Numerical Analysis
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Lecture-32
Nonlinear Equations: Newton-Raphson's Method

Hi, we are discussing iterative methods for approximating an isolated root of a non-linear equation. In this we have covered bisection method, Regula Falsi method and secant method. In this lecture we will discuss Newton-Raphson's method.

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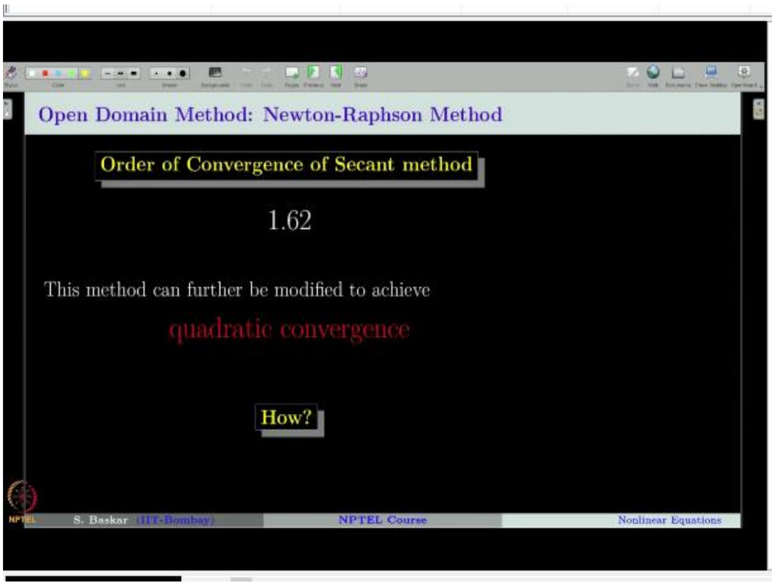

Let us first get a motivation for the Newton-Raphson's method starting from the secant method. If we recall in the last class, we have proved that the order of convergence of the secant method is 1.62. Now the question is can we modify this method to get a quadratic convergence? The answer is yes. Now the question is how to do that?

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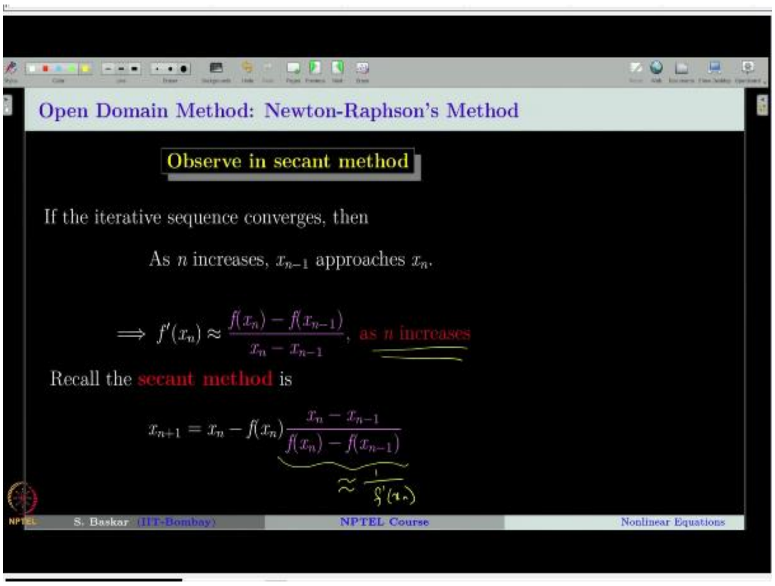

Let us quickly recall from the secant method you can see that for a large n if the sequence is converging then you can see that x_{n-1} will approach x_n. In that case the expression given like this can be more closer to $f'(x_n)$, because as n increases you can see that x_{n-1} is approaching to x_n and therefore this is actually equal to $lim\, x_{n-1} \to x_n$. Therefore, as n increases this term will be more like $f'(x_n)$.

Now look at the formula for secant method. You can see that the secant method involves 1 by this term, you can see that is what is, sitting here 1 divided by this term is what is given here. Therefore, you can see that this term is more behaving like $1/f'(x_n)$ for large n provided the sequence is converging. Now the question is why not we replace this term by $f'(x_n)$ itself right from the first term of the sequence. That is instead of having this term you just replace that by $1/f'(x_n)$.

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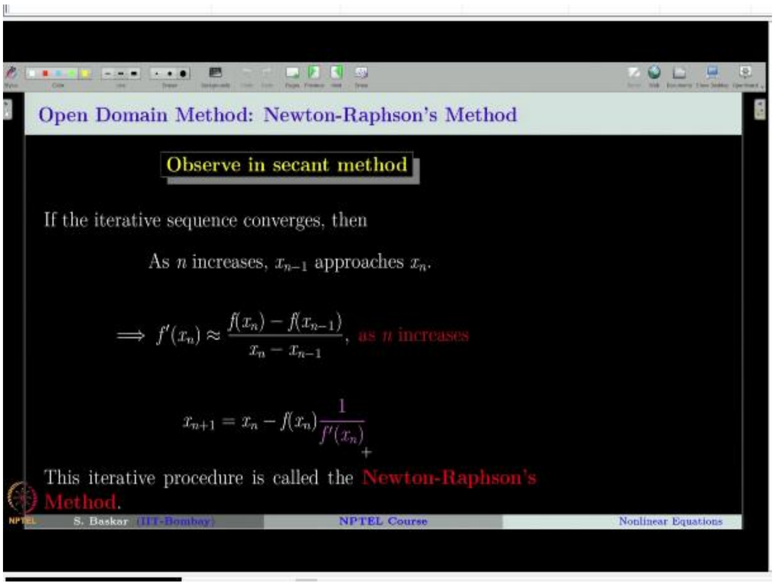

And that is what is called the Newton-Raphson's method. Newton-Raphson's method is highly preferred in the practical applications mainly because of the two reasons. One is that the Newton-Raphson's method has quadratic convergence. We will also prove this later in this lecture. Also, Newton-Raphson's method gives us a process called linearization process. This is particularly very important because in many applications we come across complicated non-linear models.

One good idea for these models is to first look for the linearized model of those models and then recovers the non-linearity through an iterative procedure. A Newton-Raphson's method is precisely based on this idea. Therefore, understanding Newton-Raphson's method is very important.

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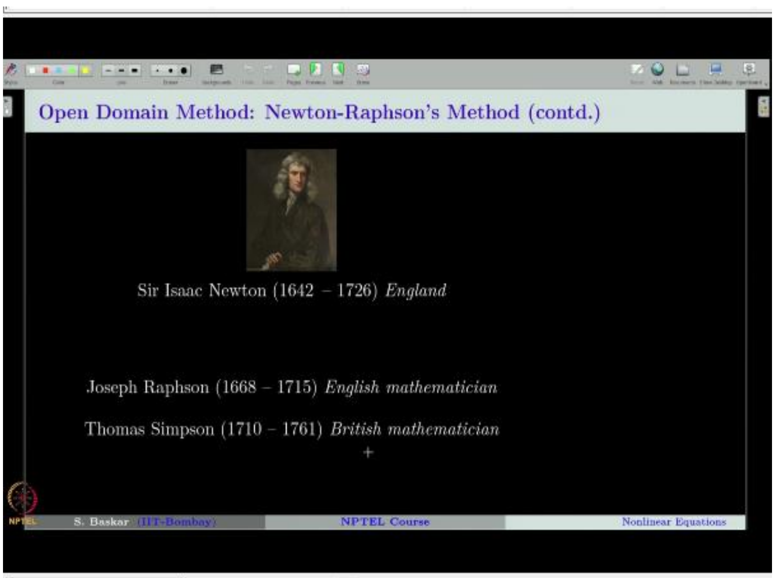

Newton-Raphson's method is first introduced by the well-known scientists Sir Isaac Newton. He formulated this method for obtaining roots for polynomials, he first started with an initial guess and then formulated sequence of error corrections and this way he obtains the roots of polynomials. He introduced this procedure both for numerical computation and also for algebraic calculations.

In the case of algebraic calculations, he in fact showed that he can recover the Taylor polynomial of a given non-linear function, but his procedure is quite complicated and that was further simplified by a mathematician called Joseph Raphson, not much is known about this mathematician, but he gave a much simpler idea of the method which was introduced by Newton.

But he also formulated the method for polynomials. Both Newton and Raphson never connected this method to calculus and gave a closed form of the formula that we have shown in the last slide, it was Thomas Simpson who connected this idea to calculus and gave the formula that we showed in the previous slide and he also introduced this method to solve any non-linear equation.

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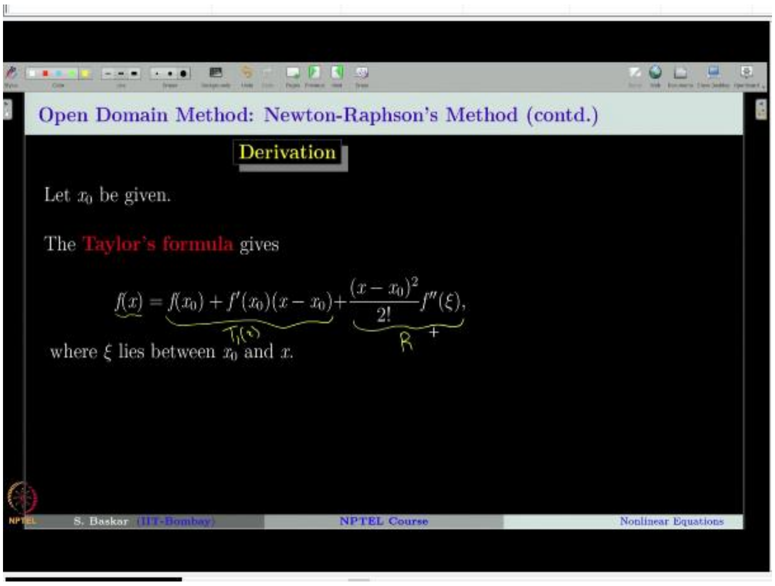

With this small historic note let us go to give a formal derivation of the Newton-Raphson's method. Let us start with the initial guess x_0. Now around that point x_0 let us write the Taylor's formula for the function $f(x)$. If you recall $f(x)$ where x is in a small neighborhood of x_0 can be written as the Taylor polynomial of order 1, $T_1(x)$ plus the remainder term.

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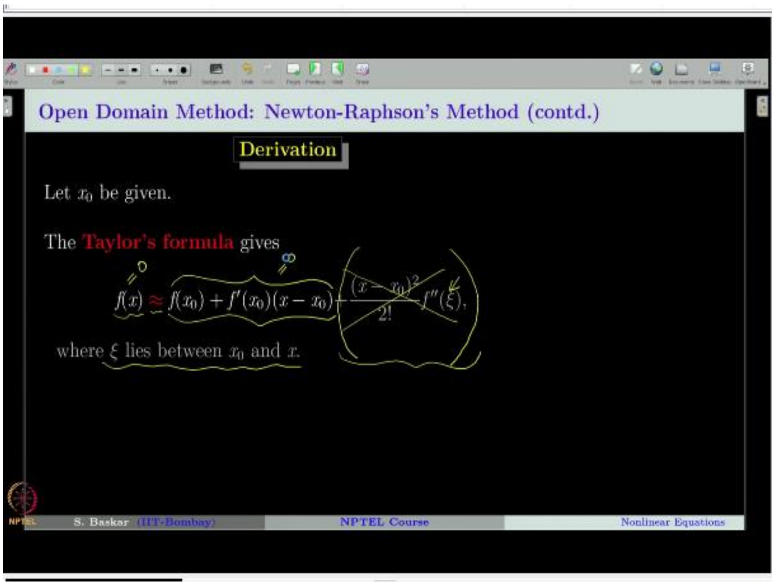

Now if you just neglect this remainder term then you can see that $f(x)$ is well approximately equal to this linear polynomial. So, this is what we meant by linearization of a non-linear function. The non-linear term is inbuilt in this expression which is unknown because this ξ is a unknown quantity lying between x_0 and x but the non-linearity is inbuilt in this expression which is neglected.

In that way we are only taking the linear part of the function f and this idea is what we call as the linearization. Now the idea is instead of looking for the root of the equation $f(x) = 0$, we will first look for the root of this polynomial $T_1(x) = 0$. You can find the root of this polynomial without any effort.

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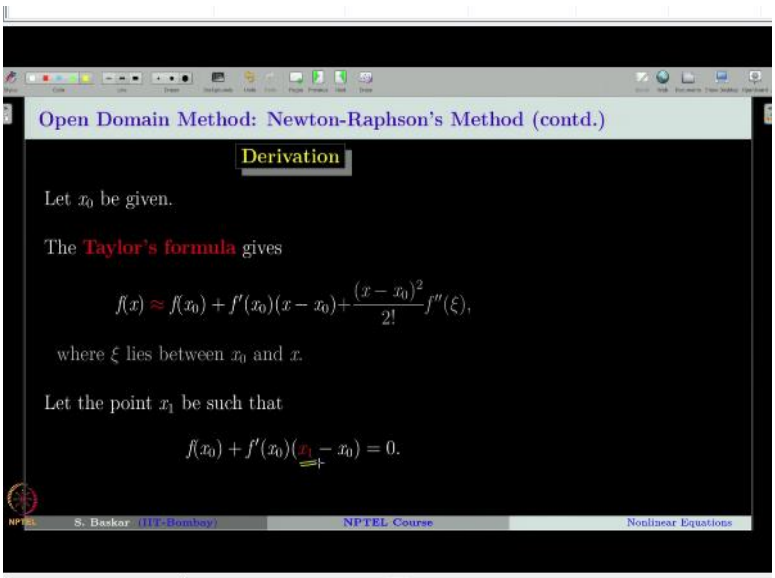

Let us call the root of this polynomial by x_1.

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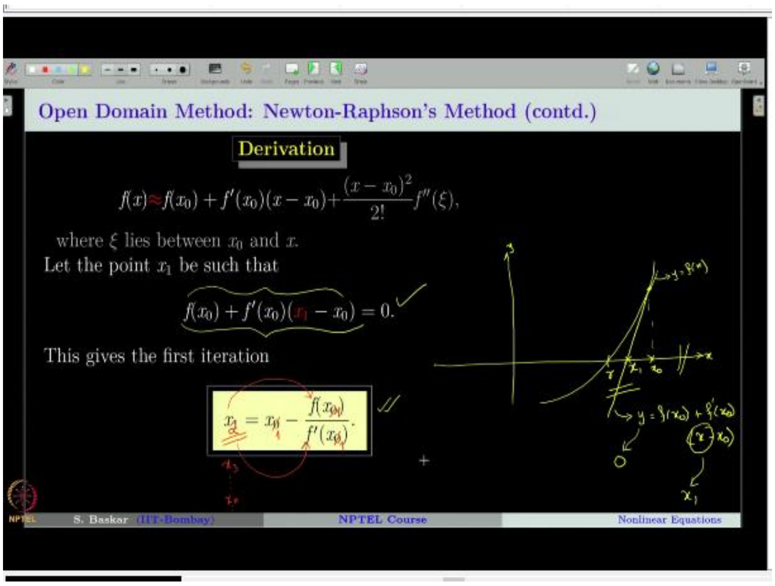

And this x_1 can be written as $x_0 - \frac{f(x_0)}{f'(x_0)}$, this can be obtained by directly rewriting this linear polynomial equation. That is how we get the first term of our iteration sequence x_1. Now from here you know how to find the next term of the iteration that is x_2; how will you find it? You will plug in this x_1 into the right-hand side expression replacing x_0 by x_1 and you will get x_2.

Once you get x_2 again you plug in x_2 on the right-hand side get x_3 and like that you can now generate the sequence x_n. Let us see the geometrical interpretation of the Newton-Raphson method. Let us take a function *f* whose graph looks like this. This is $y = f(x)$ and it has a simple root *r* here. Now what we are doing is we are starting our initial guess as x_0 and what we are doing is instead of going along this non-linear function we are going along this straight line.

What is this straight line? This straight line is precisely the tangent line at the point x_0. So, this tangent line if you recall, it is given by$y = f(x_0) + f'(x_0)(x - x_0)$. So, this is the equation of this tangent line and what you are doing precisely is taking $y = 0$, that is how you are getting this equation and correspondingly, this *x* is taken as x_1. What is the geometrical meaning of this? This is precisely the point of intersection of this tangent line with the *x*-axis.

That is x_1 is precisely the point of intersection of this tangent line with the *x*-axis. That is the geometrical interpretation of this formula. Now if you generalize this you will get the formula for Newton-Raphson's method.

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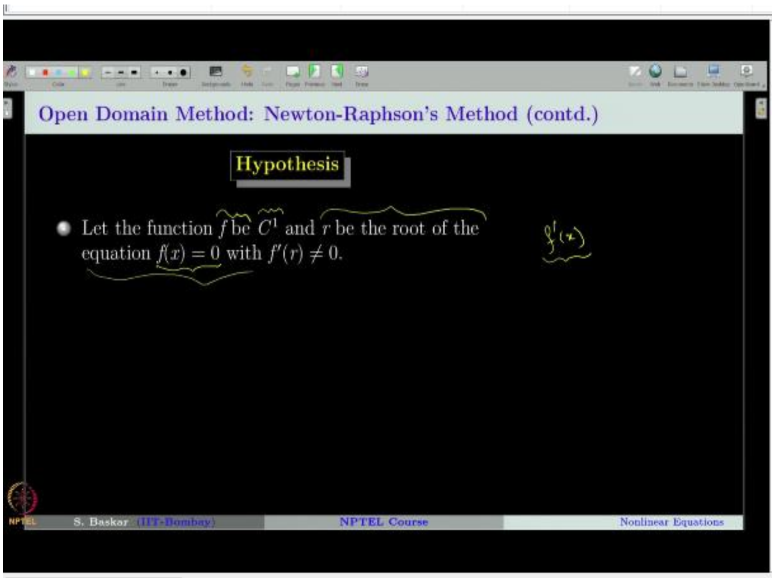

Let us put the Newton Raphson's method in the form of an algorithm. For that the inputs are the function f which defines the equation $f(x) = 0$ and since we want $f'(x)$ is involved in our formula, therefore we will assume that f' exists and for the theoretical reason we also assume that f' is also continuous function. Also, generally we are interested in computing the simple root of our equation. This is also just for the sake of theoretical purpose.

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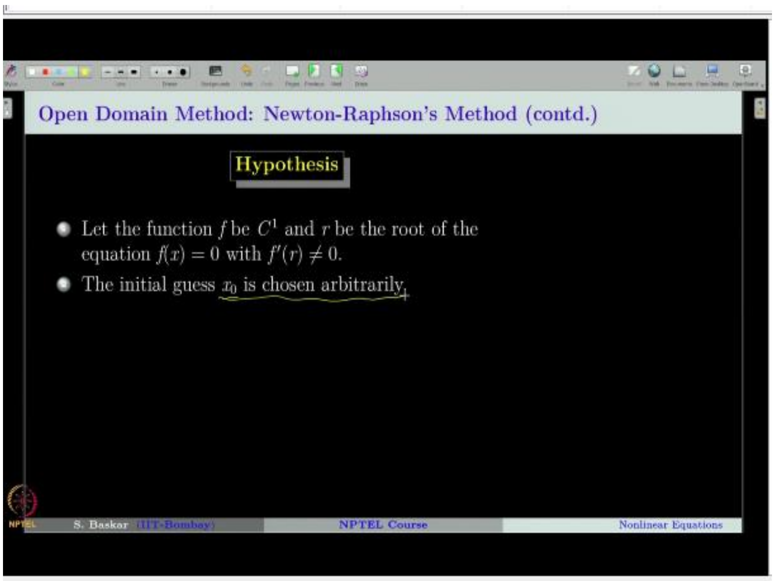

And then we have to take the initial guess that is a real number x_0. Generally, from the implementation point of view x_0 has to be chosen arbitrarily but theoretically we have to choose x_0 sufficiently close to the root r in order to have the convergence, but that cannot be achieved practically because we do not know r. Therefore, this condition is only theoretically feasible however practically we have to choose x_0 arbitrarily.

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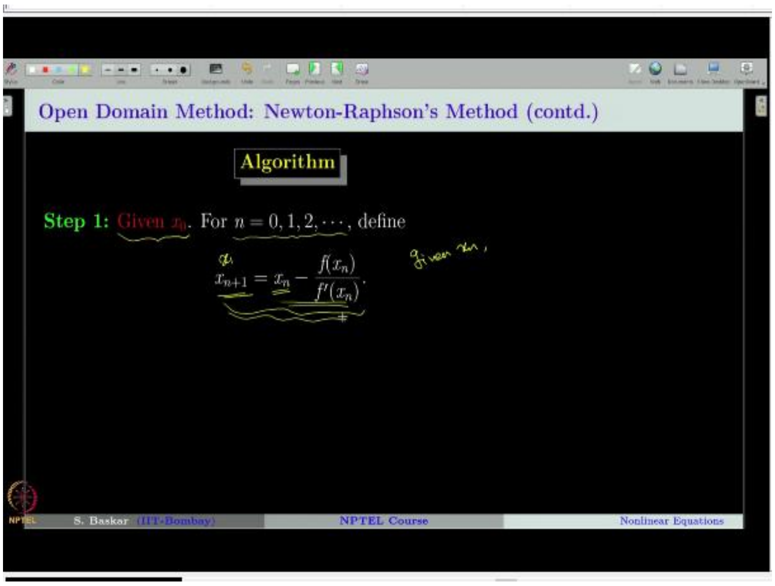

Once we are given f and x_0 then the algorithm goes like this. For any given x_0, you will compute the iterations using this formula. That is, given x_n you will find x_{n+1} using this formula, where n runs from 0 to infinity. So, if you take n = 0 you have $x_0 - \frac{f(x_0)}{f'(x_0)}$ and that gives you x_1. Then once you get x_1 you will take n = 1. Therefore, this expression will give you x_2 then and so on.

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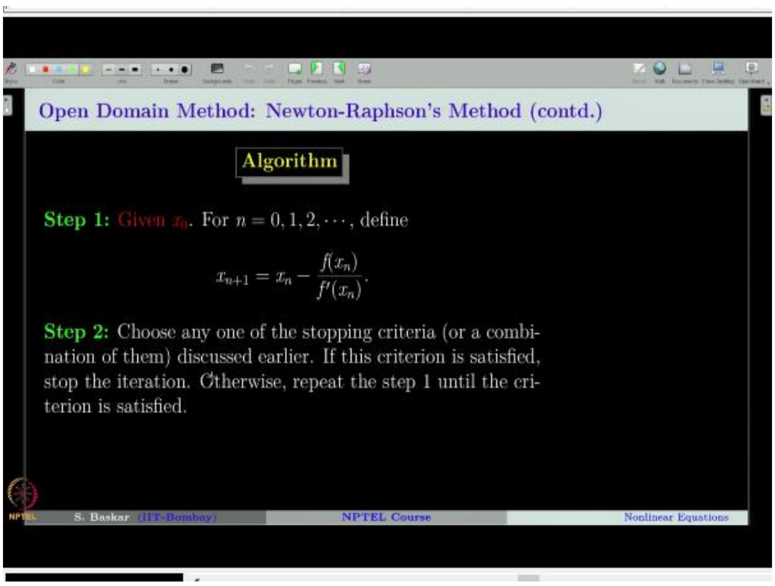

So, in this way you can generate a sequence for any given n and on a computer this procedure can go on without n, but practically we have to stop the iteration at some n. For that we need a stopping criterion. If you recall while discussing regula falsi method, we have discussed some stopping criteria in order to break our iteration computationally. One can follow any one of the stopping criteria we have introduced in that lecture.

For instance, a relative Cauchy criterion is preferred of course along with the residual error criteria also, but practically it is often enough for us to check the relative Cauchy criteria to stop the iteration.

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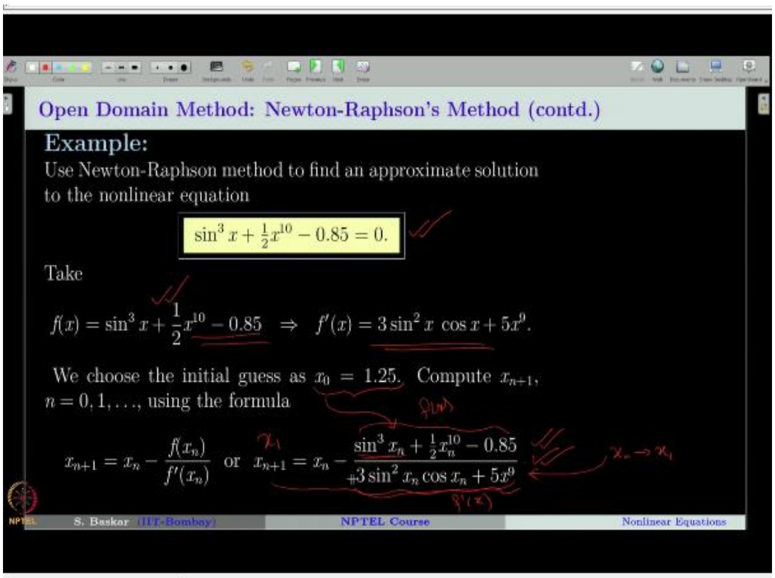

Let us take an example. Let us take this equation $sin^3 x + \frac{1}{2}x^{10} - 0.85 = 0$. Here $f(x)$ is nothing but this expression. Once this is given, we have to first compute $f'(x)$. Remember $f'(x)$ is involved in our formula. Therefore, we have to compute first $f'(x)$ and then also to start the iteration we need an initial guess. I have chosen in this example the initial guess x_0 as 1.25.

Now we are ready to set up our iterative procedure recall the Newton-Raphson's method formula that generates the iteration sequence is given by this. In our example this can be written like this; where this is our $f(x)$ and the denominator is $f'(x)$. So, that is what we have substituted from this expression with $x = x_n$ and we got the iterative formula for the given equation using Newton-Raphson's method.

Now let us go to compute the iterations; for that first take x_0 and plug in x_0 into this expression with n = 0. Remember sine x_0 is computed with radians and that will give us x_1.

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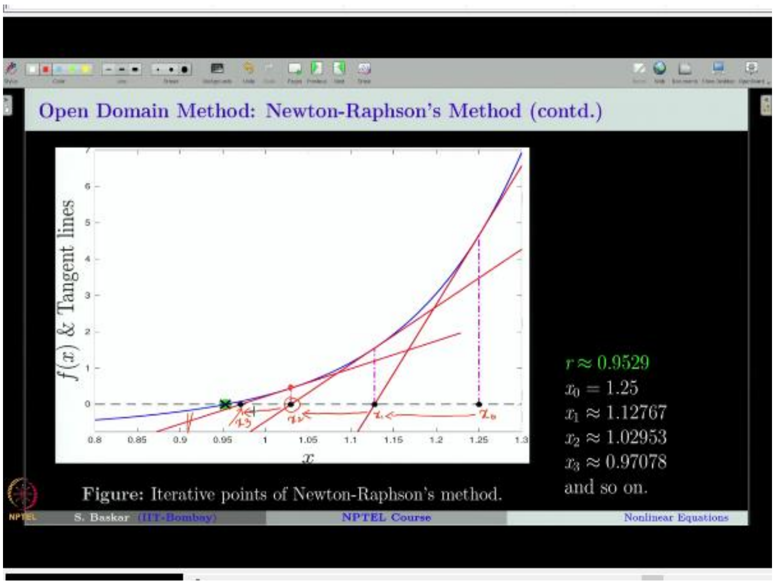

Let us also see this geometrically. The graph of the function f given like this is shown in blue solid line and the root that we are interested is highlighted here in green colour and its value is approximately 0.9529 and remember we decided that we will start our iteration with x_0 equal to 1.25 and it is here. With all this we can now compute x_1 using this expression and the value of x_1 is given like this.

And let us see the geometrical interpretation of this x_1, we started with x_0 and you have to take this point and draw the tangent line at this point. If you recall we have already mentioned this geometrical interpretation in our previous slide and what is x_1? x_1 is precisely the point of intersection of this tangent line with the x-axis and that is given by this point. Once you get x_1 you go back to that expression.

Now replace x_n by x_1 in this expression and that you can put the value of x_1 into this expression and you can use a calculator or computer to obtain the value of this expression with x_1 is equal to 1.12767 and that gives you x_2 as approximately 1.02953. Geometrically what is happening? This was x_0 where we started from there we approached to x_1 and now to get x_2 you have to take this point, draw the tangent line and see the point of intersection of that tangent line with the x-axis and that is your x_2.

This is the geometrical viewpoint but analytically you simply take x_1, substitute in the formula and get the value x_2 by computing the expression. Once you get x_2 again substitute x_2 into the Newton-Raphson's formula for our example and that gives you x_3. What is x_3? Again, you take this point x_2, draw the tangent line which is given like this and see the point of intersection of this line and that is your x_3.

Now you see you started from x_0 you got x_1 and from there you got x_2 and from there you got x_3, you can see how this sequence is converging towards the root of the equation. So, that is the geometrical viewpoint of the Newton-Raphson's method in this particular example.

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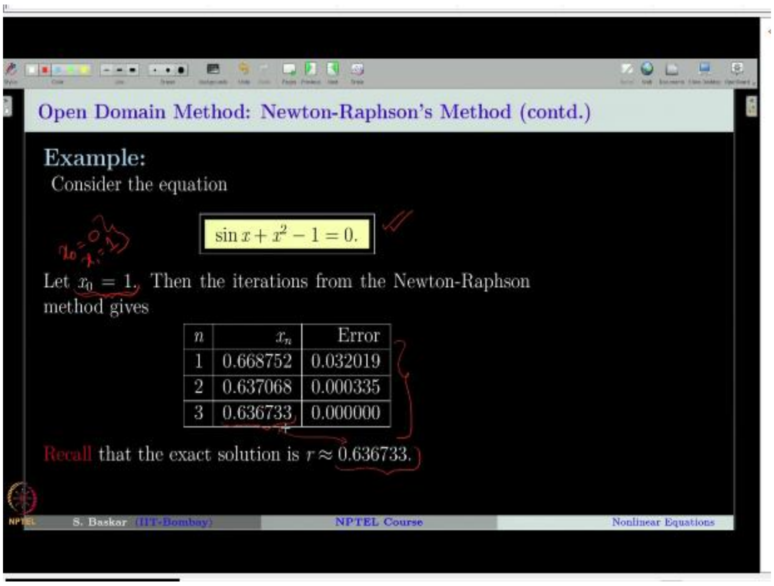

Let us go to the next example. Let us take the equation $sin\, x + x^2 - 1 = 0$ and let us take the initial guess as x_0 = 1. If you recall we have taken this example to compute a simple root using secant method also, just to compare Newton-Raphson's method with secant method I am considering the same example and if you recall in secant method, we have taken x_0 = 0 and x_1 = 1.

Recall in secant method we have to give two initial guesses. Therefore, we have chosen the initial gases like this, but in Newton-Raphson's method it is enough to give one initial guess therefore I am taking x_0 as 1. Now let us see how Newton-Raphson's method sequence goes on. I have computed first three iterations of the Newton-Raphson's method in this example starting with x_0 = 1.

You can see that in the third iteration we got the root exactly up to around 6 significant digits. One can see that the root of this equation is up to 6 significant digits is given like this and we have captured the root exactly up to this accuracy very well with just three iterations.

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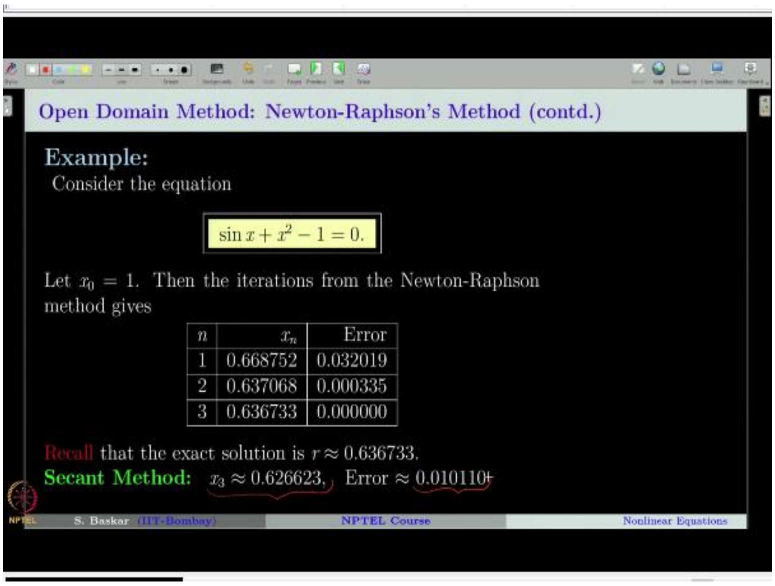

If you recall in secant method the third iteration gave us this value which is still slightly away from the root. So, this is what we meant by saying that Newton-Raphson's method is slightly faster than the secant method. If you recall we have proved that the order of convergence of secant method is 1.62. In fact, we can prove that the order of convergence of Newton-Raphson's method is 2.

That is Newton-Raphson's method has quadratic order of convergence and in this example also you can see that Newton-Raphson's method is going little faster than the secant method.

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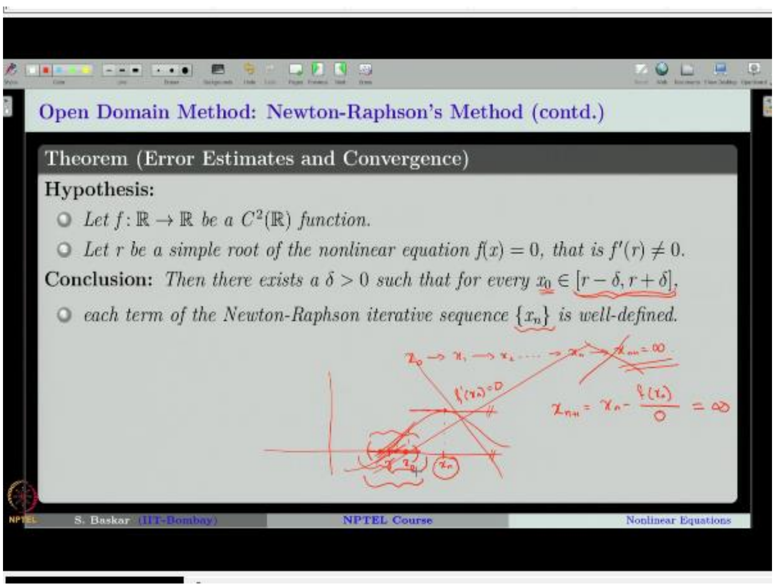

With this, let us now state the convergence theorem for Newton-Raphson's method. Let f be a C^2 function, recall to set up the Newton-Raphson's iteration we just need $f'(x_n)$. That is, we just need the first derivative of the function f, but for the convergence we need one more order of smoothness of *f* that is why we have assumed that *f* is a C^2 function and also we will assume that the root *r* which we are interested in is a simple root.

It means what? It means $f'(r) \neq 0$. If these two conditions hold for our function *f* then the theorem concludes that there exist a small neighbourhood of the root *r*. That is what we mean by saying that there existed δ such that $[r - \delta, \mathrm{r} + \delta]$ is a small neighbour δ neighbourhood of *r* and if you start your iteration that is if your initial guess is taken in this small neighbourhood remember we have to find this neighbourhood δ.

There exist means we have to find this δ neighbourhood, such that if you start your iteration with your initial guess x_0 in this neighbourhood then everything will go nicely. That is what the theorem says. What is mean by everything will go nicely let us see. First thing is each term of the Newton-Raphson's sequence is well defined, what is mean by this? What happens if you started with a x_0 then you went to x_1.

Then you went to x_2, like that you kept on going, at some x_n you see that your x_{n+1} is infinity then what happens? Then the Newton-Raphson's iteration sequence is not well defined. The theorem says that if you start your initial guess pretty close to the root then this will never happen. That is what it says. The question is when such situation will happen? Let us see suppose we are working with a function *f* whose graph is like this.

You can see that it is a nice smooth function; at least visually you can see that and suppose we are interested in capturing this root *r*. Now you started with some x_0 say and you started computing x_1, x_2 and so on at some *n* say x_n is this then what happens, your $f(x_n)$ is this and now x_{n+1} is nothing but the point of intersection of the tangent line with *x*-axis. What is this tangent line? Tangent line is parallel to the *x*-axis.

And therefore, it never intersects *x*-axis. That means you can never get x_{n+1} at all if you face such a situation. This is a geometrical illustration. What is happening analytically; you see in this case you will have $x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$. What is $f'(x_n)$ here that is equal to 0. So, this becomes 0 here and that will give you infinity. This is how the Newton-Raphson's iteration sequence may fail to exist.

The theorem says that, I can find a small neighbourhood of r as long as I start my initial guess within that neighbourhood then such a situation will never occur. That is what it says. Suppose you see if you restrict yourself to this neighbourhood and start your x_n here, x_0 here then you see your x_1 will be somewhere here and your x_2 will be somewhere here and your x_3 will be somewhere here.

Everything will lie in this interval only. That is all your x_n's will lie in this interval only and they all will be well defined. So, that is what we are going to see. So, as long as you start your iteration with the initial guess x_0 in a small neighbourhood like this then your Newton-Raphson's iteration sequence will be well defined.

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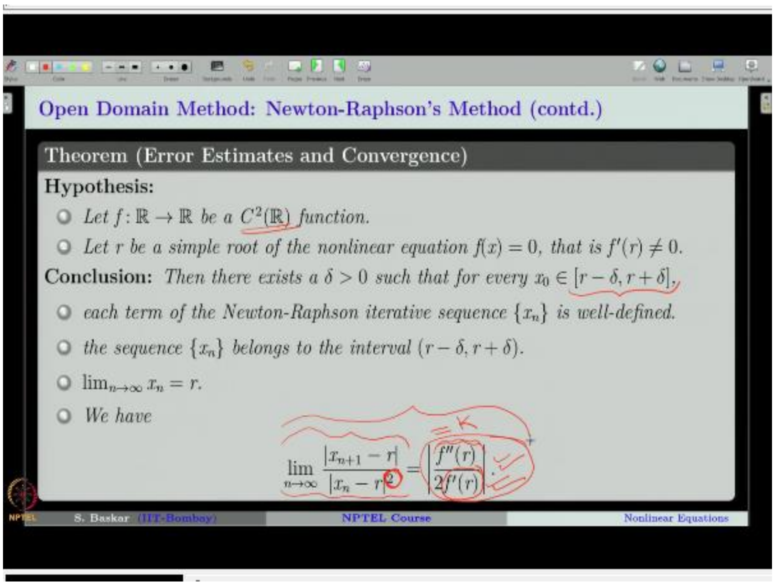

The sequence x_n will remain within this interval. This is what I have shown geometrically in the previous slide and the important point is that the sequence will also converge to the root and finally this gives us the quadratic convergence of the Newton-Raphson's method. It means what, this quantity is equal to constant, some finite number that is what we have to show in order to conclude that the sequence converges with this order.

You can see now here it is 2, therefore the order is 2. Why this term is finite because we assumed $f'(r) \neq 0$ and f'' is a continuous function. That is what we assumed here that f is a C^2 function and we are always restricting to a closed and bounded interval therefore this is a finite quantity, this is also non zero finite and therefore this entire thing will be finite.

So, if you prove this expression, it precisely means that the Newton-Raphson's iterative sequence converges quadratically. I hope you understood the statement of the convergence theorem of the Newton-Raphson's method. We will prove this theorem in the next class, thank you for your attention.