Numerical Analysis
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Lecture – 03
Mathematical Preliminaries: Order of Convergence

Hi in this lecture we will study the notion of Big Oh and Small oh of a sequence and also for a continuous function. We will also see the definition of order of convergence which is very important in Numerical Analysis.

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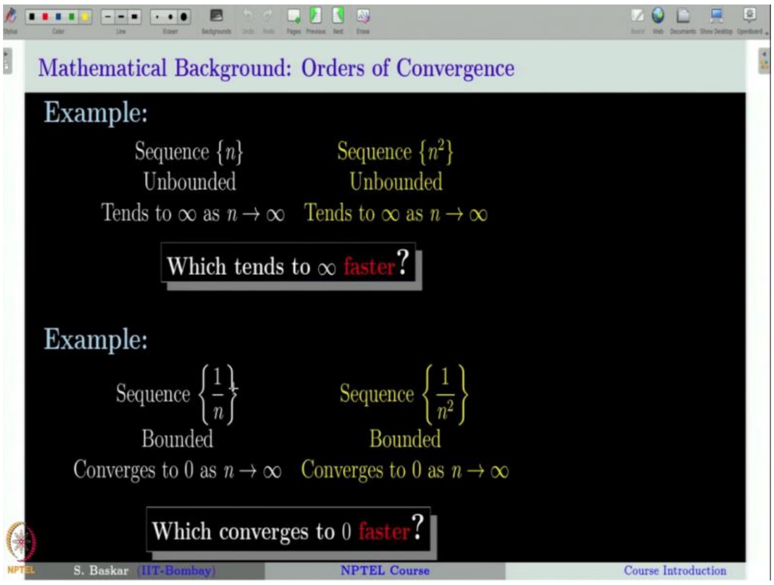

Well, let us start with an example consider these two sequences, sequence $\{n\}$ and sequence $\{n^2\}$. Both are unbounded sequences and now you can also see that both these sequences tend to infinity as n tends to infinity. Now, if you ask the question which of these two sequences will tend to infinity faster. Obviously, you can immediately tell that n^2 will go to infinity faster than n. Let us take another example.

Now we will consider two sequences $\{\frac{1}{n}\}$ and $\{\frac{1}{n^2}\}$. Now these two sequences are bounded sequences and both will tend to 0 as $n \to \infty$. Now again we will ask the question which tends to 0 faster? Obviously, you can see that the sequence $\{\frac{1}{n^2}\}$ goes to 0 faster than the sequence $\{\frac{1}{n}\}$. Therefore, we have now intuitively developed the feeling that if two sequences are converging to some limit same limit, then one may go faster than the other. This is the basic idea of this Small oh and Big Oh because we are interested in measuring who goes faster to the limit than the other.

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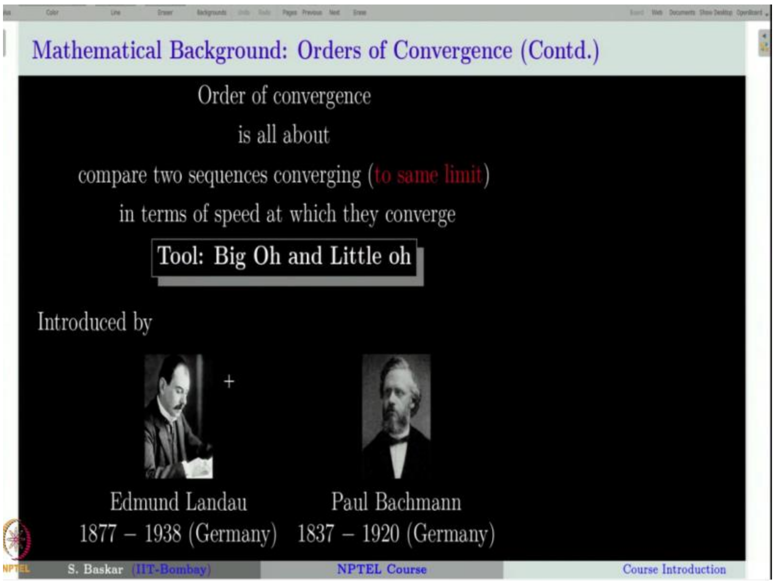

That is the question of order of convergence and it is all about comparing two sequences that are converging to the same limit in terms of the speed at which they converge and this is the basic idea of Big Oh and Small oh or little-o introduced by Edmund Landau and Paul Bachmann.

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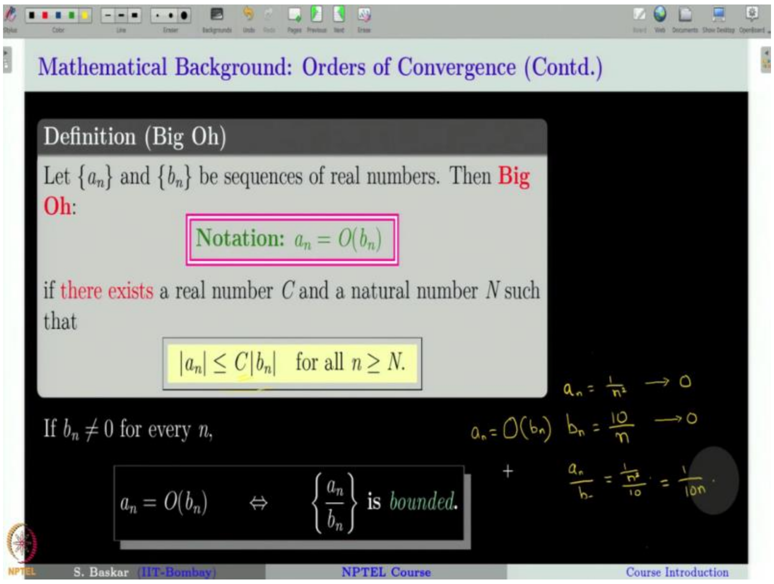

Let us see the definition of Big Oh in the context of sequences. Assume that we have two sequences; sequence $\{a_n\}$ and $\{b_n\}$ both are sequences of real numbers then we say that $a_n = O(b_n)$ if there exist a constant C and the natural number N, such that $|a_n|$ is always less than or equal to constant times $|b_n|$ and this should happen at least for sufficiently large N that is what we meant by saying that there exist a natural number N such that the condition holds for all $n \geq N$.

Now what this definition tries to say? Well, we will just put this definition in a different form and see what it means. We can do this if all b_n at least for sufficiently large N are not equal to 0 then you can see that $\frac{|a_n|}{|b_n|} \leq C$ that is equivalent to saying that the sequence $\{\frac{a_n}{b_n}\}$ is bounded. Let us take a simple example let $a_n = \frac{1}{n}$ and $b_n = \frac{1}{n}$.

We know that both these sequences are going to 0. Therefore, you may just think that $\frac{a_n}{b_n}$ may go to infinity because b_n is going to 0, but that is not going to happen obviously because when b_n is going to 0 simultaneously a_n is also going to 0 and at what speed both of them are going in the same speed. Of course, you can see it with simple calculation this is equal to 1/*n* divided by 10/*n* that is equal to 1/10.

And that is the constant that is appearing here. In this particular case you can in fact say that a_n is equal to $10|b_n|$ there is no need to put mod in this particular example and that is what this says. Therefore, a_n in this example is a Big Oh of b n. Similarly, you can also take another example where a_n is n^2 and b_n is say $\frac{10}{n}$ then you have $\frac{a_n}{b_n} = (n^2)/\left(\frac{10}{n}\right)$ that is equal to $\frac{n}{10}$.

And in fact this is going to 0. Therefore, you can always bound it by any number say for instance 1 also you can take as C and say that for sufficiently large N, $\frac{|a_n|}{|b_n|} \leq 1$. So, in this case also you can say that $a_n = O(b_n)$. In fact in the second example you can say something more than merely what happened in the first example.

In the second example a_n is in fact going much faster than b_n that is the basic idea of Small oh.

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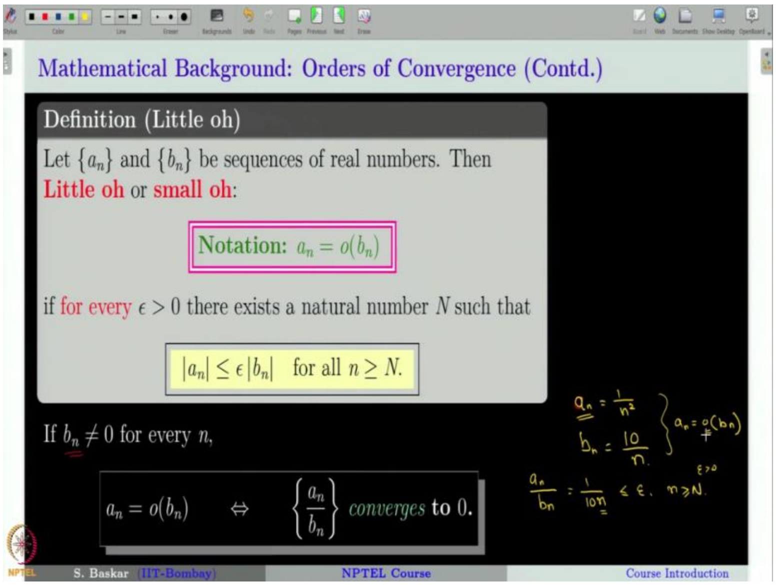

Let us define Small oh now. Again you have two sequences $\{a_n\}$ and $\{b_n\}$. Now we say that $a_n = o(b_n)$, if for every $\epsilon > 0$ there exist a natural number N such that $|a_n| \leq \epsilon |b_n|$ for all $n \geq N$. So, in this case what we are doing is we have a sequence a_n and we are saying that you give me any ϵ, see that is more important you give me any ϵ, I am giving you guarantee that my a_n the sequence will become less than that small number that you gave into b_n at least for sufficiently large N that is what we are saying it may not happen right from the first term of a_n, but you go after 10 or 15 or some sufficiently large small n then this will surely happen that is what we are saying and you are pretty confident about whatever ϵ that somebody gives you that is what the definition says you give me any ϵ, I can pack this sequence in this.

But still you are saying that I can make my a_n something less than or equal to ϵ times this. So, you will have that confidence only when that $\frac{a_n}{b_n}$ is converging to 0 then only it can happen. So, that is what the definition says that your sequence $\frac{a_n}{b_n}$ is intuitively if you see it is converging to 0 that is what in a particular case when all this b_n's are not equal to 0 at least for sufficiently large N.

This is equivalent to saying that $\frac{a_n}{b_n}$ is in fact converging to 0. If you recall in the last slide we have taken an example where $a_n = \frac{1}{n^2}$ and $b_n = \frac{10}{n}$. You can see that a_n is going to 0 much faster than b_n. So, if you take $\frac{a_n}{b_n}$ you will see that it is $\frac{1}{10n}$. Now you give me any $\epsilon > 0$, I can always find a sufficiently large N such that this is less than or equal to ϵ for all $n \geq N$.

Therefore, in this case $a_n = o(b_n)$. So, the notation Small oh means a_n has to surely go faster to the limit than b_n then only you will say that $a_n = o(b_n)$ whereas in Big Oh either a_n should go faster or equally as fast as b_n then also you can say that $a_n = O(b_n)$ that is a subtle difference between Big Oh and Small oh.

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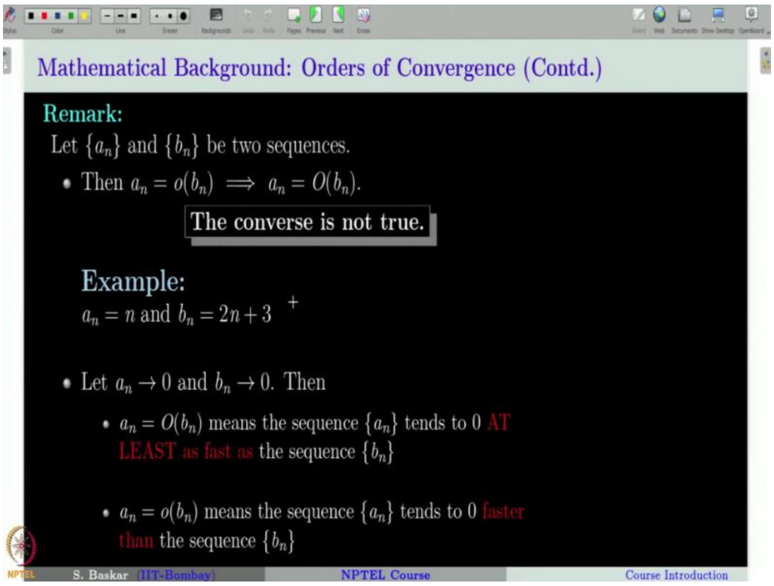

So this is what we will remark, you have two sequences a_n and b_n. You can see that $a_n = o(b_n)$ means what a_n is definitely going faster than b_n that also means a_n is going as fast as b_n because going faster than b_n is something more than that, therefore, a_n is also big Oh of b_n. So, if a_n goes faster than b_n then it is also $a\_n = O(b_n)$ because we are demanding more stronger conditions here than this that is why if this happens this will surely happen, but the converse is not true.

You can take many examples the first example that we have given in the previous slide is also an example where $a_n = O(b_n)$ because a_n and b_n are going at equal speed that is if you recall we have taken $a_n = \frac{1}{n}$ and $b_n = \frac{10}{n}$. Remember, in this case b_n is always staying ahead of a_n. So, it is not the position that matters it is the speed at which they tend to 0 matters.

You can see that $\frac{a_n}{b_n} = \frac{1}{10}$ and that is a fixed number. Therefore, this cannot go to 0. Therefore, in this example $a_n = O(b_n)$, but definitely $a_n \neq o(b_n)$. The same idea is also given in this example. We are taking $a_n = n$ and $b_n = 2n + 3$. Now, let me summarize what we will learn from Big Oh and Small oh notation. We have two sequences in particular let us take both the sequences are converging to 0.

Then $a_n = O(b_n)$ means the sequence $a_n \to 0$ at least as fast as the sequence b_n that is what is meant by Big Oh and small oh means that $a_n \to 0$ definitely faster than the sequence b_n this is Small oh. So, I hope you have understood the notation of Big Oh and Small oh and the subtle difference between them. Once you understand this you can also extend this idea to continuous functions also.

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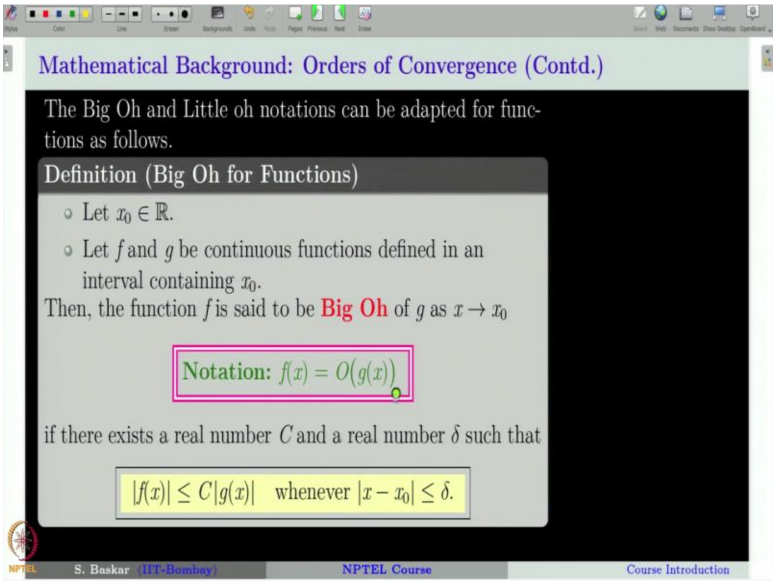

Let us give this two definitions in terms of functions. Consider a point $x_0 \in R$ and f and g are continuous functions defined in a small neighborhood of x_0 then we say that f is Big Oh of g as $x \to x_0$ this is very important all this oh notations are defined as x tends to something. Therefore, you should always write this where this x tending to in order to compare one function with the other function.

Notationally we will write $f(x) = O(g(x))$ as $x \to x_0$ that also one should write. The definition is almost similar as we did with the sequences when can we say this that when there exist a constant and a real number δ such that $|f(x)| \le C|g(x)|$ whenever $|x - x_0| \le \delta$. What it means, I do not want this condition to happen everywhere on the real line, in a small neighborhood of x_0 if it happens that is enough because we are only worried about $x \to x_0$.

Therefore, as you go closer and closer to x_0 this should happen If that is the case then we say that $f(x) = O(g(x))$.

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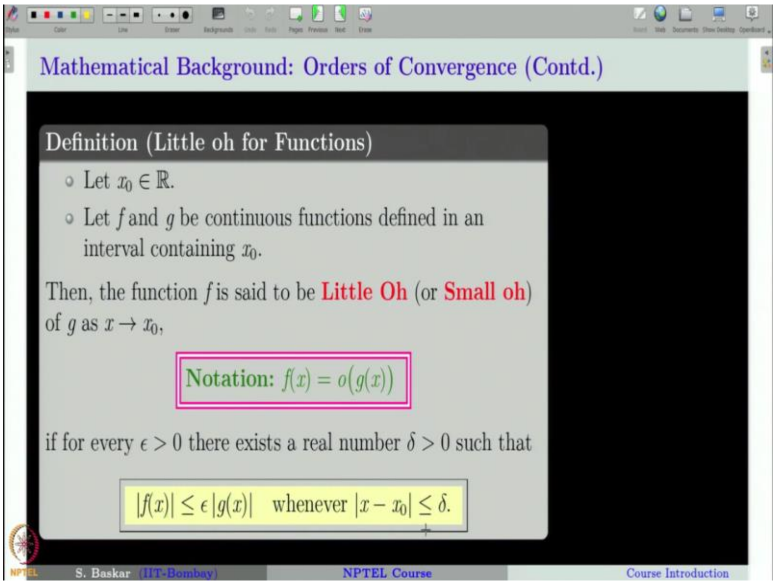

Let us similarly define the notion of Small oh again. Let x be some real number and f and g are continuous functions defined in a small neighborhood of x_0 then we say that f is Small oh of g as $x \to x_0$. Notationally we write as $f(x) = o\big(g(x)\big)$. When this happens, if for every $\epsilon > 0$ there exist a real number $\delta > 0$ such that $|f(x)| \leq \epsilon|g(x)|$ whenever $|x - x_0| \leq \delta$.

Again, this condition should happen in a small neighborhood of x_0 that is what is important here and again you can see that the role of this ϵ which says that $\frac{f(x)}{g(x)}$ should tend to 0 as $x \to x_0$ and that is provided if $g(x) \neq 0$ otherwise that is what this inequality means.

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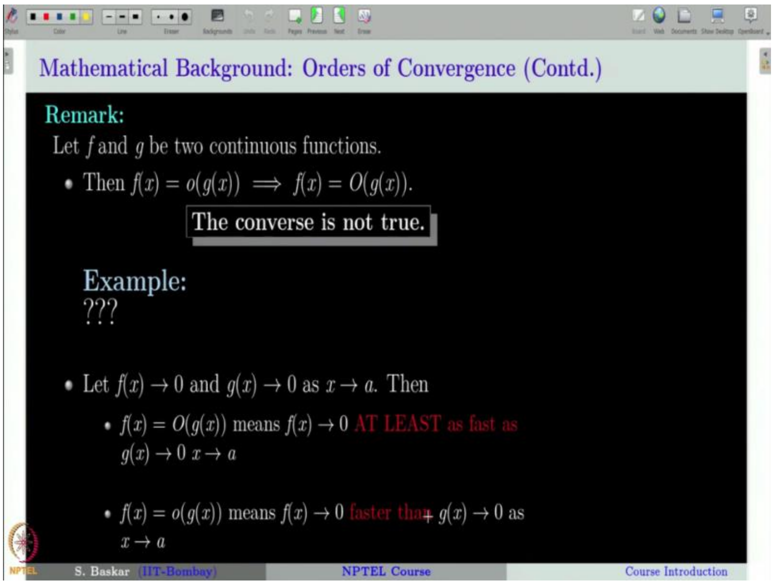

Again the same kind of remark that we made with sequence will hold here. We have two functions f and g which are continuous functions defined in a small neighborhood of x_0 then you can see from the definitions of Big Oh and Small oh that $f(x) = o(g(x))$ will always imply that $f(x) = O(g(x))$, but again the converse is not true. You can take examples in a similar way as we did with the sequences.

So, I leave it to you to think about various examples for why the converse is not true. Again let us summarize, suppose you have two functions f and g and we know that $f(x) \to 0$ and also $g(x) \to 0$ as x tends to some point say a then we say that $f(x) = O(g(x))$ means $f(x) \to 0$ at least as fast as $g(x) \to 0$ as $x \to a$ that is what is meant by Big Oh and what is meant by Small oh?

Small oh means f should go to 0 definitely faster than $g(x)$ going to 0 as x tends to some point say a or x_0 in the previous definition we have taken. These notations are very important in numerical analysis and why in particular we are more interested in taking whether a sequence or a function tending to 0 because we will be studying errors for various methods.

And we wish our errors to go to 0. Now, our interest is to see how fast this error goes to 0 as some parameter tends to something. This is what we will be interested in the subject and therefore this notations will come quite often and it is very convenient for us to use this notations than every time telling what they actually do.

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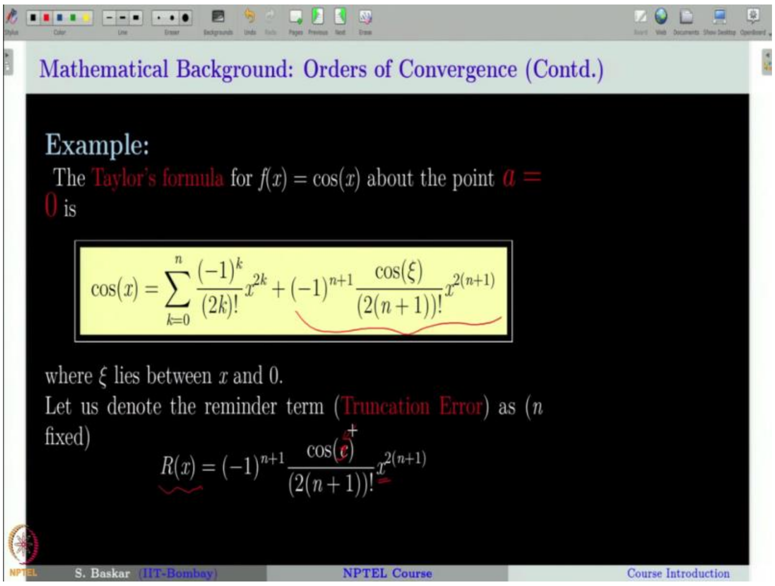

Let us take an example. Well this example is familiar because we have discussed in our previous lecture. Let us take the function $f(x) = cos(x)$ and let us try to write the Taylor formula for cos x around the point $a = 0$. If you recall this is given by the Taylor polynomial given by this plus the remainder term given like this. Now, if I want to use the Taylor polynomial instead of cos x then what is the error that I am committing in that representation. Well, that is precisely this remainder term or the truncation error.

This is what we have seen in the last class. Unfortunately, this is unknown expression because of this quantity ξ which generally we do not know, but we only know that ξ lies between x and 0. So, that makes this expression to be unknown. Now our interest is to understand how fast does this error goes to 0 for that let us just define this remainder term or the truncation error as a function of x.

Let us call this as $R(x)$ and sorry this maybe ξ here so $R(x)$ is a function of x in fact this ξ also depends on x as x changes the ξ will also change. Therefore, it is really a complicated function.

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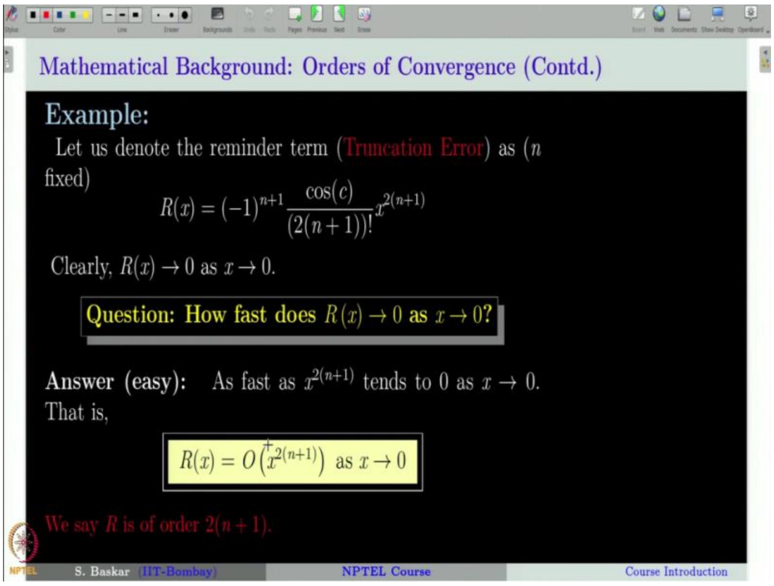

Now our question is as x tends to 0 what will happen to $R(x)$. Well, it is very clear that as x tends to 0, $R(x)$ will also go to 0. Why? Because although this term cos ξ is unknown, but definitely we know that it is bounded. Therefore, this quantity is some bounded quantity it means it remains constant and this term is going to 0. The first one is bounded and the second this one is going to 0.

Therefore, the entire thing will go to 0 as $x \to 0$ that is quite clear. Now, our interest is to see how fast it goes to 0. Well, that is not a very difficult question because we can understand how this term goes to 0. So, you can immediately say that $R(x) \to 0$ as fast as $x^{2(n+1)} \to 0$ as $x \to$

0. So that is what we meant by saying that $R(x)$ this is the function just you compare this with the definition.

We had two functions f and g and we say that $f(x) = O(g(x))$. The same instead of f here I have R and R is Big Oh of what this function. So, just take this as $g(x)$. So, $R(x) = O\left(X^{2(n+1)}\right)$ as $x \to 0$. This is something like you are comparing your error with something which you know very well how it behaves. See this you know how it behaves intuitively you can imagine how fast it goes to 0. Now what you are saying is my truncation error in Taylor series is going as fast as this function goes to 0 as x tends to 0.

Often we say that the function R of is of order 2(n+1). This kind of words are quite often used in numerical analysis to say how our error goes to 0, how fast the error goes to 0. Suppose, you say that my error is going to 0 with order say 5 it means your error is Big Oh of x^5 or similarly you can interpret in terms of the Small oh also.

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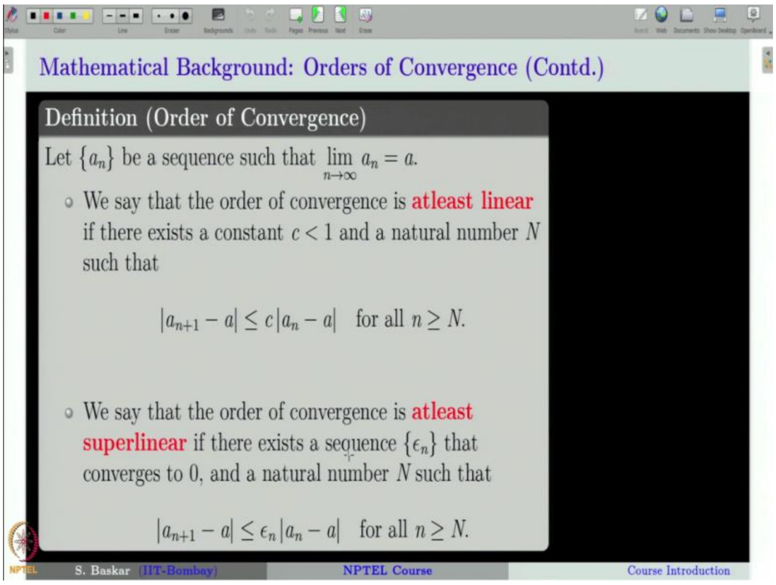

Finally, we will quickly define the notion of order of convergence you have a sequence $\{a_n\}$ and you know that the $\{a_n\}$ is converging to some limit say a then we say that the order of convergence. Again you can see that the order of convergence is something to understand how fast this sequence is going to a. So, we say that the sequence going to a with order of convergence at least linear, if we can find a constant $c < 1$ and the natural number N such that $|a_{n+1} - a| \leq c|a_n - 1|$ for sufficiently large n that is $n \geq N$ that you form. So, this is what is called at least linearly. So, it means the sequence is converging at least of order 1 that is what it means. Similarly, you can also say that the order of convergence of this sequence is at least super linear if there exist ϵ that converges to 0.

And the natural number n such that $|a_{n+1} - a| \leq \epsilon_n |a_n - a|$ for all $n \geq N$ that is for sufficiently large N onwards this condition should hold.

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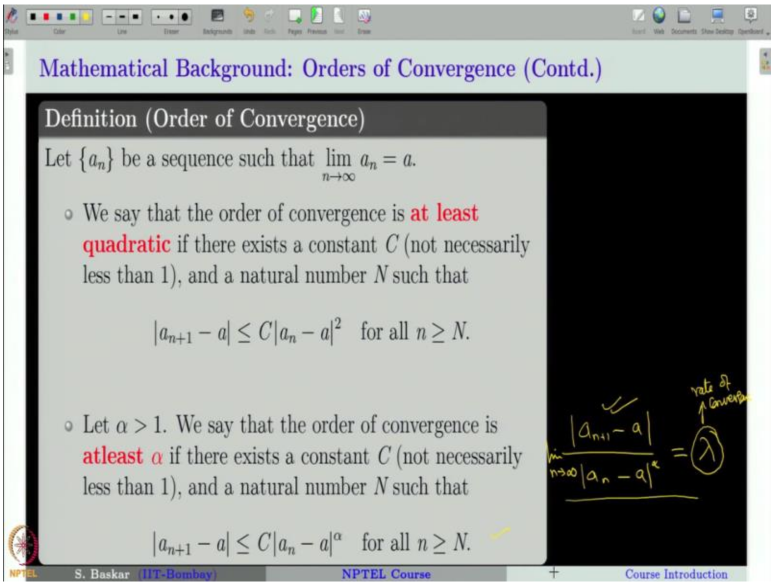

Similarly, you can define for at least quadratic convergence and this can further by generalized to any α. We say that the order of convergence of this sequence is at least α that is very important it is converging at least up to this speed that is what it means. If you can find a constant C and a natural number n such that this condition holds. You see this $|a_{n+1} - a| \leq C|a_n - a|^{\alpha}$ for all $n \geq N$.

Often we will also use the notation $lim_{n\to\infty} \frac{|a_{n+1}-a|}{|a_n - a|}$ and that is may be some constant say λ because if this is happening of course to the power of α so this is also one equivalent definition for this. In this case often in some books people call this as rate of convergence. In some books even order of convergence is otherwise called as rate of convergence.

Therefore there is no standard usage of these words. We will come across such expressions in non linear equations and with this I thank you for your attention.