Amazon
Numerical Analysis
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Lecture – 28
Nonlinear Equations: Bisection Method

In the last lecture, we had an overview of what we are going to do in the current chapter on nonlinear equations In today's class, we will introduce our first method for approximating an isolated root of a nonlinear equation. The method is called the bisection method and this method is a bracketing method.

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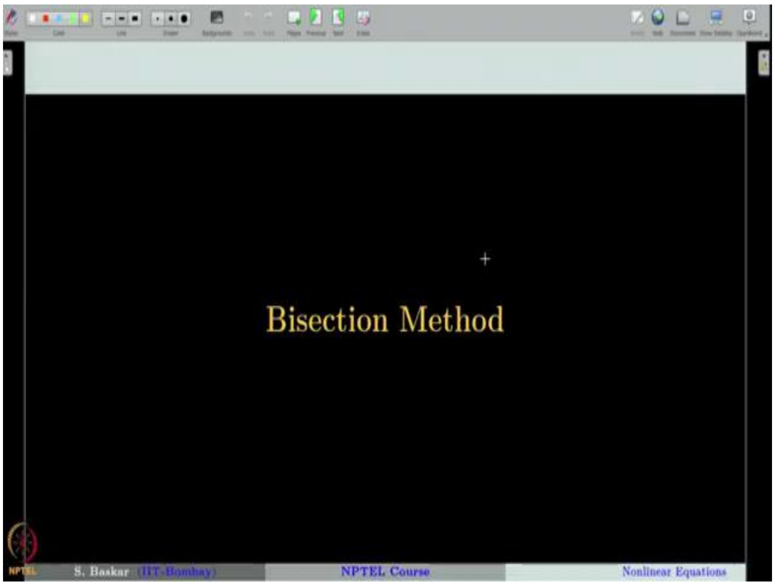

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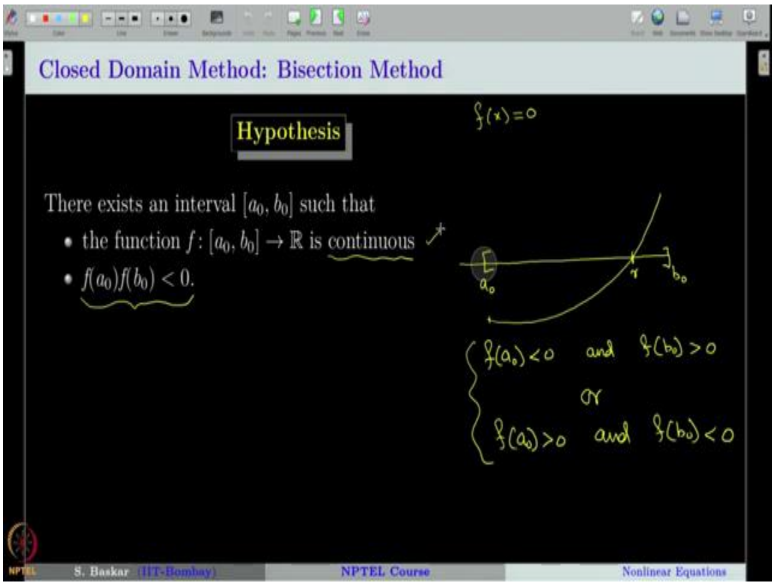

Let us explain the bisection method, suppose we have a function $f(x)$ and we are interested in finding an isolated root of the function $f(x) = 0$. Suppose the graph of the function f is given like this. Then we are interested in capturing the root or for the equation $f(x) = 0$. We have to first locate this root in an interval, say $[a_0, b_0]$. How we can do? Well, you have to see if $f(a_0) < 0$ and $f(b_0) > 0$.

Recall that we always assume the function f to be a C^1 function in particular, it is also a continuous function. Therefore, if you have $f(a) < 0$ and $f(b) > 0$ then by intermediate value theorem, you can say that there exists a point say r such that $f(r) = 0$, that is the idea. In fact, you can also equivalently look for a_0 and b_0 such that $f(a_0) > 0$ and $f(b_0) < 0$.

Any of these two conditions is enough for us to say that there exists a point r in the interval $[a_0, b_0]$ such that $f(r) = 0$. These two conditions in fact, can be checked equivalently by looking for an a_0 and b_0 such that $f(a_0)f(b_0) < 0$. Therefore, we will impose two conditions on f and its domain $[a_0, b_0]$.

That is, f is a continuous function defined on the interval $[a_0, b_0]$ such that $f(a_0)f(b_0) < 0$. So, this is the main disadvantage of the bisection method, as we discussed in the last class that you have to give this interval $[a_0, b_0]$ as an input to the method. How will you find this $[a_0, b_0]$? Well, there is no general way or general algorithm to find this a_0 and b_0.

One has to do it by trial and error either manually or even if you want to do it on a computer, it has to be done with some trial and error only that is the main disadvantage of any bracketing method. By bracketing method, we mean to say that you have to locate a root in an interval and that interval has to be given as an input to your method. Therefore, bisection method is a bracketing method. Now, once you give a_0, b_0 as an input and also the function f as an input.

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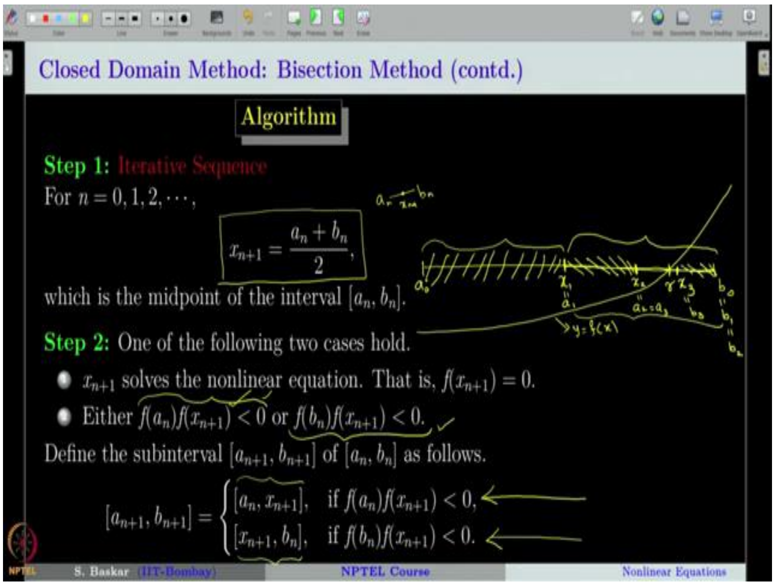

Then the iterative procedure has to be now set up. How are we going to set up? Let us see. So, we have already located the root r in the interval $[a_0, b_0]$. Now, what we have to do is? We have to find x_1 which is the first term of our iterative sequence. How will you find x_1? x_1 is taken as the midpoint of the interval $[a_0, b_0]$. And that is given by $x_1 = \frac{a_0+b_0}{2}$.

And now, once you have this then you take the value $f(x_1)$ and then see in which part of these two intervals does the sign change happens. Either, it is happening between $[a_0, x_1]$ or $[x_1, b_0]$. That is the next step. So, in that way you may have one of the following situations that is your $f(x_1)$ itself may be 0, in which case x_1 itself is a root of your equation and your job is done. Therefore, you can stop your iteration.

Suppose it did not happen, that is $f(x_1)$ is not equal to 0 then you look for whether $f(a_0)f(x_1) < 0$ or $f(x_1)f(b_0) < 0$. If this happens then you take a_0 to x_1 as your new interval. If not, then you take the other piece as the new interval. Let us see how it works? You initially have a_0 and b_0, suppose this is the graph of your equation and the function $f(x)$ is given like this.

And we are interested in capturing the root *r* which is lying in the interval $[a_0, b_0]$. In the first iteration you will take x_1 as the midpoint of $[a_0, b_0]$. And you will check the sign change of the function in the interval $[a_0, x_1]$ that is this condition you will check. If that is satisfied then you will take the first condition and declare a naught x_1 as the interval for your second iteration.

Otherwise, you will check for whether the sign change happens in the second part of the interval that is from $[x_1, b_0]$. That is, you will check this condition if that happens then you will take the second interval, as the interval for your next iteration. You can note that either this or this only will happen. You will never get a situation where both happens or you will never get a situation where none of these two happens.

You can think why it is like that? I leave it to you to do that. Once you have the interval say in this particular example your interval for the next iteration will be a_1 which is equal to x_1 and b_1 which is equal to b_0. For the next iteration again, you will go to find the midpoint of $[a_1, b_1]$ and call it as x_2. That is what, in general, we are writing like this. For any *n* you already got a_n and b_n.

Now, for the next iteration, you will find x_{n+1} as the midpoint of the interval, $[a_n, b_n]$. And once you do that you will see where the sign change happens. That is now you have broken the interval into two parts. One is $[a_n, x_{n+1}]$ and another one is $[x_{n+1}, b_n]$. Now, you will check where the sign change happens and once you find that interval in which sign change happens, you will discard the other interval and take only that interval which has the sign change.

And then you will go for the next iteration, where you will find the midpoint of the interval that is obtained in the previous iteration. And then again you will see the sign change in this example, the sign change is again not happening in this interval. The sign change is happening in this second part of the interval.

Therefore, a_2 is taken as x_2 and b_2 is taken as b_1. Now again, you will go to find the midpoint of the interval $[a_2, b_2]$ and suppose that happens to be this one that is called as x_3. And now you will again find the sign change this time sign change happens in the first part of the interval.

Therefore, a_3 is taken as a_2 and b_3 is taken as x_3 and you discord this part of the interval. And you have only this interval in your next iteration and the iteration goes on like this.

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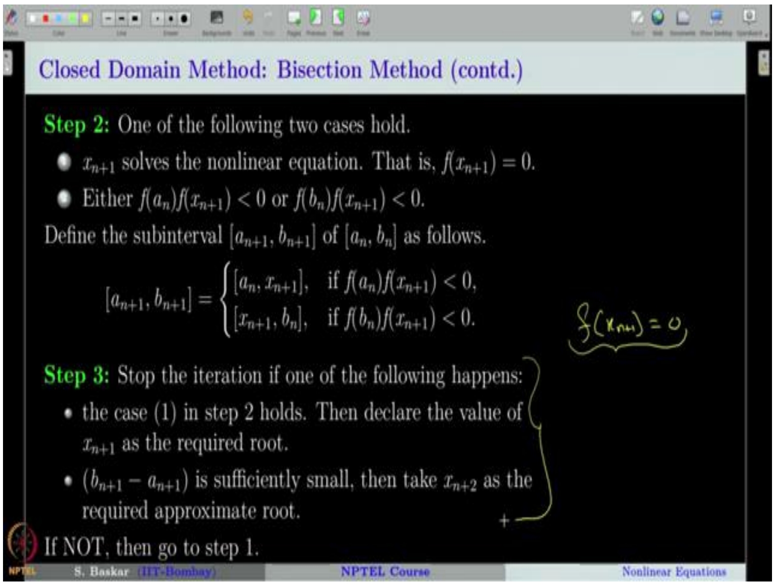

Now, the next step is to check whether you have to stop the iteration at any *n*th iteration or not. You can clearly see that at every iteration the length of the interval is reducing. Therefore, there is a very natural criteria for us to stop the iteration, what you do is, you check the length of the interval. If the length of the interval is less than some pre-assigned positive quantity which is called the tolerance parameter.

That is, you take some ϵ say 10^{-2} or 10^{-3} or whatever it is. And then check for the length of the interval. Once you finish the iteration at the *n*th stage then you have the interval $[a_n, b_n]$, you look for this number. If this number is less than ϵ then surely the difference between x_{n+1} which is the midpoint of the interval $[a_n, b_n]$ minus the root *r* that will be surely less than ϵ.

Therefore, you can always check for this condition. That is a good and natural stopping criteria in the bisection method. This is because in bisection method, the length of the interval at every iteration is reducing. That is why we can impose this condition as the stopping criteria. Therefore, if $b_{n+1} - a_{n+1}$ is sufficiently small then take the midpoint of the interval $[a_{n+1}, b_{n+1}]$ and declare that as the required approximate root.

So that is the stopping criteria that we are imposing. If that is not happening again, go to the step one and continue this process till either $f(x_{n+1})$ becomes 0, remember on a computer this may not happen at all. Therefore, you may have to stop your iteration only with this stopping criterion.

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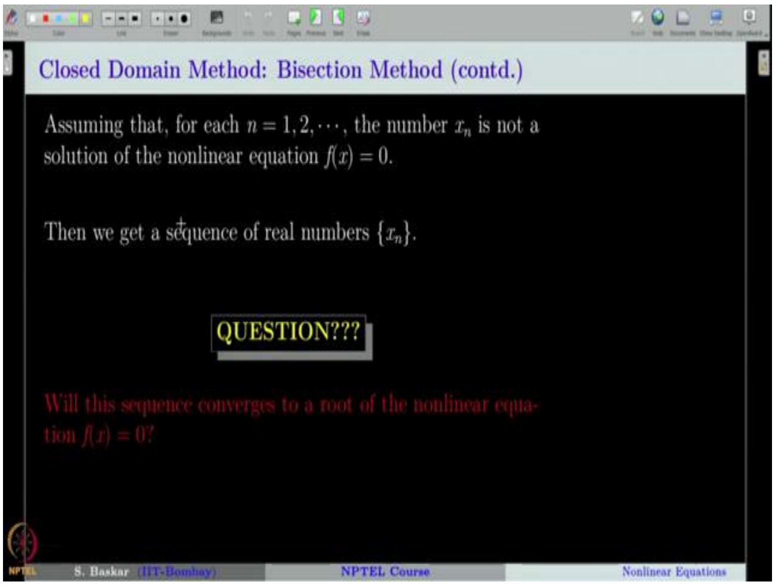

Now, assuming that at every iteration x_n is not coinciding exactly with a root, at least as a computation on a computer this may not happen because of the rounding errors. In that way, generally, we will land up generating a sequence of real numbers. Now the question is, will this sequence converge to a root of the nonlinear equation, $f(x) = 0$? That is the question. So, to answer this question, we have to go for the convergence analysis on the sequence generated by the bisection method.

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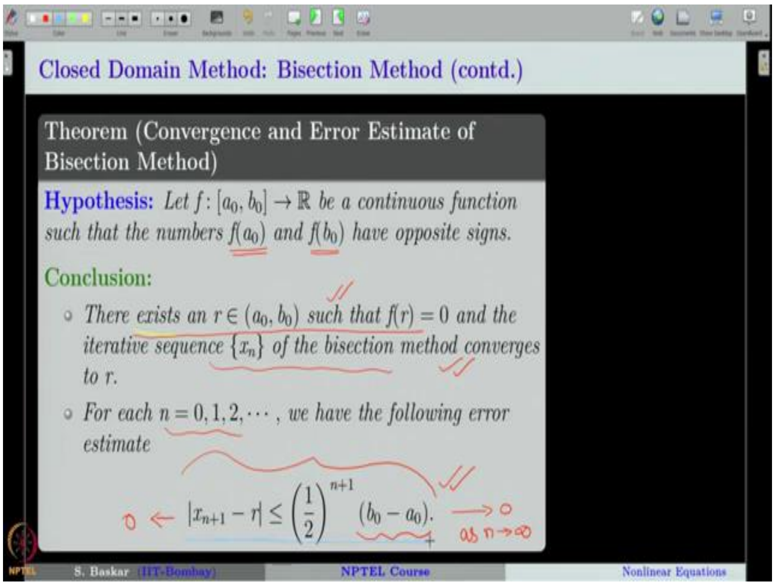

Let us state this result in the form of a theorem. Let f be a continuous function defined on an interval $[a_0, b_0]$, where a_0 and b_0 are chosen in such a way that $f(a_0)$ and $f(b_0)$ have opposite signs. We now know why we are imposing this condition because by this we can ensure that there is at least one root in between the points, a_0 and b_0 because *f* is a continuous function.

The conclusion is that, there exists *r* in the interval $[a_0, b_0]$ such that $f(r) = 0$. And this is direct consequence of the intermediate value theorem. Because *f* is continuous and $f(a)$ and $f(b)$ are of opposite sign. Therefore, this is directly proved from the intermediate value theorem. And further the iterative sequence of the bisection method will always converge to *r*.

This is what in the previous lecture, we told that the bracketed methods, once you obtain an interval $[a_0, b_0]$, then the sequence generated by this bracketing methods will always converge. So that is what we listed as an advantage of bracketing methods. Here you can see that the sequence x_n will always converge that is what the conclusion says and in fact you also have an error estimate here.

You can see that $|x_{n+1} - r| \leq \left(\frac{1}{2}\right)^{n+1} (b_0 - a_0)$ and this will happen for each *n* = 0, 1, 2 and so on. Now, how to prove this theorem? Well, you can clearly see that once you prove this estimate then the convergence comes very easily. Why? Because by taking *n* tends to infinity, you can see that $\left(\frac{1}{2}\right)^{n}$ goes to 0 and this is a fixed number.

Therefore, the right-hand side goes to 0, as *n* tends to infinity. And that shows that of course, by the sandwich theorem that the left-hand side will also goes to 0, as *n* tends to infinity. Therefore, we have to only prove this inequality and that will in fact prove the first part of the theorem.

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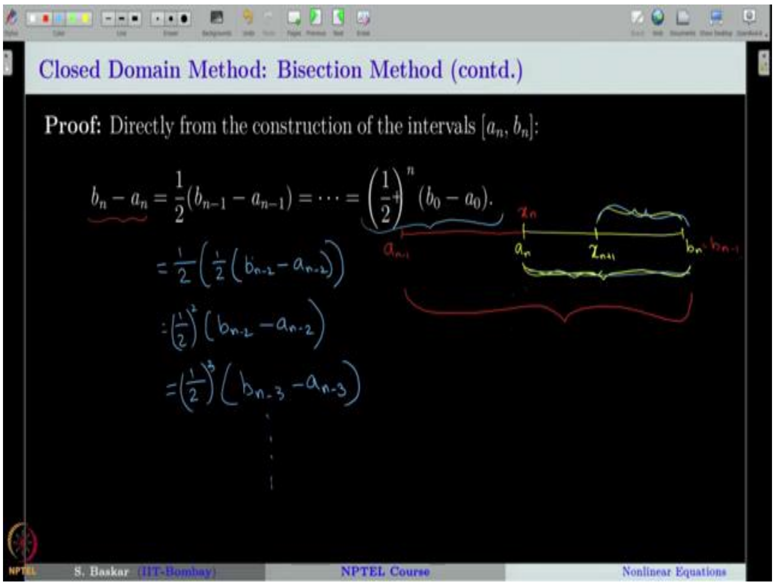

Let us see, how to prove this inequality? It is not very difficult. You just take the length of the interval $[a_n, b_n]$. You can see that at every iteration you have $[a_n, b_n]$ and you are bisecting this interval. That is, you are defining x_{n+1} as the midpoint of the interval $[a_n, b_n]$. And that is how you are obtaining the next interval. That is, you are either choosing this as your next interval or you are choosing this interval as the interval for your next iteration.

Therefore, the length a_n, b_n similarly, would have come from your previous iteration, a_{n-1}, b_{n-1} by cutting that interval into half. That is how you would have got your x_n and depending on where the sign change happens. You would have picked one piece of the interval a_{n-1}, b_{n-1} and named it as a_n, b_n. Therefore, $b_n - a_n$ should be half of $b_{n-1} - a_{n-1}$ that is this interval length will be half of the length of this interval.

Similarly, the length of this interval will be half of the length of this interval and it goes like that. That is why we are writing $(b_n - a_n) = \frac{1}{2}(b_{n-1} - a_{n-1})$ and that will be actually, half of this is nothing but half of $(b_{n-2} - a_{n-2})$ that is nothing but $\left(\frac{1}{2}\right)^2 (b_{n-2} - a_{n-2})$. And that again you can apply this idea to get $\left(\frac{1}{2}\right)^3 (b_{n-3} - a_{n-3})$ and you can keep on going like this.

This kind of idea is always used in our discussions. You once find an expression and then you can recursively apply that expression. And once you do that where will you reach? You will reach at $(b_0 - a_0)$ and that will have $\left(\frac{1}{2}\right)^n$. So that is the idea. Let us see how it goes?

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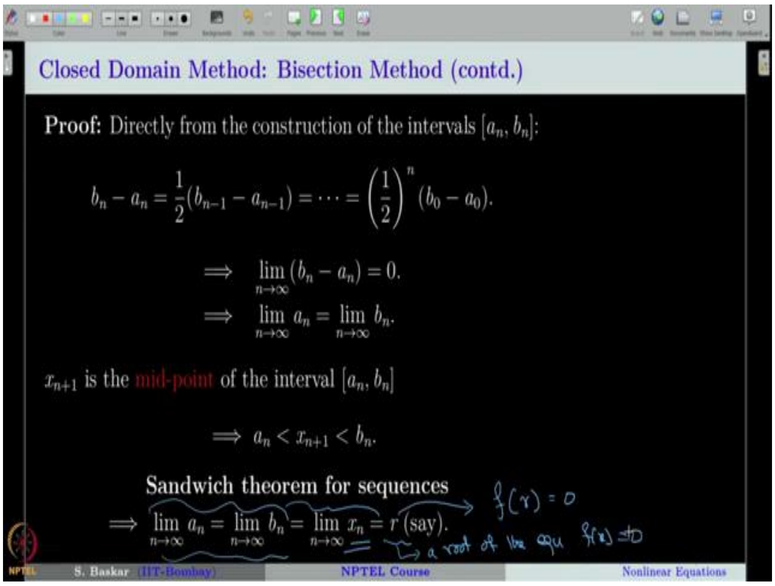

Now, take the limit on both sides. You can see that $\lim_{n\to\infty}(b_n - a_n)$ is nothing but $\lim_{n\to\infty}\left(\frac{1}{2}\right)^n (b_0 - a_0)$. As I told this expression goes to 0, as n tends to infinity. So, therefore, $\lim_{n\to\infty}(b_n - a_n) = 0$. That implies that $\lim_{n\to\infty} a_n = \lim_{n\to\infty} b_n$. Remember, we have always chosen x_{n+1} as the midpoint of the interval (a_n, b_n).

This is how we have chosen x_{n+1}. And now we will see that the left-hand side goes to some limit, say r as n tends to infinity. Similarly, the right-hand side limit also goes to r as n tends to infinity. Why they go to the same limit? Because of this fact. Now, you apply the sandwich theorem and we get that the $\lim_{n\to\infty} a_n = \lim_{n\to\infty} b_n$, this we have already shown.

And now, this path comes from the sandwich theorem which says that $\lim_{n\to\infty} x_n$ will also exists and that will be equal to the limit to which these two sequences are converged that is r. Now, what we have to prove that this r is the root of the equation. That is what we have to prove.

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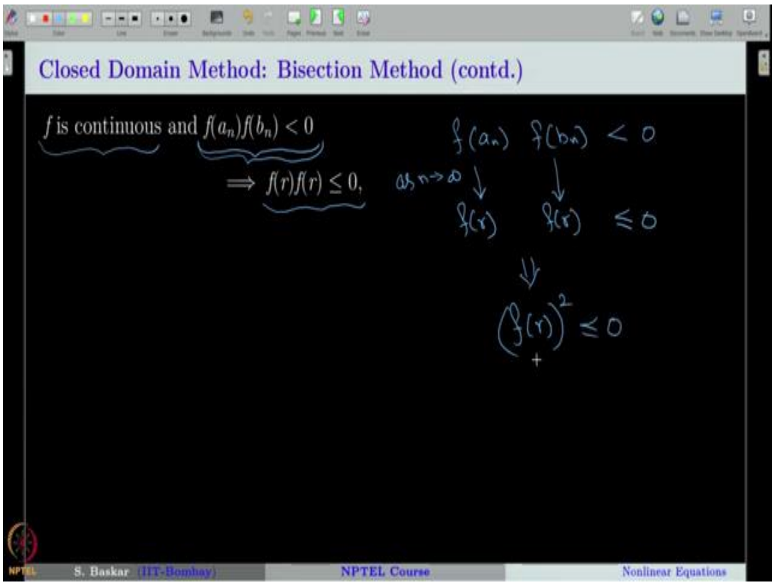

Recall that f is a continuous function this is one of the hypotheses that we have taken in our theorem and also at every iteration we are checking this condition. Therefore, for every n, $f(a_n)f(b_n) < 0$. This is how we have built the algorithm. This is coming from our basic assumption and this is coming from the way we have built the algorithm. Therefore, as you take the limit, you can see that $f(a_n)$ will converge to $f(r)$, why?

Because f is a continuous function similarly, $f(b_n)$ will also converge to $f(r)$ as n tends to infinity. If the product $f(a_n)f(b_n) < 0$ then the product $f(r)f(r)$ will also be less than or equal to 0 that is what we are writing here. Now, what is this? This is nothing but $\left(f(r)\right)^2 \le 0$. Remember, f is a real valued function. Therefore, $f(r)$ is a real number and its square should always be non-negative.

So that shows that $f(r)$ should be 0. So, what we proved is that? The limit to which x_n converges which we named as r is a root of the equation $f(x) = 0$, that is what we have proved.

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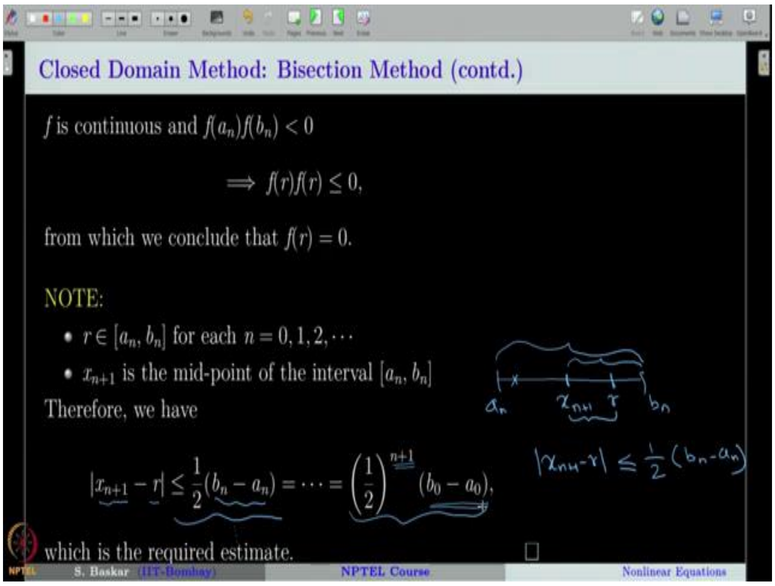

Now, what all we have obtained? We obtained an interval $[a_n, b_n]$ at every iteration. And we made sure that at least one root of our equation lies in the interval. And once you get a_n, b_n then you go to find the midpoint of that interval and that is taken as the iteration term of the sequence at the $(n+1)$th stage. So, this is how the iteration went. Therefore, at every stage $[a_n, b_n]$ you see you have at least one root lying in this.

And then you went to find the midpoint of this interval x_{n+1}. Therefore, the distance between x_{n+1} and r that is $x_{n+1} - r$ is surely less than or equal to half of this interval's length. Because of the interval's length is this and you always make sure that r lies in $[a_n, b_n]$. Therefore, r is somewhere in this interval either it may be this side or that side wherever it is once you bisect $[a_n, b_n]$.

Then the distance between that point that is, the midpoint and the root r will be less than half times the length of the interval $[a_n, b_n]$. So that is what I am writing here. The distance between $|x_{n+1} - r| \le \frac{1}{2}(b_n - a_n)$. Now, once you have this now you go on applying this recursively just like what we did in the previous slide? And you can obtain that $|x_{n+1} - r| \le \left(\frac{1}{2}\right)^n (b_0 - a_0)$.

This is precisely what we wanted to prove? As the error estimate for our bisection method sequence. So, here and this is proved now and that completes the proof of this theorem. In fact, there is another result, very interesting result, which says that we can obtain n, the number of iterations that are needed for us to compute in order to get our approximation close to the root up to the required accuracy that n can be obtained from this error estimate itself. This is a very interesting result and we will discuss this result in the next class. Thank you for your attention.