,Numerical Analysis
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Lecture-35
Nonlinear Equations: Fixed-Point Iteration Methods (Convergence) and Modified Newton's Method

In this lecture, we will continue our discussion and prove the convergence theorem for fixed point iteration method. We will also continue our discussion on some modified versions of Newton-Raphson method in order to achieve quadratic convergence in the Newton-Raphson method when the root is not a simple root that is if it has multiplicity something greater than 1.

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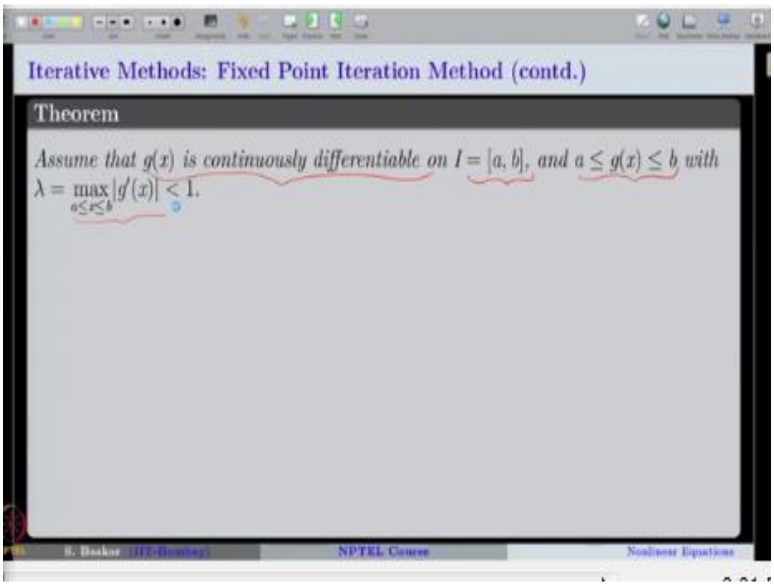

Let us start, our discussion with a convergence theorem of the fixed point iteration method. If you recall in the last class, we have listed 3 assumptions that we want to make in order to choose a good iteration function. We will put those assumptions as the hypothesis of our theorem. The first assumption is that the function g which we want to choose as an iteration function should be a self map on an interval of our interest.

And also the function g should be continuously differentiable, that is g should be continuous and it is derivative g' should exist and it is also continuous on the interval $[a, b]$. And the third assumption was the contraction principle that is the maximum of the absolute value $\max_{\backslash a \le x \le b} |g'(x)| < 1$. So, these are the 3 hypothesis that we will impose.

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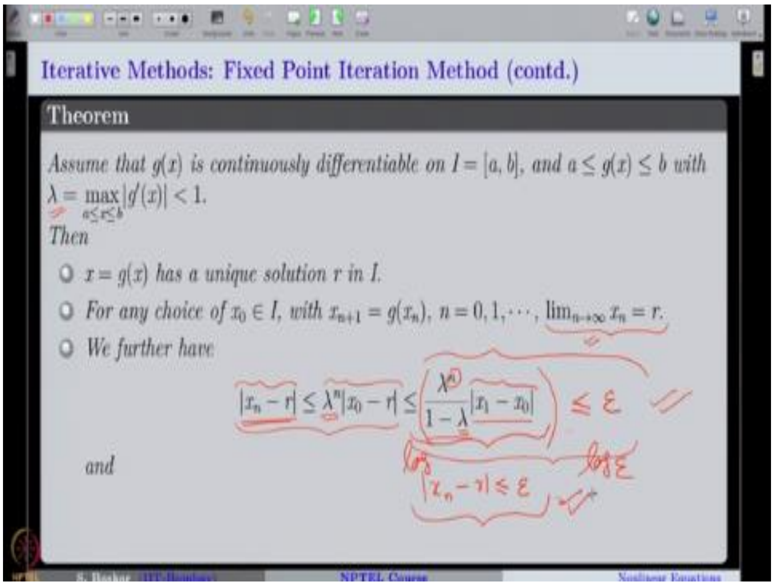

And the theorem says that if we impose these 3 conditions on our iteration function then the iteration function g will have a unique fixed point in the interval $[a, b]$. The existence of the fixed point comes from the assumption that g is a self map and g is continuous on the interval $[a, b]$, whereas the uniqueness comes from the assumption that the function g is a contraction map. This is not very difficult for you to see, I will leave it to you to prove this.

The second conclusion of the theorem is that if you choose your x_0 inside the interval $[a, b]$ then all the terms of the sequence will belong to the interval $[a, b]$ and the sequence x_n will converge to the fixed point r as n tends to infinity. The Third conclusion is rather interesting, the conclusion says that you can write $|x_n - r|$, this is the absolute error is less than or equal to $\lambda^n |x_0 - r|$.

Observe that if you prove this inequality then you will achieve the convergence because λ is less than 1 therefore λ^n will go to 0 as n tends to infinity. Therefore, convergence will come immediately once you prove this inequality and not only that is another interesting inequality that can be derived from here. And that says that the absolute error is in fact less than or equal to $\frac{\lambda^n}{1-\lambda}|x_1 - x_0|$. What is interesting in this?

You can see that the right hand side involves all the terms which are known to as at least after computing x_1. You see λ is also known to us here and therefore this term is completely known to us. Suppose if we want our iteration term x_n to be very close to r, for instance we want the accuracy of our approximation in such a way that the absolute error is say less than or equal to some ϵ, where ϵ is some tolerance error which we choose.

Then what you do is you can impose that condition here, that is this term is less than or equal to ϵ, in that way you can achieve this accuracy. And now you see in order to achieve this accuracy, you can see how many iterations are needed for us to compute and that information you can get without explicitly computing the iteration terms. If you recall such a result was obtained in the bisection method also. How did we get such an n?

You can take log on both sides and from hear you can get an inequality for n that is we will get the integer n greater than or equal to some expression which can be computed fully without explicitly computing the iterations. That is the interesting part of this inequality, you should recall how we did it in the case of bisection method, come back you can also do it here and get a n which says that we have to perform at most that many iterations in order to achieve this accuracy, that is very interesting.

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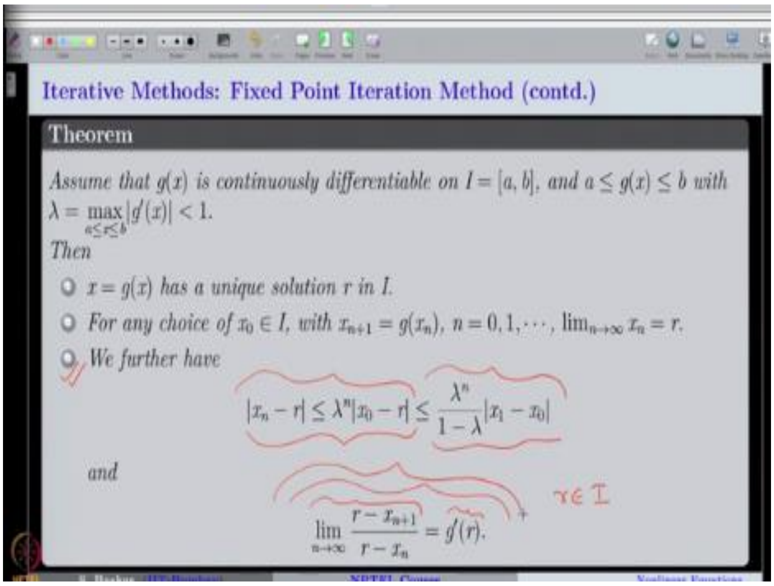

And finally, the theorem also concludes that $\lim_{n\to\infty} \frac{r-x_{n+1}}{r-x_n} = g'(r)$. If you recall this is one way of defining linear convergence as long as this is a finite number. Note that, r belongs to the interval $[a, b]$ which is a closed and bounded interval and g' is a continuous function on this closed and bounded interval, therefore $g'(r)$ should be a finite number. Therefore, this expression in fact tells that the fixed point iteration method will in general have at least linear convergence. Let us go to prove this theorem.

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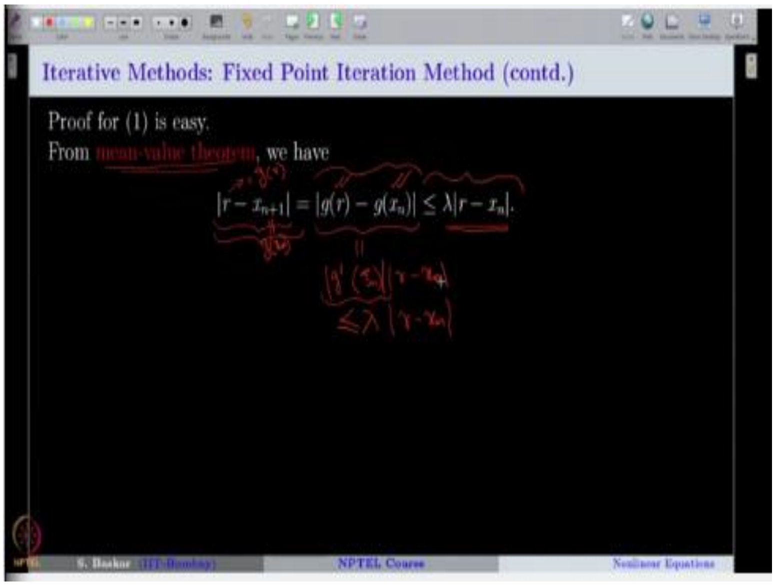

The proof of this theorem is not very difficult; in fact as I told the first part is very easy, it is a very simple exercise in your calculus course, therefore I will leave it to you as an exercise. Let us try to prove the second result; in fact the second result can be achieved if we prove the inequality in the third part of the conclusion. Therefore, we will try to prove this inequality, for that we have to use the mean value theorem.

Before using mean value theorem first let us start with the absolute error $|r - x_{n+1}|$ since r is the fixed point you can write $r = g(r)$ that is what I am writing in the first place here. And the way the iteration method is defined you can see that $x_{n+1} = g(x_n)$, that is why I am writing $g(x_n)$ here. Therefore, your absolute error is precisely equal to $|g(r) - g(x_n)|$.

Now, I will use the mean value theorem which says that I can find a ξ_n such that this term is equal to $|g'(\xi_n)|(r - x_n)$. So, that is what I am going to use and of course there is a modulus here. And now go back to our statement, we have taken the maximum over the interval $[a, b]$ of

$|g'(x)|$ and we call that as λ. Therefore, I can replace this by λ and I will get this to be less than or equal to $\lambda|r - x_n|$.

And that is what I am writing here, this term is less than or equal to $\lambda|r - x_n|$. So, we got this, now what is the next idea? Whenever you see such an inequality you immediately go to apply this inequality recursively.

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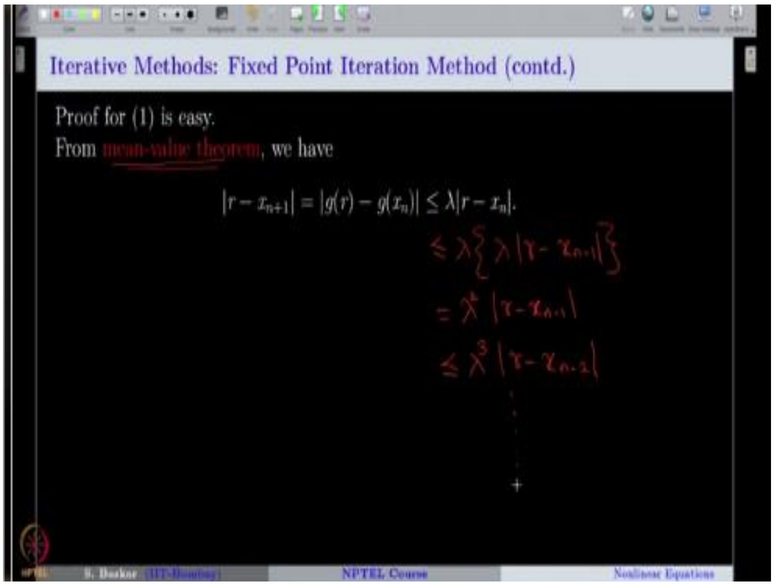

That is what you do is, you can write this as less than or equal to $\lambda\lambda|r - x_{n-1}|$. That is applying the same inequality for $|r - x_n|$ will give you this that is equal to $\lambda^2|r - x_{n-1}|$. Again, apply this inequality to $|r - x_{n-1}|$, that gives you $\lambda^3|r - x_{n-2}|$ and you can keep on going like this till you hit the last term that is $|r - x_0|$.

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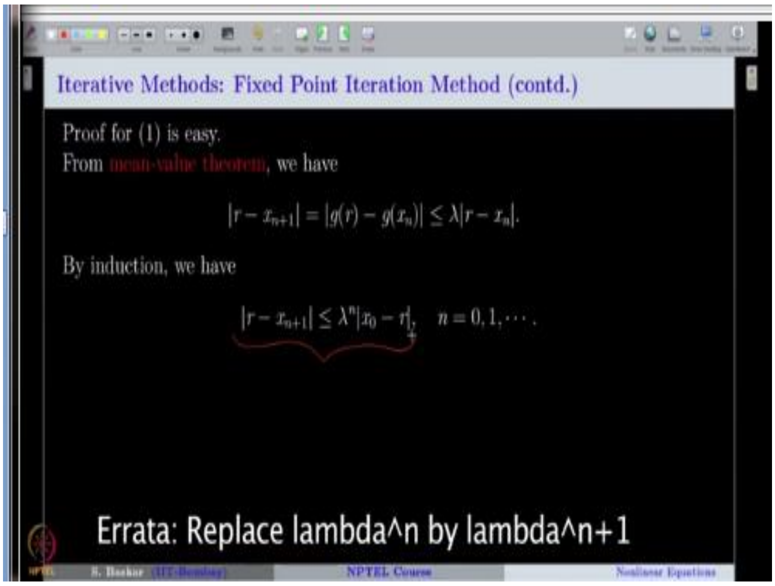

And therefore, you can get this inequality $|r - x_{n+1}| \le \lambda^{n+1}|x_0 - r|$. If you recall that is what we wanted to show as the first part of the inequality in the third conclusion. Let us now try to derive this part of the inequality again that is not very difficult.

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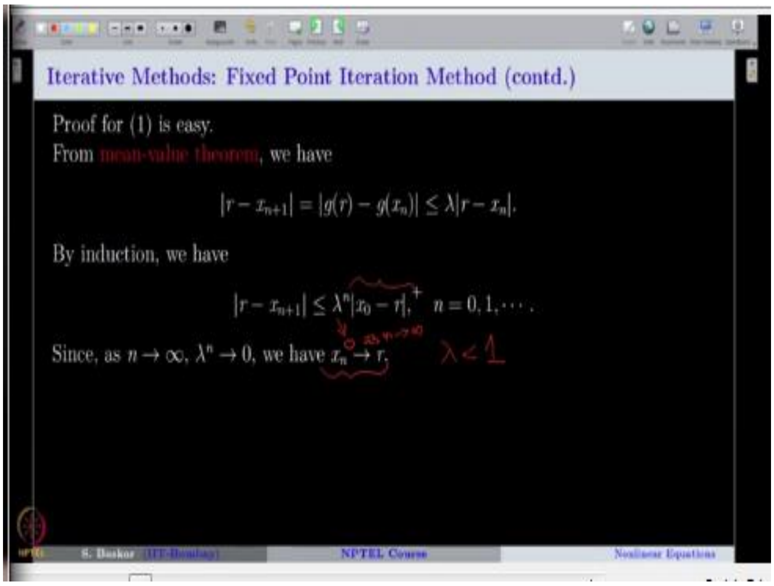

Before going to derive that we will also observe that λ is less than 1, that is the contraction property of the iterative function g. And that says that λ^n goes to 0 as n tends to infinity and that gives us also the convergence and that also completes the second part of the conclusion. So, let us prove the last part of the inequality, for that let us take the right hand side $|x_0 - r|$.

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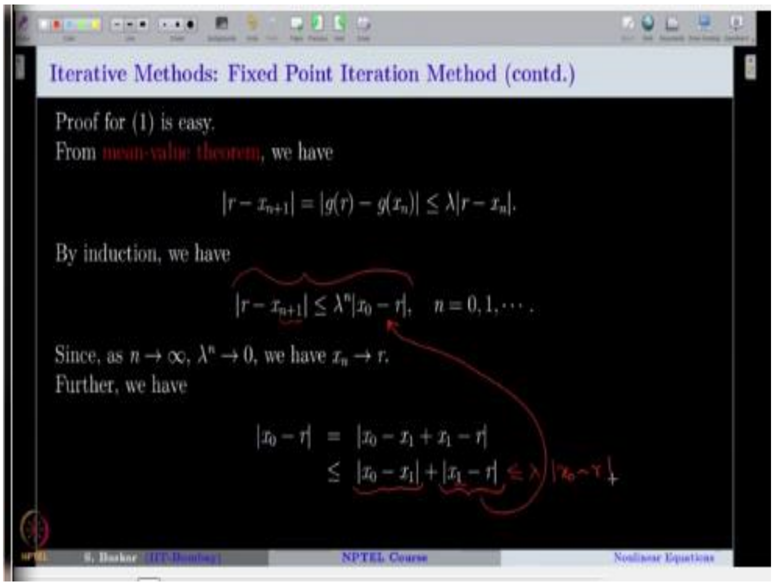

And what we do is, we will add and subtract x_1 in that and then we will use the triangle inequality to write like this. And now you see this is already there in our right hand side, whereas this term needs to be rewritten but that is not very difficult because we already have an estimate of this term from here. So, we will try to use this idea into it with n + 1 replaced by 1, then what we will get? This is less than or equal to $\lambda|x_0 - r|$.

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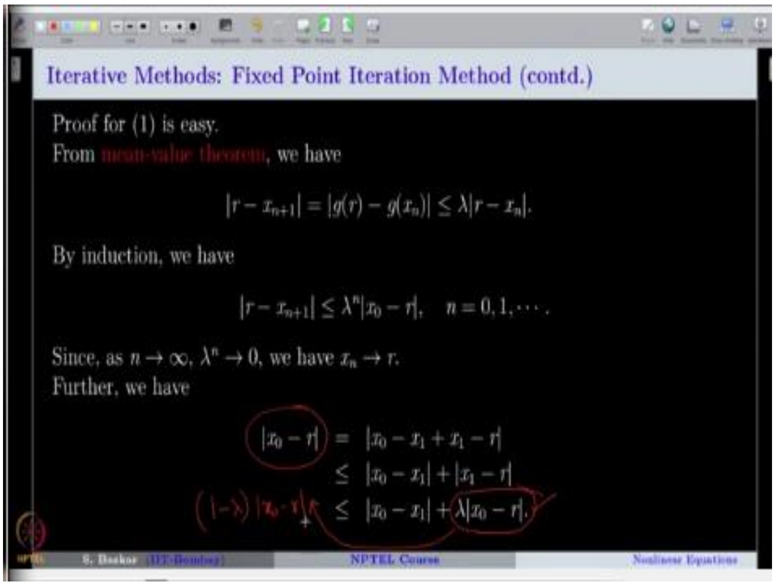

So, that is what we get here. Now you see you have $|x_0 - r|$ on the left hand side you have $\lambda|x_0 - r|$ on the right hand side, you take it to the left hand side you will have $(1 - \lambda)|x_0 - r|$.

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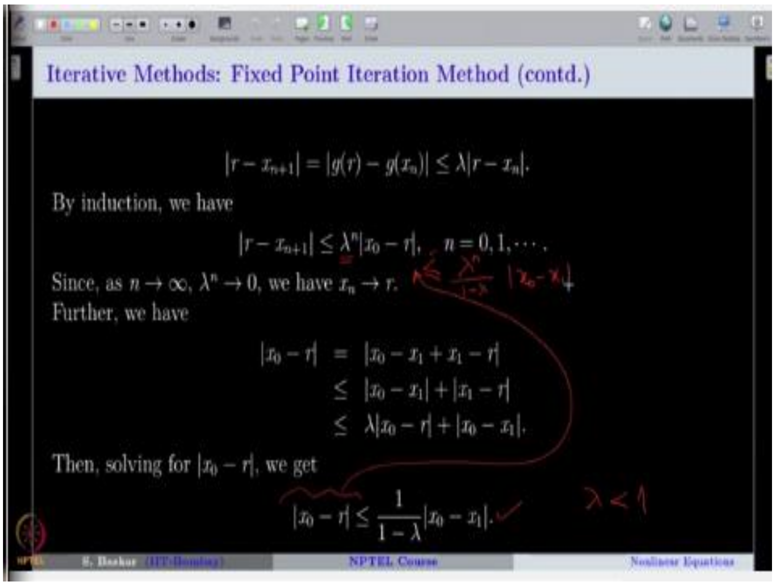

So, that is more or less what we wanted to show, therefore that can be brought to the right hand side why because λ is less than 1, therefore when you bring it to the right hand side the inequality will not get disturbed. So, we got this now what you do is instead of this you put the right hand side estimate for this. That gives us λ^n which is already there, now $|x_0 - r|$ we will write as less than or equal to $\frac{1}{1-\lambda}|x_0 - x_1|$.

And this is precisely what we wanted to show in the third conclusion; you can see this is what we wanted. Therefore all remains for us to now show the order of convergence, let us see how to prove the order of convergence.

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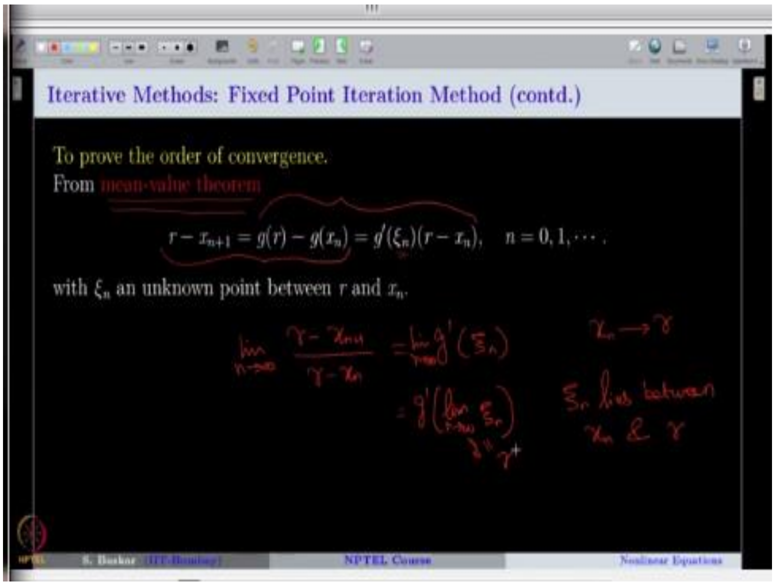

That is also not very difficult; again you go back to our definition of the fixed point iteration that gives you the first equality. And from this to this we will use the mean value theorem, this is what precisely we have done at the beginning of this proof, the same exercise we are doing once again and we get this unknown ξ_n lying between r and x_n. Once you get this you can see that $\frac{r-x_{n+1}}{r-x_n} = g'(\xi_n)$.

We have already proved that x_n converges to r and ξ_n lies between x_n and r. Now you use the Sandwich theorem and that will tell us that when we take limit n tends to infinity we will have $g'\left(\lim_{n\to\infty} \xi_n\right)$, remember g' is a continuous function and now this converges to r because of the Sandwich theorem.

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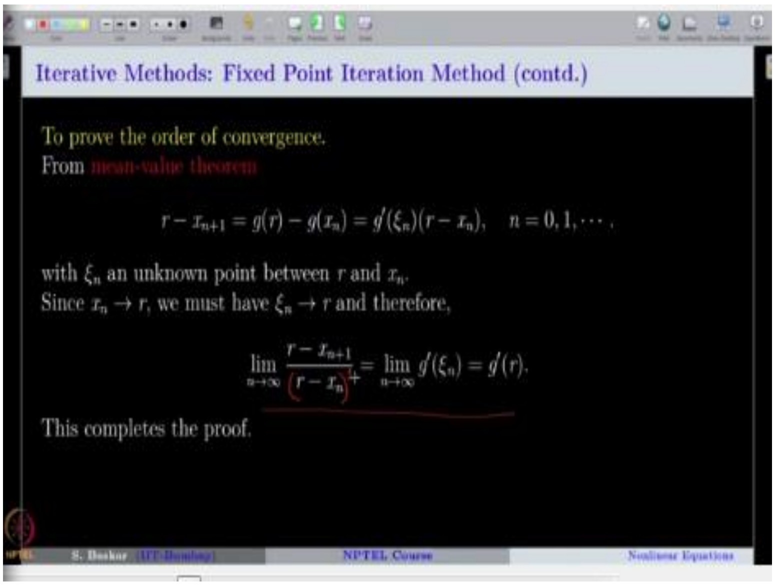

And that will immediately give us what we want in the order of convergence path. So, this also shows that the order of convergence of the fixed point iteration method is in general 1, that is it will have at least linear order of convergence.

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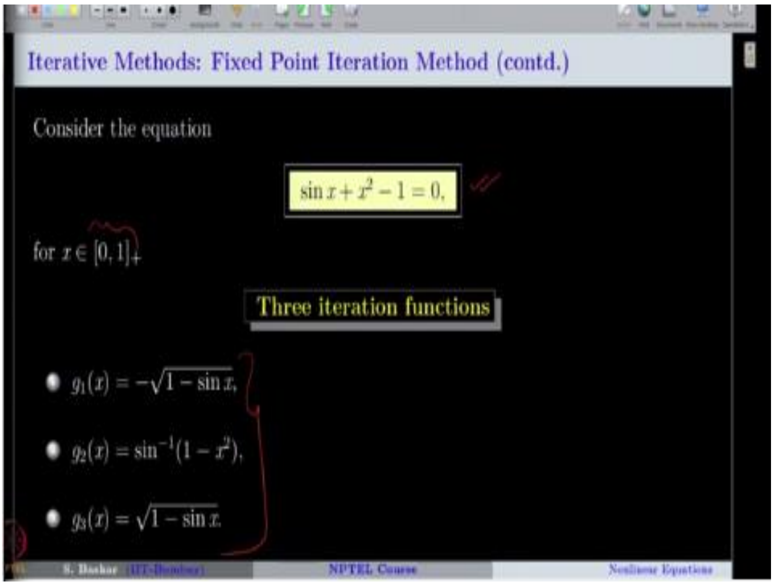

Let us try to see some examples. Let us take this equation $\sin x + x^2 - 1 = 0$. You can see that there is a root of this equation in the interval $[0,1]$. If you recall we have computed a root of this equation in the interval $[0,1]$ using bisection method, secant method and also Newton-Raphson method. Now let us try to formulate a fixed point iteration method for this equation and see whether it converges or not?

But now for the fixed point iteration method we have to choose a good iteration function. Let us propose these 3 iteration functions for this equation and see which one will be good for us to capture a root of this equation in the interval [0,1].

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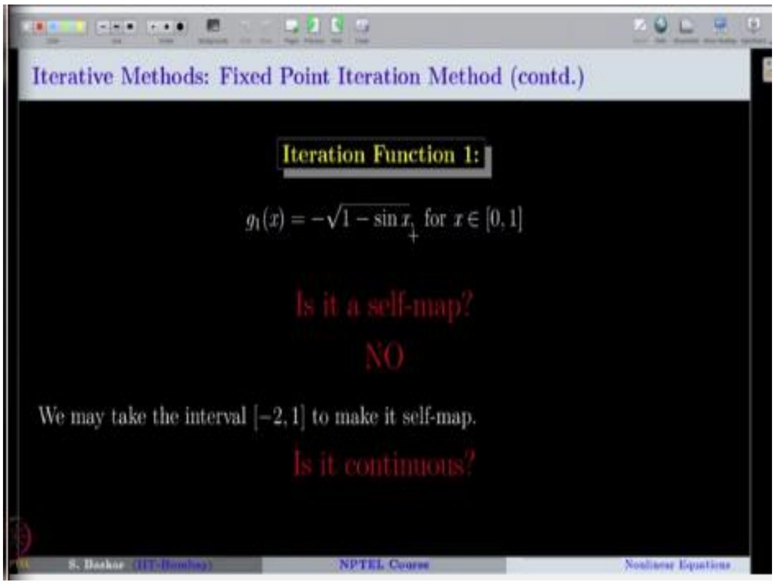

Let us take the first choice, $g_1(x)$ and we are interested in this equation. Now in order to see whether g_1 is a good iteration function or not we have to check 3 properties, one is the self map whether g_1 is a self map or not? You can see that definitely g_1 is not a self map because you choose any point in this interval and put that into this expression you will get a negative value, therefore g_1 is not a self map.

However that is not a very serious problem in going for g_1 because you can always extend the interval on the negative side little bit in order to make it a self map. But then you have to also check whether it is continuous function which is also not very difficult for you to check. And also you have to check whether it is a contraction map or not, I will leave it to you to check that.

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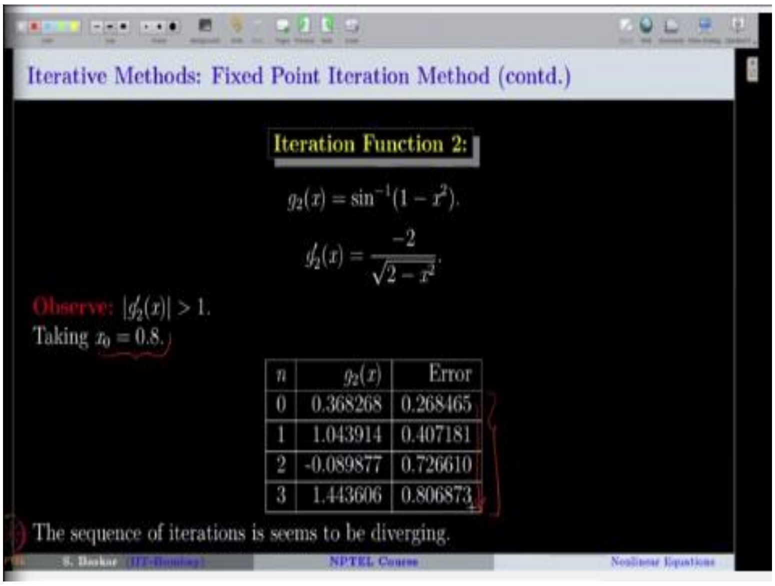

Let us go to the next choice g_2, and again we want to check all the 3 assumptions let me directly go to check whether it is a contraction map or not. Remember, in order to check whether g_2 is a contraction map or not we have to first differentiate g_2 with respect to *x* and the derivative of g_2 is given like this. And you can clearly observe that $g_2'(x) > 1$ that is absolute value of $|g_2'(x)| > 1$ for *x* in the interval [0, 1].

That shows that g_2 is not a contraction map in the interval $[0,1]$. Let us still go to compute few iterations with g_2 as the iteration function. Let us start our iteration with x_0 = 0.8, I have shown here first 3 terms of the iteration sequence. You can see that the error is gradually increasing as you go on computing the iterations. In fact, if you go on computing the terms greater than 3 you will see that the error increases more rapidly and that gives us a feeling that the iteration sequence seems to be not converging to the fixed point of the function g_2.

It means the corresponding fixed point iteration method is not converging in this case, that is not surprising because g_2 is not a contraction map.

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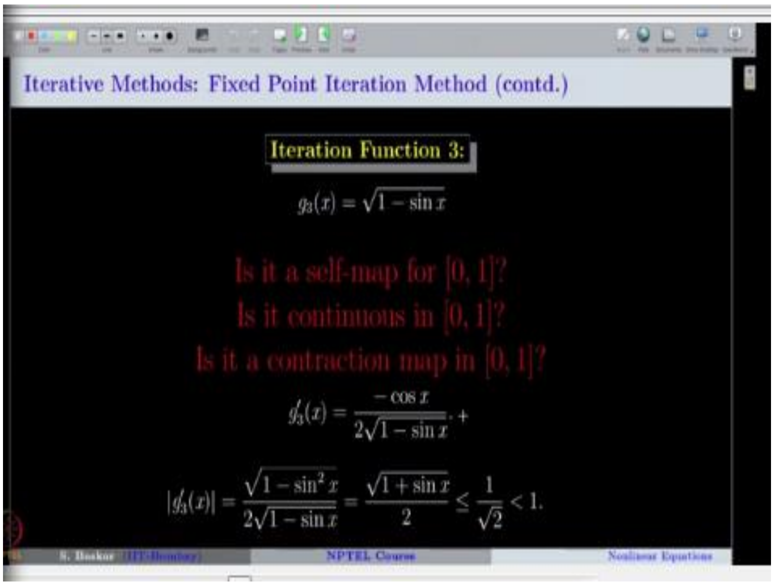

Let us now consider our last choice g_3 and see whether this is a good iteration function for us or not? You have to check first whether it is a self map or not? You can see that it is a self map and also it is a continuous function, you can easily check that. And finally, the question is it a contraction map on the interval [0, 1]? For that we have to first compute $g_3'(x)$ and that is given like this.

And from here we can see that $g_3'(x) \leq \frac{1}{\sqrt{2}}$ for any x in the interval $[0,1]$. That shows that g_3 is a contraction map in the interval $[0,1]$ therefore it satisfies all the hypothesis of our convergence theorem which implies that the fixed point iteration sequence with g_3 as the iteration function will surely converge if you choose any x_0 in the interval $[0,1]$.

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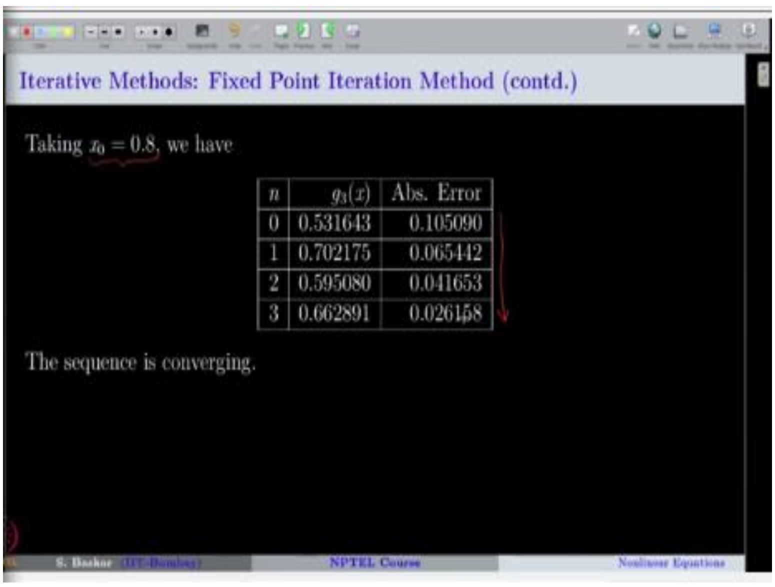

Let us take our x_0 as 0. 8 and you can see that the error is gradually decreasing at least up to 3 iterations that I have shown here. In fact, I have computed it for more iterations and I have observed that the error is gradually decreasing. That gives us a feeling that the sequence in this case is converging to a fixed point of the iteration function g_3 in the interval [0,1]. With this our discussion on fixed point iteration method is over.

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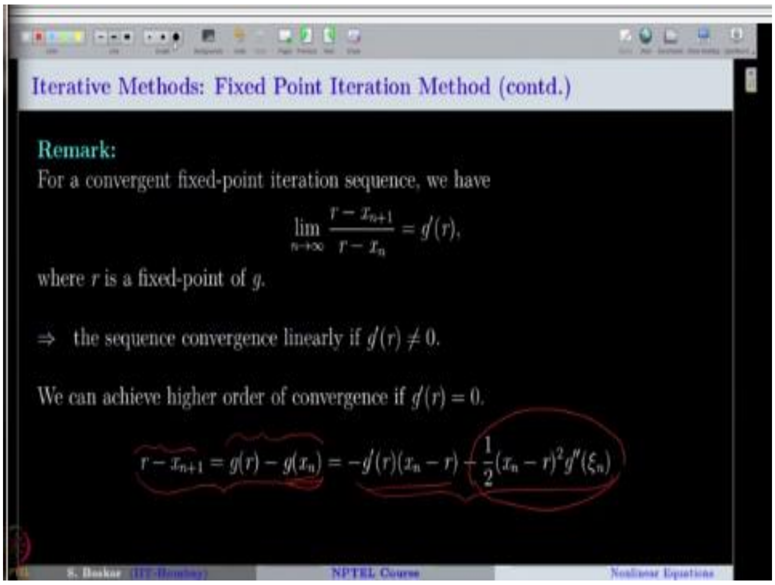

Let us list some of the key takeaways of the fixed point iteration method. First thing is fixed point iteration method in general will provide you at least linear convergence, especially when $g'(r) \neq 0$. What happens if $g'(r) = 0$? Then we can expect a higher order of convergence. Why it is so? Again, take our derivation using mean value theorem, now you use the Taylor expansion instead of applying the mean value theorem.

That is you take the error $r - x_{n+1}$ and that can be written as $g(r) - g(x_n)$ and from here you can see that $g(x_n)$ can be written in the Taylor's formula with Taylor polynomial of degree 1 plus the remainder term.

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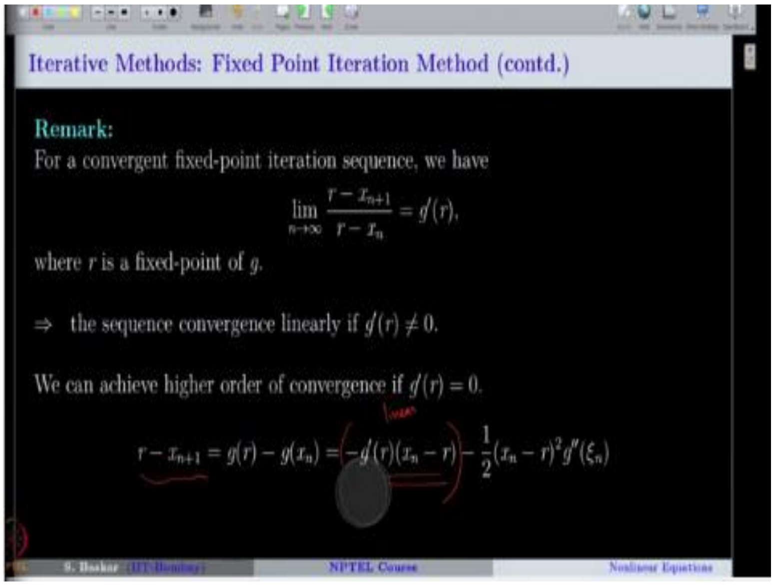

Once you do that you can see that when $g \neq 0$ then the error is actually dominated by this term and hence you have linear convergence.

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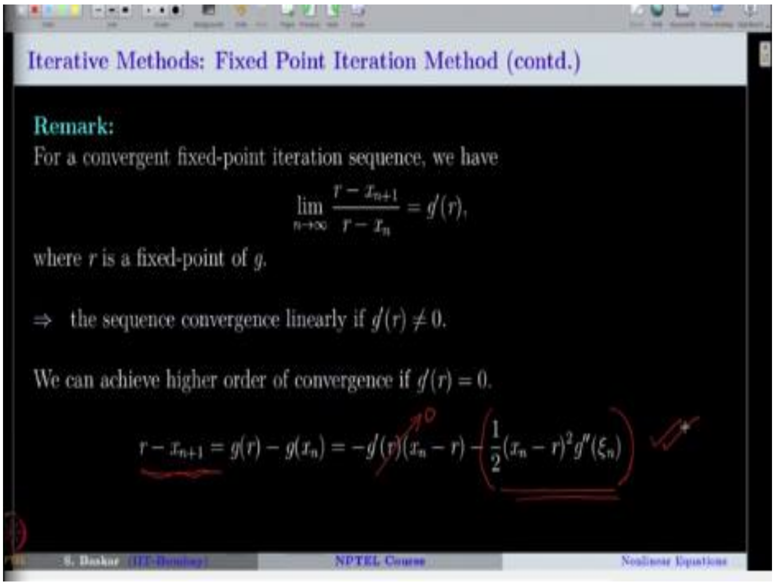

On the other hand, if g' is 0 then what happens? You can see that this term vanishes and then this term will remain as the dominating term in your truncation error which will show that the error is actually having a quadratic convergence. Therefore, in your fixed point iteration method if your g in addition to the 3 assumptions that we made if it is also happens to be that $g'(r) = 0$ then we will have at least quadratic convergence, that is what we are seeing from this expression.

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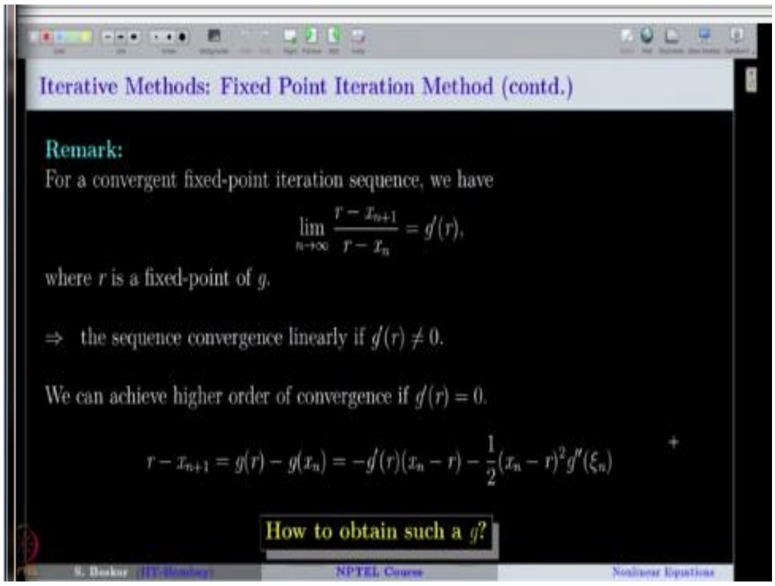

Now the question is how to obtain a g such that $g'(r) = 0$? Let us see how to do that.

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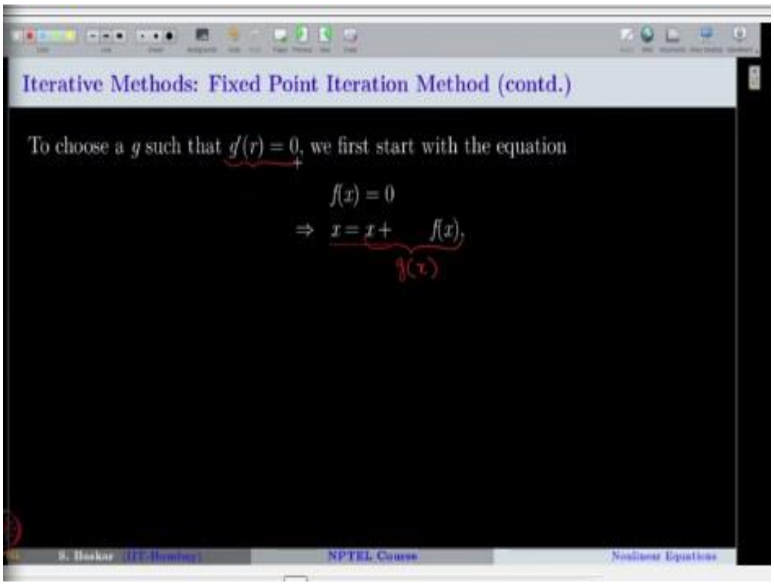

Let us start with our basic equation $f(x) = 0$. Now what you do is you add x on both sides, now this will define an iteration function $g(x)$ but what I want? I am not now interested in any $g(x)$ but I want that g such that $g'(r) = 0$.

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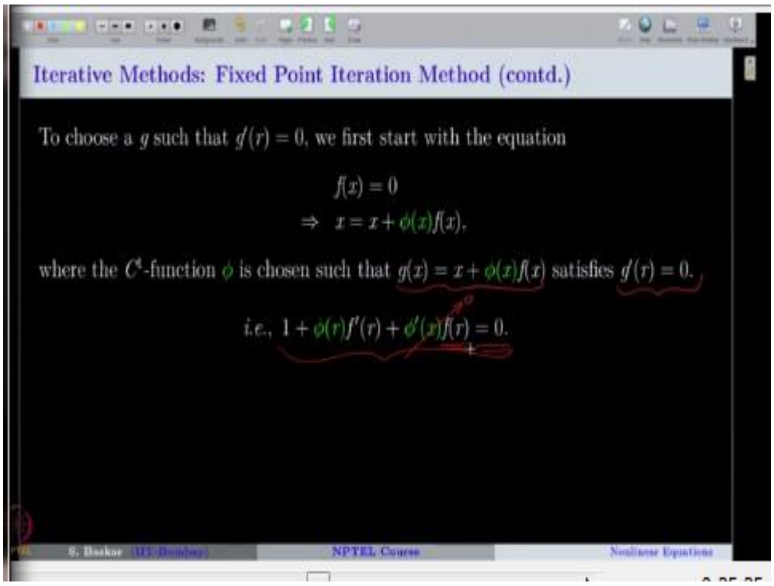

For that what I will do is I will introduce a function $\phi(x)$ here and then I will try to choose this function in such a way that $g'(r) = 0$. Let us see how to do that? I want to choose my $\phi(x)$ such that the iteration function $g(x)$ will give me the condition that $g'(r) = 0$, why? That will give me quadratic convergence in the fixed point iteration method, so that is the interest for us; let us see how to achieve that.

For that first we will differentiate this function g with respect to x and then put $x = r$ in that expression and equate it to 0, that will give us this expression. What I am doing? I am just differentiating g with respect to x, then putting $x = r$ and then equating to 0 in order to achieve this condition. In this you can notice that this term will be 0 because r is a root of the equation $f(x) = 0$.

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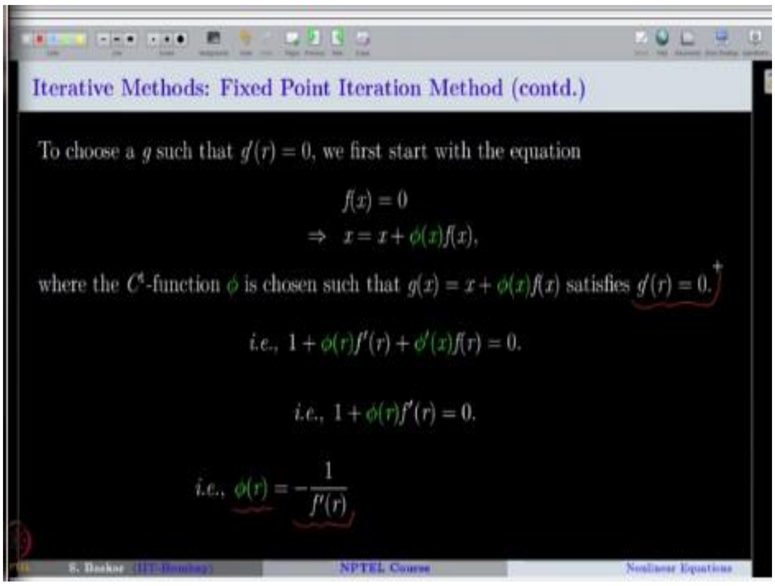

Therefore, this third term will vanish and we will have $1 + \phi(r)f'(r) = 0$. And that will give us $\phi(r)$ to be $-\frac{1}{f'(r)}$, remember this will go only when $f'(r) \neq 0$ that is when r is a simple root of our equation $f(x) = 0$, that we have to keep in mind. But when you take $\phi(r) = -\frac{1}{f'(r)}$, then we will get $g'(r) = 0$ which is of our interest.

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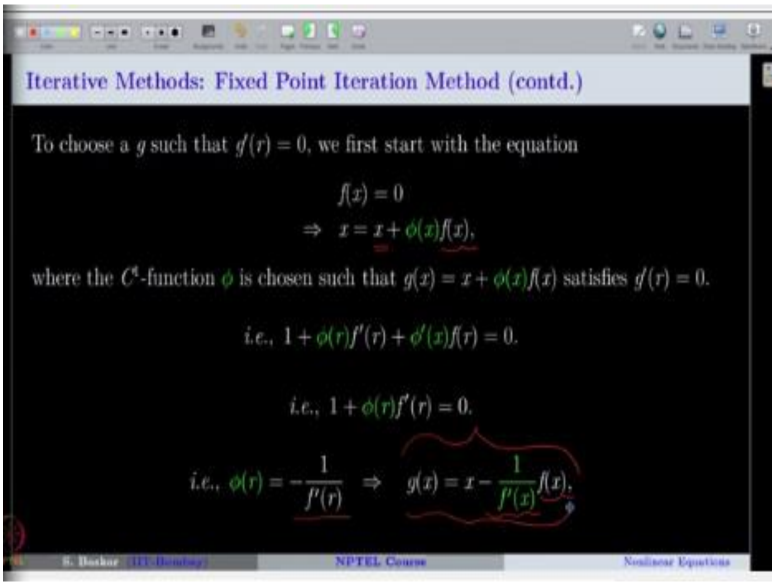

Therefore, this will motivate us to define our iteration function like this. That is $x + \phi$ is now $-\frac{1}{f'(r)}$ therefore it is $-\frac{1}{f'(x)}f(x)$. If you define your iteration function like this and if $f'(r) \neq 0$ then you will get quadratic convergence. This is another point of view where we made our fixed point iteration method to converge quadratically by appropriately choosing the iteration function. Remember, the convergence will be quadratic provided the sequence converges, that is more important.

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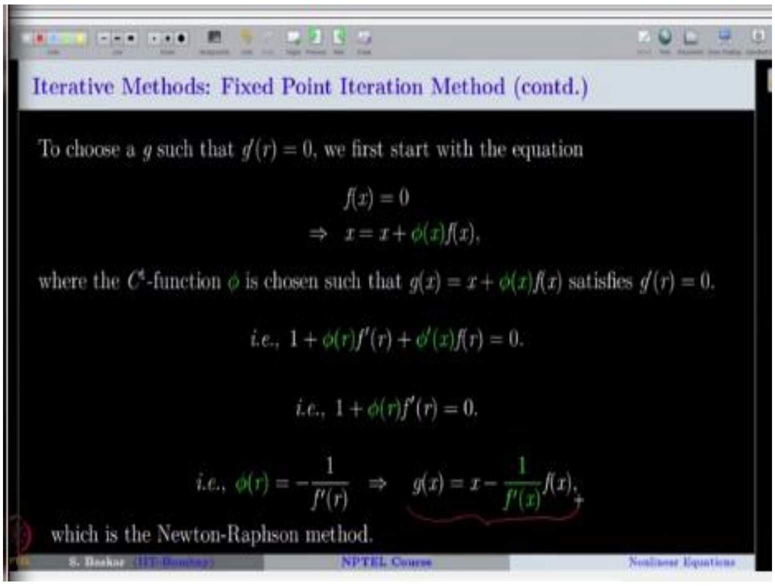

This is not something new to us because this is what precisely we learned as Newton-Raphson method.

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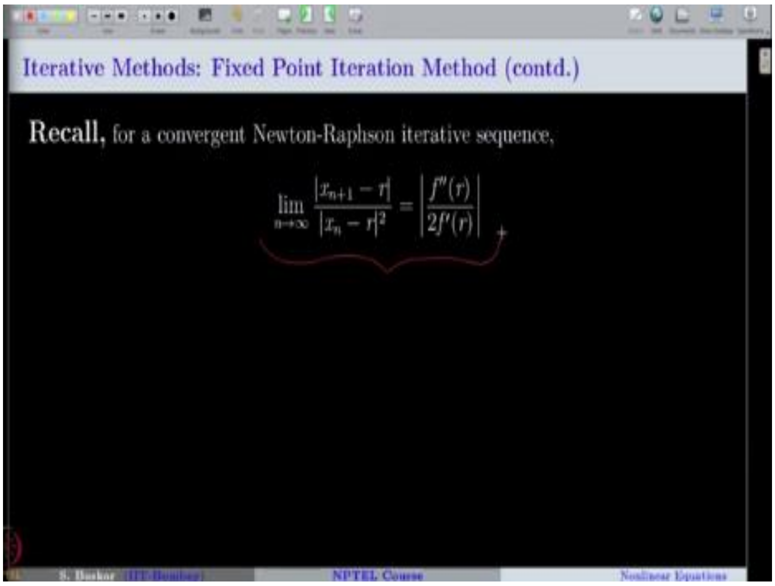

In fact, in the convergence theorem of Newton-Raphson method we have derived this expression which also tells us that the sequence with iteration function like this, that is the Newton-Raphson method will converge quadratically. You can see that this is what we have derived in that theorem. We are just putting a different point of view of what we studied in our previous classes that is all, we are not doing anything new, this is just another point of view. But why am I putting this point of view? There is a purpose for that.

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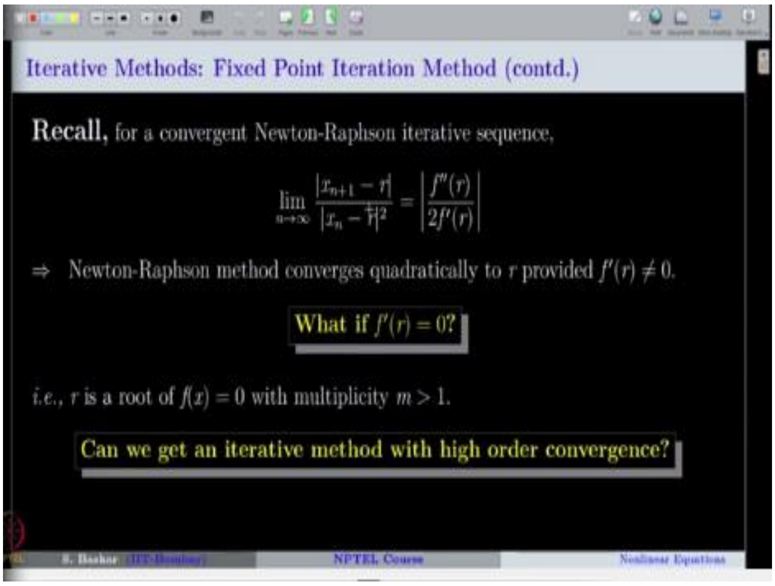

So, what we conclude? Therefore is that if $f'(r) \neq 0$ that is if r is a simple root of our equation then the Newton-Raphson method converges quadratically, we know it very well. The question now is what if $f'(r) = 0$? That is the question. Now that is to say that if r is a root of the equation $f(x) = 0$ with multiplicity some integer m which is greater than 1, then what happens? Will the Newton-Raphson method converge quadratically?

No, it may not converge quadratically. Therefore, now our next question is can we get an iterative method with higher order convergence? That is can you define another iterative method that can give you quadratic convergence when our root is with multiplicity something m greater than 1? That is our next question.

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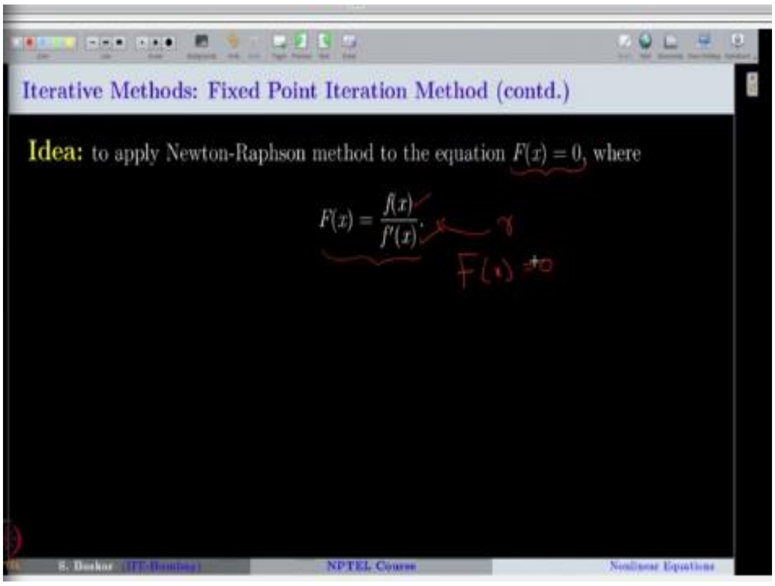

That is a nice idea to do this, the idea is look for the root of this equation $F(x) = 0$, what is F? F is given by $\frac{f(x)}{f'(x)}$. Now when you define your function F like this what is the advantage? Let us see, first you can see that r is also a root of the equation $F(x) = 0$, that you can easily see. Now the next question is what is the multiplicity of r for the function $F(x)$?

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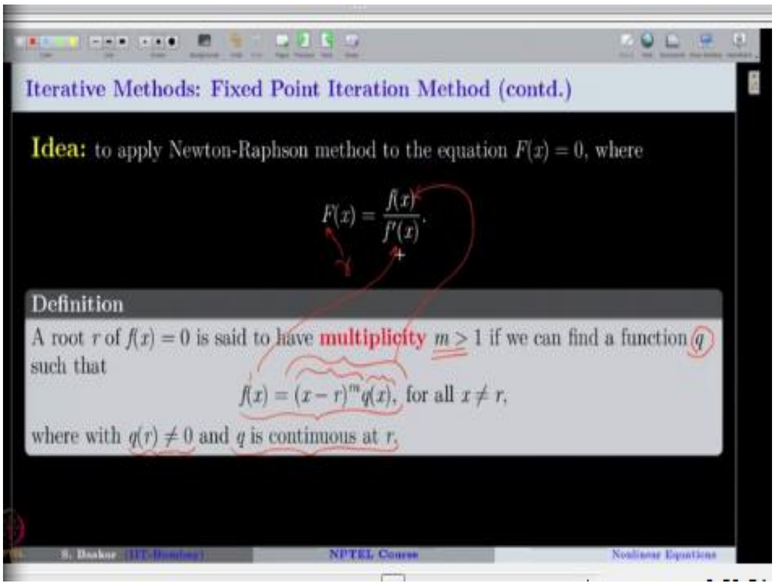

Let us recall what is the definition of multiplicity? A root r of the equation $f(x) = 0$ is said to have multiplicity m if we can find a function q such that $f(x)$ can be written as the product of this multiplicity. Remember, this is the multiplicity part and the rest of the part of the function $f(x)$ is written as $q(x)$, where q is such that $q(r) \neq 0$ and also q is continuous at r. So, this is the definition of multiplicity of a root.

What is interesting now is, to check what is the multiplicity of r for this function F? That is our aim, for that what we will do is we will take this expression and substitute in this place and also we differentiate f and substitute that in this place and see what happens?

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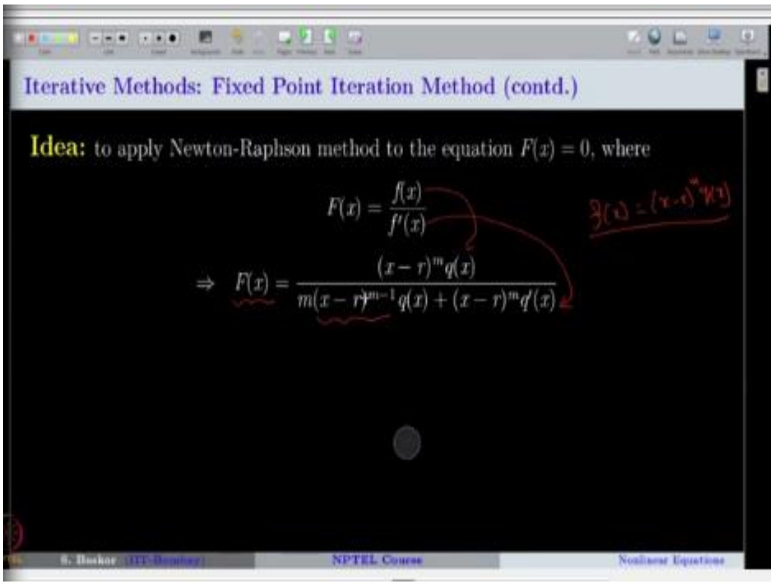

Then $F(x)$ can be given by, remember we are taking $f(x) = (x - t)^m q(x)$, that is what we have put therefore $f(x)$ is given like this. Now I am differentiating this and putting that expression in the place of $f'(x)$ in the definition of F. And that gives us this expression, you can see that you can pull out this $(x - r)^{m-1}$ and then cancel it with the numerator term.

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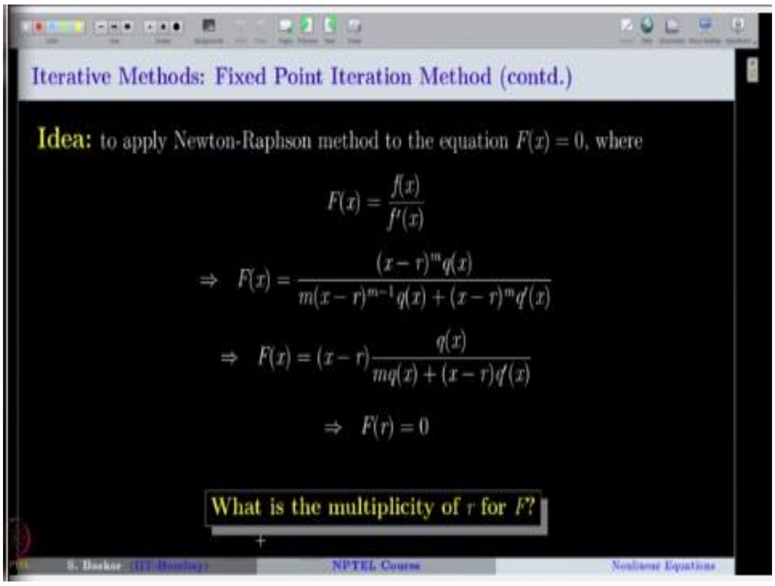

And you will get $F(x)$ equal to this expression. Now from here you can see that $F(r) = 0$, that is r is the root of this equation $f(x) = 0$. And what is the multiplicity?

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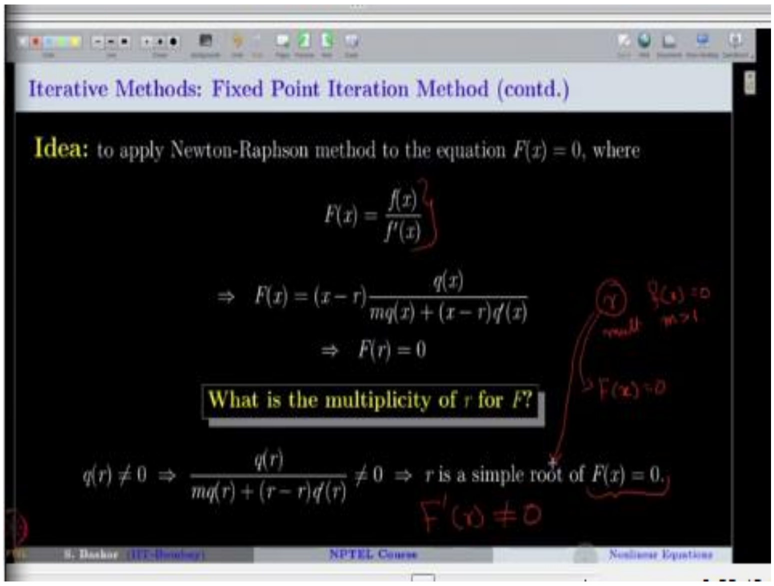

The multiplicity can be seen very easily. Since $q(r) \neq 0$, remember this is coming from the definition of multiplicity of r and that tells us that this term is not equal to 0. Therefore, you can see that $F'(r) \neq 0$ and that says that r is a simple root to the equation $F(x) = 0$. So, what we achieved here? We assume that r is a root of the equation $f(x) = 0$ with multiplicity m then you define $F(x) = \frac{f(x)}{f'(x)}$.

Then we have seen that r will also be the root of the equation $F(x) = 0$ and this will be a simple root of that equation. Now instead of applying the Newton-Raphson method to $f(x)$, you can also apply the Newton-Raphson method to $F(x)$.

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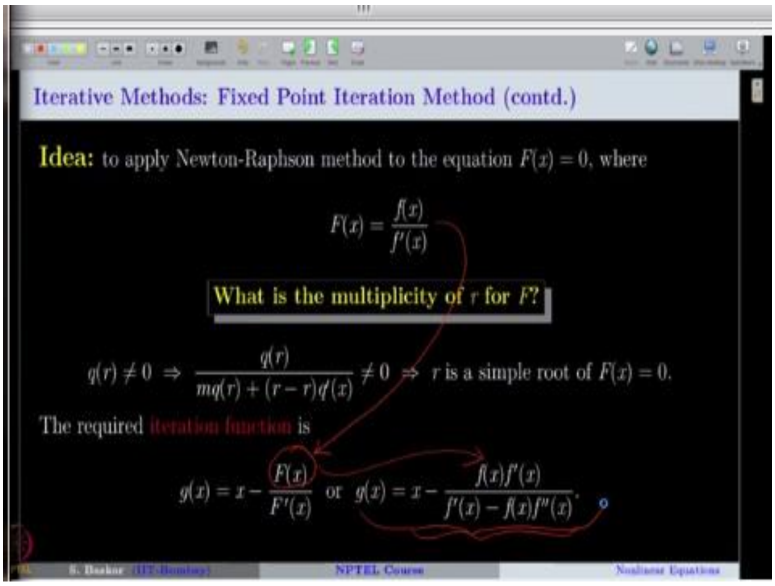

So, that is the idea, therefore you take the iteration function now as like this. Remember, this is the Newton-Raphson iteration function applied to $F(x)$ now, that is the idea. And that can be in fact written like this, that is I am putting $F(x) = \frac{f(x)}{f'(x)}$. And then just writing it here to get the precise expression for the iterative function in terms of f that is what I am trying to do.

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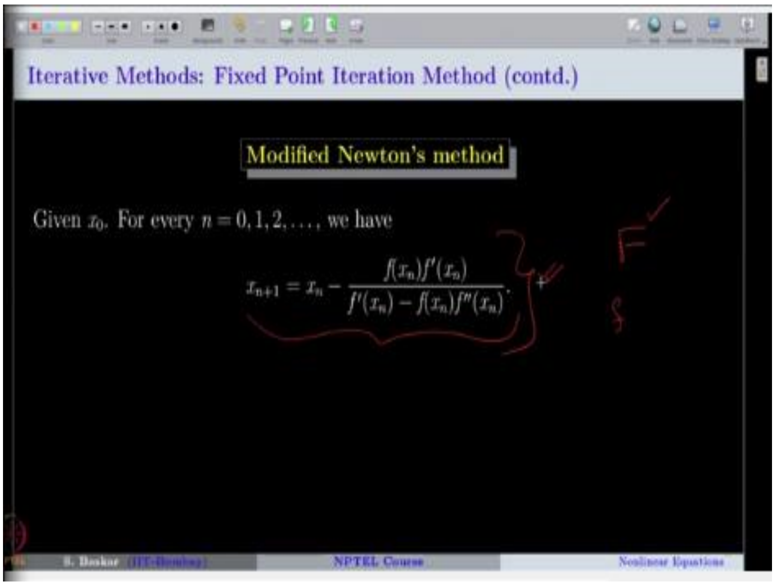

Therefore, I will now define my first modified Newton's method as this; this is my modified Newton's method. Remember, we have not done anything, we have just applied the Newton-Raphson method to F instead of f and that is how we have defined it here. Since r is a root of F and it is a simple root, you can immediately see that this iterative sequence if it converges, it will converge to r with quadratic order of convergence.

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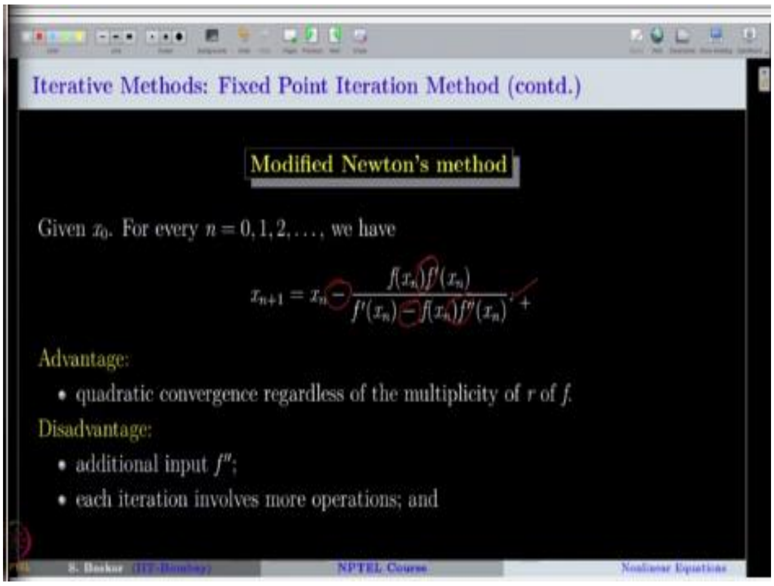

Now let us see what is the advantage of this? The advantage is that this sequence will converge quadratically even if the root r has some multiplicity greater than 1. But there are some disadvantages, what are they? First thing is you have to give $f''(x)$ as an input to your code because that is involved in your formula in addition to f'. And second thing is it also involves more operations than the Newton-Raphson method.

You can see that Newton-Raphson method has 1 subtraction and 1 division, in addition to the function evaluation. Of course, function evaluation is something extra, excluding that you have 1 division and 1 subtraction. Whereas here you can see that there are 2 subtractions, 2 multiplications and 1 division, so it involves more operations.

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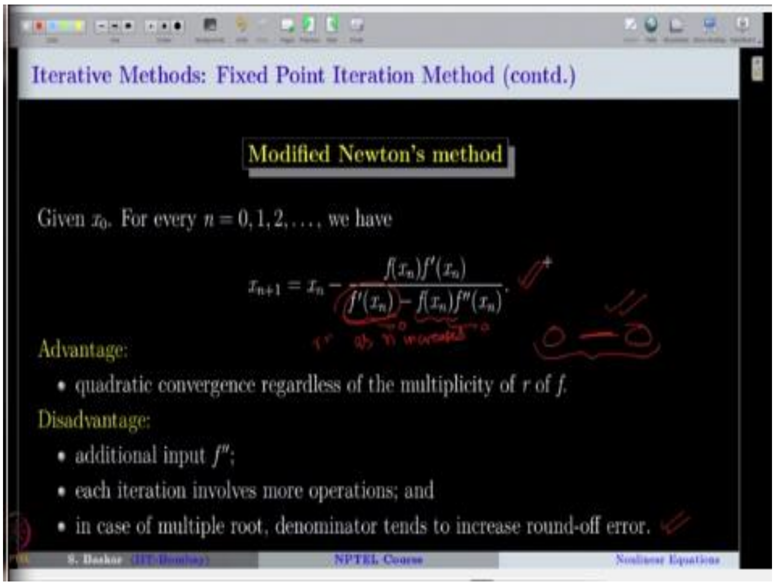

And more importantly, you can see that as n increases, this term also goes very close to 0 at the same time this will also goes very close to 0, remember why this term goes to 0? Because r is having multiplicity something greater than 1 therefore $f'(r) = 0$. So, if the convergence happens then as you go on increasing n, $x_n$will go very close to r therefore $f'(x_n)$ will go very close to 0.

Similarly, $f(x_n)$ will also go very close to 0 because r is the root of the equation $f(x) = 0$. Therefore, you have 2 terms which are going very close to each other, in fact both of them are going close to 0 and you are subtracting them. Therefore, you have the risk of having the loss of significance here, so that is going to be dangerous as we have seen in our first chapter when we do the computation with such expressions.

So, that danger is inbuilt into this formula, so these are some of the disadvantages of using this formula as an iteration sequence of our method.

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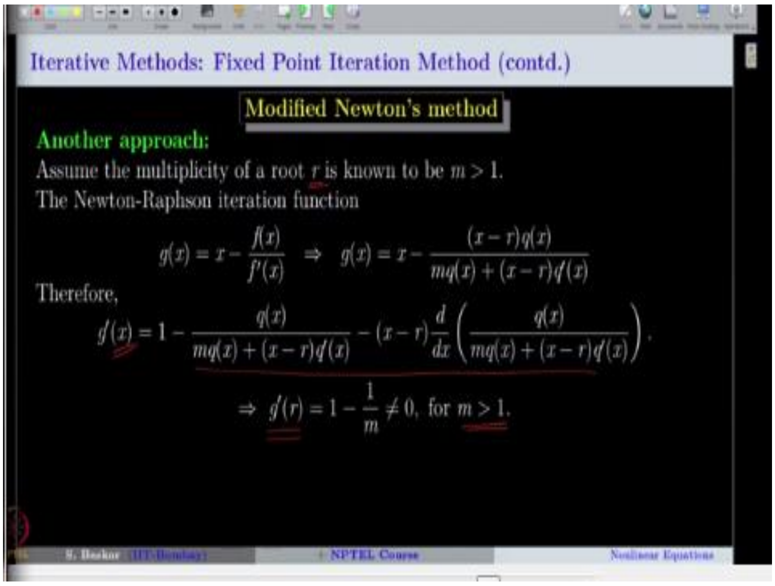

There is another idea that can avoid these disadvantages but for this we need to know what is the multiplicity of r. Therefore, in this approach we will assume that the multiplicity of r is known to us. And let us take the Newton-Raphson iterative function and since the multiplicity is m in the previous slide we saw that $g(x)$ can be written like this. And now you differentiate g and see whether it is going to be 0 or not.

If you differentiate g you get this expression and now put $x = r$ in this expression, you will see that $g'(r) = 1 - \frac{1}{m}$. Therefore, if m that is the multiplicity of r is strictly greater than 1 then $g'(r)$ is never going to be 0. So, this is also a nice way to see that if the multiplicity of r is strictly greater than 1, then the Newton-Raphson method is not going to have quadratic convergence.

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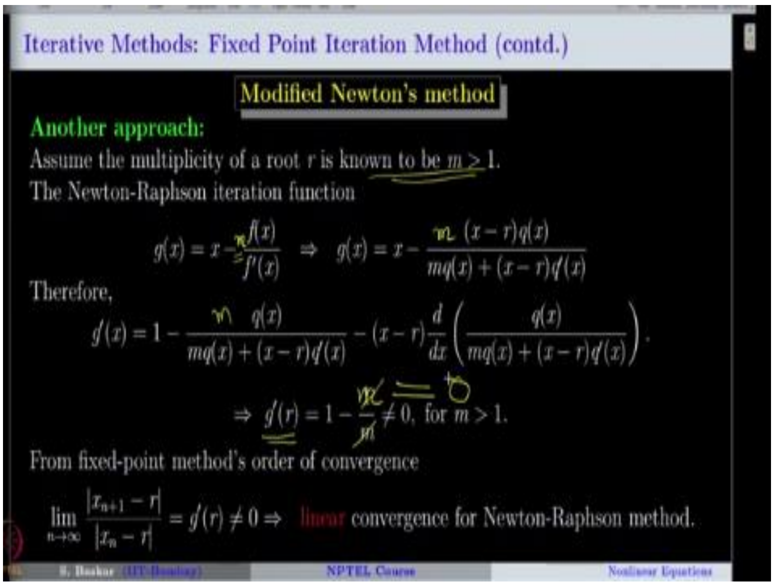

So, $g'(r)$ is not equal to 0, therefore we can only say that this Newton-Raphson method in this case will converge at least linearly, we cannot say it converges quadratically. So, that is the problem when you are working with root with multiplicity greater than 1.

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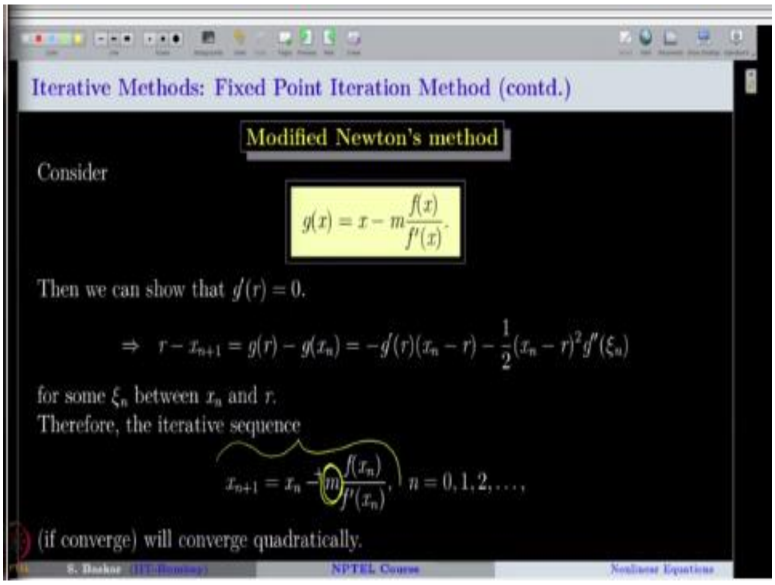

Now the idea is very clear, how to rewrite our Newton-Raphson method. You see just multiply *m* here, since we have assumed that *m* is known we can just multiply *m* here that will make $g(x)$ in this expression as *m* into this term and that will bring a *m* here. And that when you take $g'(r)$ it will bring a *m* here as well and that will get cancelled with the *m* that is already there in the denominator and that will make $g'= 0$.

Therefore, the idea is to consider this expression as the iterative function rather than the classical Newton-Raphson iterative function. That will again lead to quadratic convergence if the root *r* has multiplicity greater than 1. Because in this case you can see that $g'(r) = 0$. And again, if you recall $g'(r) = 0$ therefore we will have the quadratic convergence in this case.

Therefore, another way of defining the modified Newton's method is to take the iteration sequence with this formula where you are multiplying this *m* in the second term that is the only difference when compared to the Newton-Raphson method. But this can be done only if you know what is the multiplicity of *r*.

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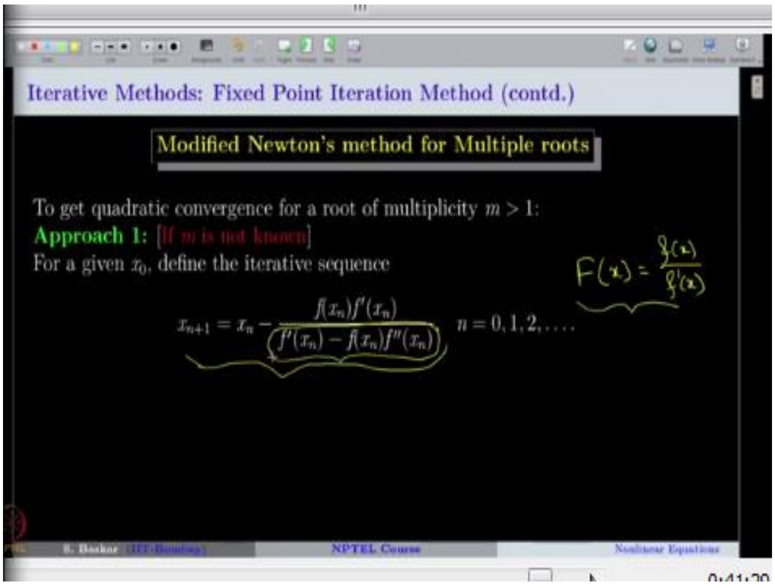

So, in the second part of our lecture we have introduced 2 approaches to modify our Newton-Raphson method in order to achieve quadratic convergence for a root having multiplicity greater than 1. The first approach is to work with $F(x)$ which is given by $\frac{f(x)}{f'(x)}$. The advantage is that we do not need to know what is the multiplicity *m*, this idea will automatically take care of it.

Now the idea is to apply the Newton-Raphson method for *F*, that is the idea and the resulting iterative method will converge quadratically, if it converges, irrespective to whatever may be the multiplicity. But it has it is own disadvantages, that we need to know F''also it involves more operations than what is involved in Newton-Raphson method. And the last disadvantage is that there is a danger of loss of significance in the denominator expression.

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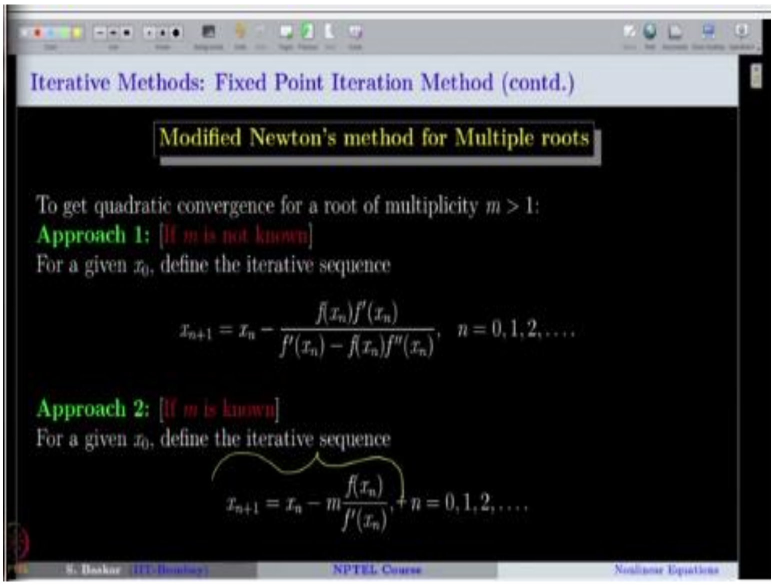

Another approach is, if you know *m* then the best way to achieve quadratic convergence is to take the iterative formula like this and that will give you quadratic convergence if the sequence converges. These are the few modifications that one can do if your root is known to be of multiplicity strictly greater than 1. There are also other modifications that one can do to achieve higher order convergence but we will restrict our discussions only to this.

You can refer some of the books like Atkinson's Numerical Analysis or Kincaid and Cheney's Numerical Mathematics or Burden and Faires Numerical Analysis books. These are some good books where you can find more modified Newton's method to achieve quadratic convergence for root with multiplicity strictly greater than 1. With this I will stop my discussion on non-linear equations here and thank you for your attention.