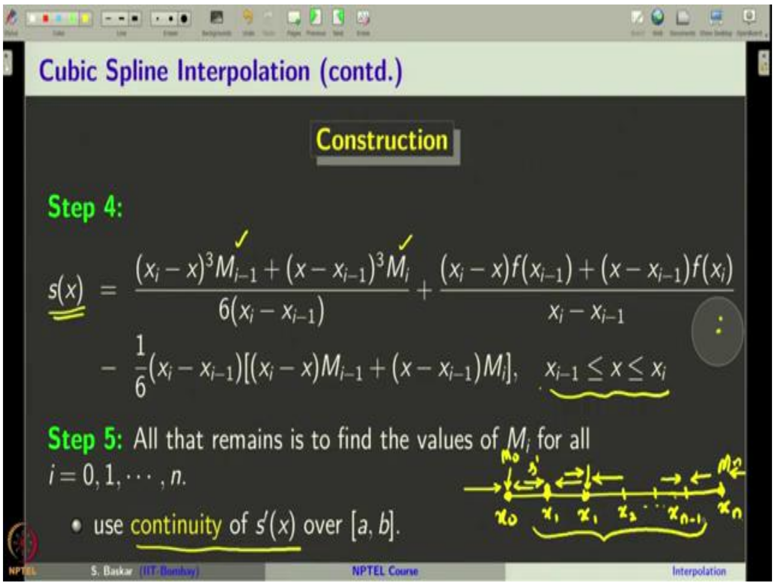

Note that you can do this at x_1 because you have the interval $[x_0, x_1]$ and $[x_1, x_2]$. You can approach from the left and you can approach from the right for the function s' to get the condition at this point. Similarly, to get a condition at x_1 you approach from left and you approach from right from this interval $[x_1, x_2]$. And that will give you one condition here. Similarly, you can get up to x_{n-1}.

Note that at the points x_0 and x_n you cannot impose this idea because we only have the polynomial approaching from one side. From the other side, you do not have a polynomial to obtain M_0 and similarly you do not have the polynomial to approach from the right in order to get M_n. So, therefore, M_0 and M_n are excluded from this idea.

Otherwise, for all i = 1 to n – 1, you can find the corresponding values or M_i using this continuity property of s'.

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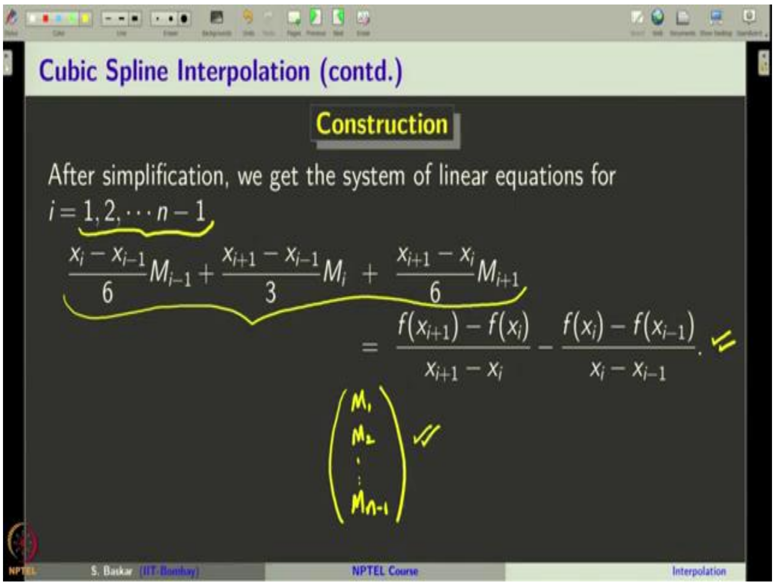

And you can see that imposing the continuity property of s' at each node $x_1, x_2, \cdots, x_{n-1}$will give us a tridiagonal system of linear equations. You can derive and see. I will not show you the derivation. It is little lengthy but very simple to do. I leave it to you to see how to bring this tri diagonal system. You can see that for each *i* you have non zero entries only on the tridiagonal positions.

Otherwise, all other entries are zeros and this is a tridiagonal system with unknowns as M_1, M_2, M_{n-1}. By solving this tri-diagonal system, we can get all these constants except M_0 and M_n. We cannot get them using this idea.

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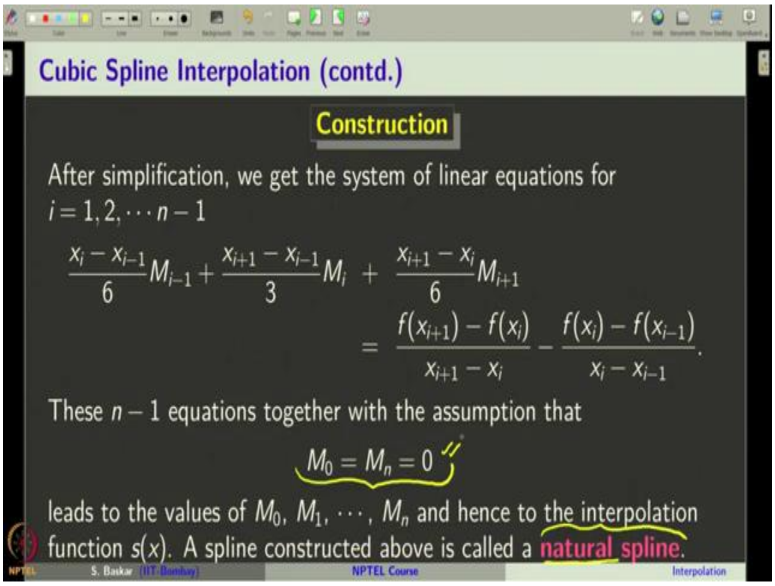

So, what we will do is we will take $M_0 = 0$ and $M_n = 0$. There is no reason for why we choose like this. This is the way we choose. The constants M_0 and M_n and we will call the resulting plane as natural spline. So, this is the construction of cubic naturals spline interpolation of a given function. Cubic spline, because we have imposed the smoothness of degree 2 at the nodes and also from the way it is constructed.

You can see that s is a piecewise cubic polynomial. And the name natural is because we have chosen M_0 and M_n like this. There are other ways to choose these constants, different way of choosing these constants will lead to different splines. But natural spline means you have to choose M_0 and M_n as 0. There is no reason for why it is like this. This is the way in the literature people have used it.

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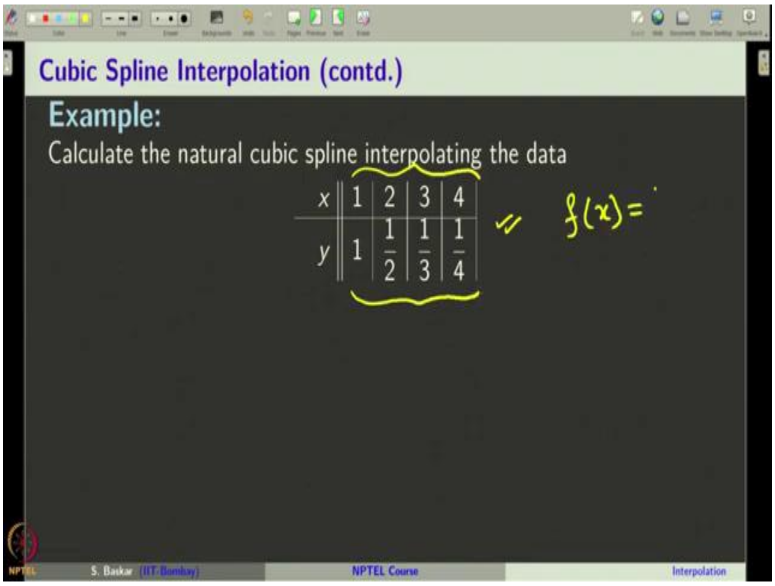

Let us see an example. Let us try to construct the natural cubic spline interpolation for the data set given to us. You can see, there are four node points and the value of the function at these node points are given like this. From here you can see that the function that we are considering is $f(x) = \frac{1}{x}$. Let us see how to construct the natural cubic spline.

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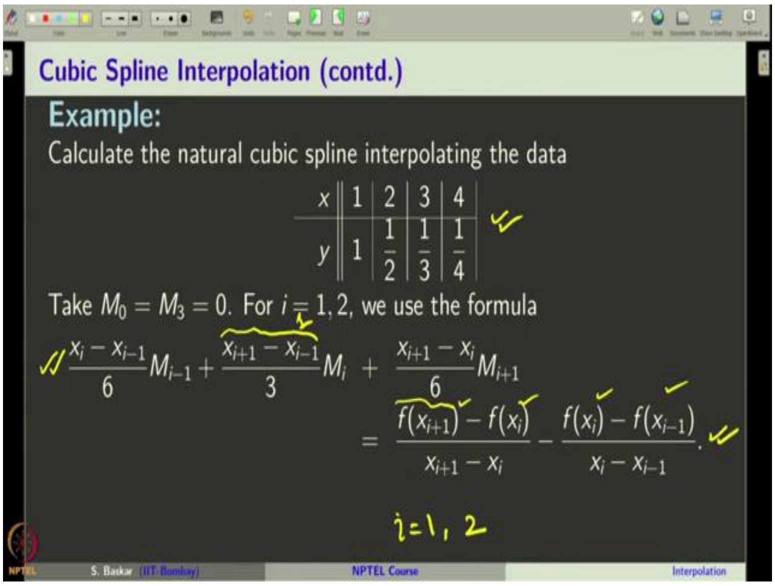

Since it is natural, first thing is you simply take $M_0 = M_3 = 0$. Here *n* = 3 because we start from x_0 then x_1, x_2 and x_3. So, *n* = 3 so, you have to take M_3 as 0 but you have to find M_1 and M_2 by constructing a system of linear equations. It is just a 2 by 2 system. Therefore, there is no clear tridiagonal structure is visible in this case. However, to construct the system you can use this formula which we have derived in our construction.

You just plug in the values of x_i's and $f(x_i)$'s, given in this data set. You can see that for each *I*, this is just one and you have the corresponding values for this Therefore, for *i* = 1 and 2 you can get two equations for M_1 and M_2.

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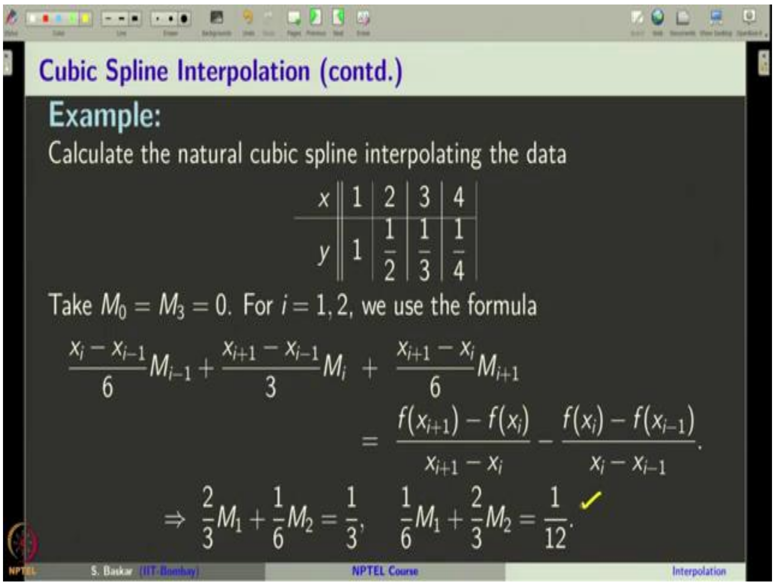

They are linear equations, given by this, for *i* = 1 and this equation for *i* = 2. You can see that it is a linear system with two equations. Since it is only 2 you do not see this tridiagonal structure in this system. However, you can easily solve this system by using Gaussian elimination method also.

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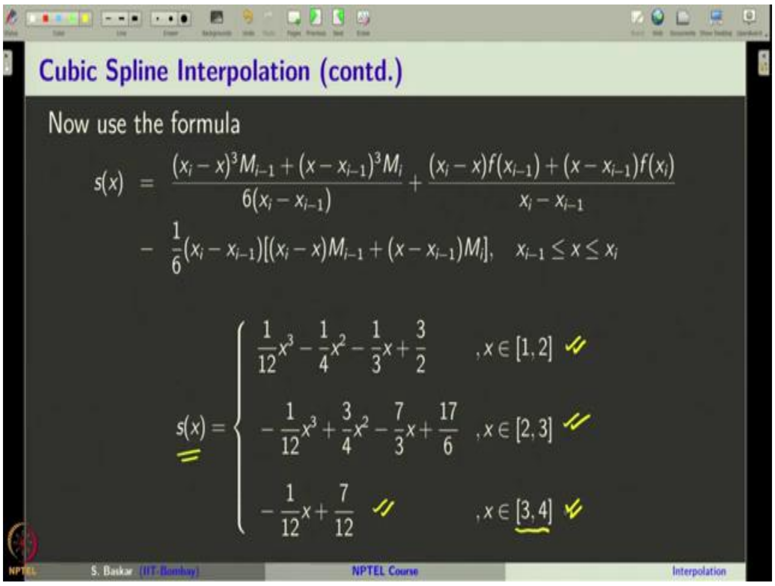

And thereby you will get the value of M_1 and M_2. Already, you have the values of M_0 and M_3. Therefore, you obtained all the constants M_0, M_1, M_2 and M_3. Now, you have to plug in those constants into the expression of the cubic spline interpolation. Remember you have to do this in each of the sub internals from $[x_0, x_1]$. You have to plug in M_0, M_1 and you will get this cubic polynomial.

And similarly, you have to do it for the interval $[x_1, x_2]$ that is $[2,3]$. Here you have to plug in M_1 and M_2 and you will get this polynomial. And similarly, in the last sub interval you get the polynomial as this. Here you can see that *s* is a piecewise cubic polynomial, here cubic means the degree is at most 3 not exactly 3. So, this is how we generally define the interpolating polynomials.

So, in the first interval it is a cubic polynomial in the second interval also, it is a cubic polynomial. But in the third interval it is only a linear polynomial but that does not matter. We want *s* to be a polynomial of degree less than or equal to 3 in each of the sub interval. Therefore, this is fine and this is the natural cubic spline interpolation for the given data set. With this, we will close this class. Thank you for your attention.