Numerical Analysis
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Lecture-24
Eigenvalues and Eigenvectors: Power Methods (Convergence Theorem)

Hi, we are learning methods for computing eigenvalues and eigenvectors. In the last class we have introduced power method. In this lecture we will learn the convergence theorem for power method.

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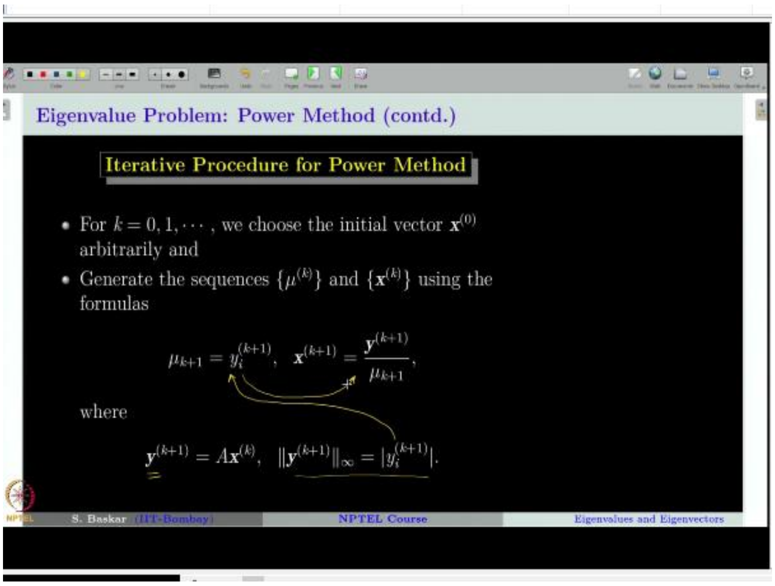

Let us quickly recall the iterative procedure for power method. We first have to choose an initial vector $\boldsymbol{x}^{(0)}$ arbitrarily and then for k = 0, 1, 2 and so on the power method generates 2 sequences; one is a sequence of real numbers and another one is a sequence of vectors and they are defined as $\mu_{k+1} = y_i^{(k+1)}$ and $\boldsymbol{x}^{(k+1)} = \frac{\boldsymbol{y}^{(k+1)}}{\mu_{k+1}}$, where $\boldsymbol{y}$ is a vector given by $A\boldsymbol{x}^{(k)}$ that is $\boldsymbol{y}^{(k+1)}$.

The vector $\boldsymbol{y}$ in the present sequence is obtained by multiplying A with the vector $\boldsymbol{x}$ computed in the previous sequence. Therefore, it is an iterative procedure. Once you get $\boldsymbol{y}$ you will find that coordinate at which the maximum norm is achieved. Then you take the minimum of such index and take that value and consider that as μ_{k+1}. Once you have μ_{k+1} you will compute $\boldsymbol{x}^{(k+1)} = \frac{\boldsymbol{y}^{(k+1)}}{\mu_{k+1}}$. This will go for all k = 0, 1, 2 and so on. So, this is the iterative procedure for power method.

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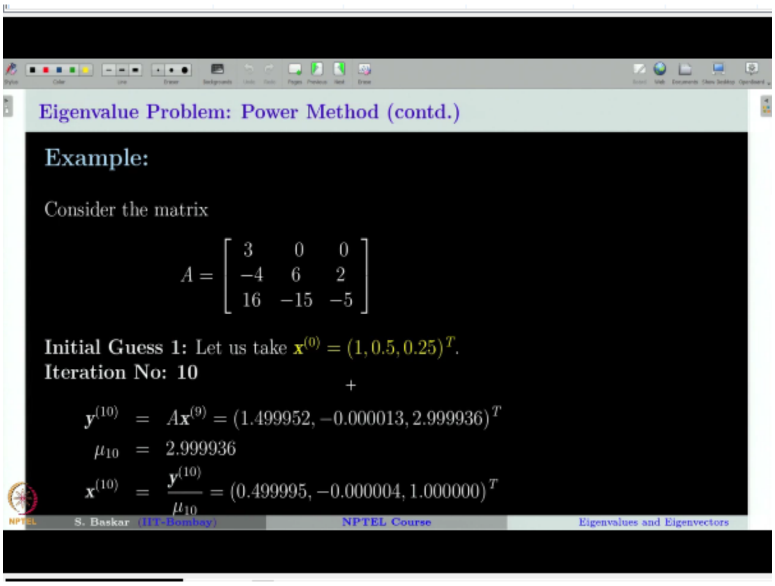

We have also seen in the last class, an example where power method was successfully applied and we have computed 10 iterations and saw that the power method sequences are seems to be converging to the dominant eigenvalue λ_1 which happens to be 3 in this example and also a corresponding eigenvector of the dominant eigenvalue λ_1.

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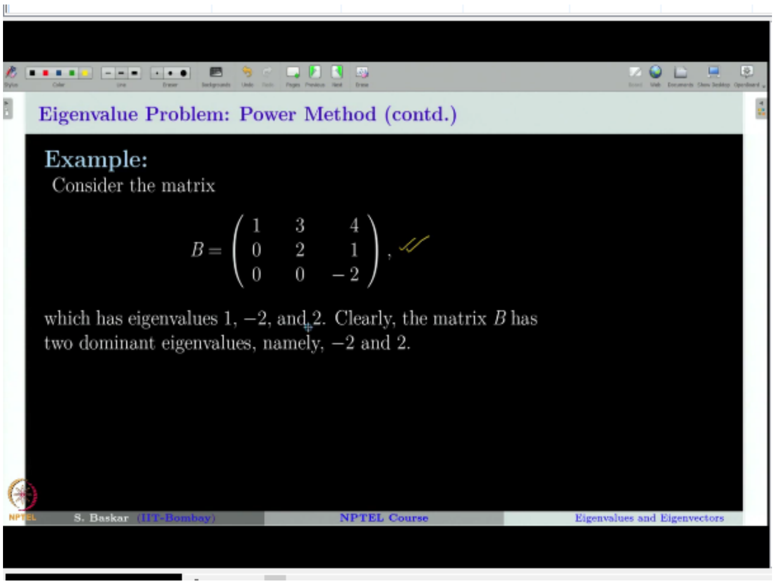

Let us take another example where power method is not successful. Now we will take matrix *B* which is given like this the eigenvalues of *B* are (1, -2, 2). You can see that *B* has 2 as the dominant eigenvalue, but it does not have the unique dominant eigenvalue because 2 and -2 both will contribute as dominant eigenvalues.

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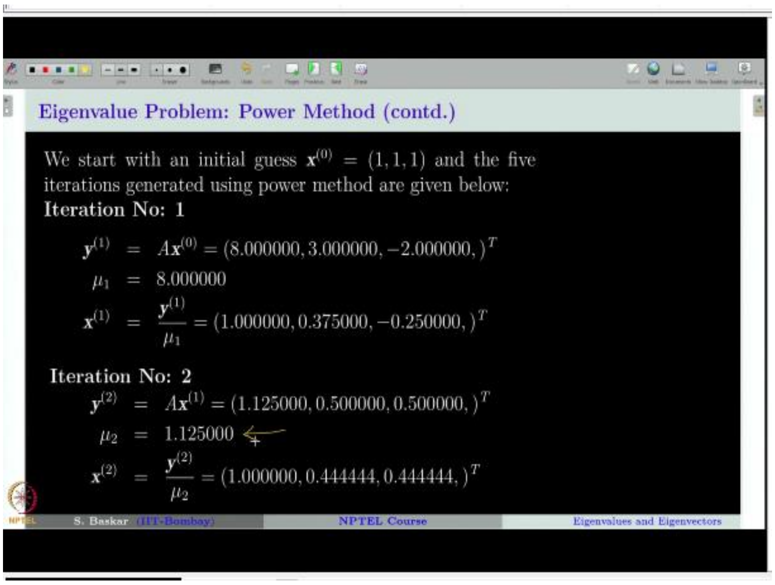

Let us use power method to compute the iterations μ_k and $\boldsymbol{x}^{(k)}$ and see how these sequences are coming out. For that we have to choose an initial guess; again let us take it arbitrarily as (1, 1, 1). The first iteration happens to be (8, 3, -2) as the $\boldsymbol{y}$ vector and you can see that the maximum norm of $\boldsymbol{y}$ is attained at the first coordinate of $\boldsymbol{y}$ and therefore that is taken as the value for μ_1.

Once you have μ_1, $\boldsymbol{x}^{(1)}$ is computed like this. Recall we are supposed to converge to either 2 or -2; let us see what is happening. The second iteration happens to be like this; let us just keep on observing how the sequence μ is coming out.

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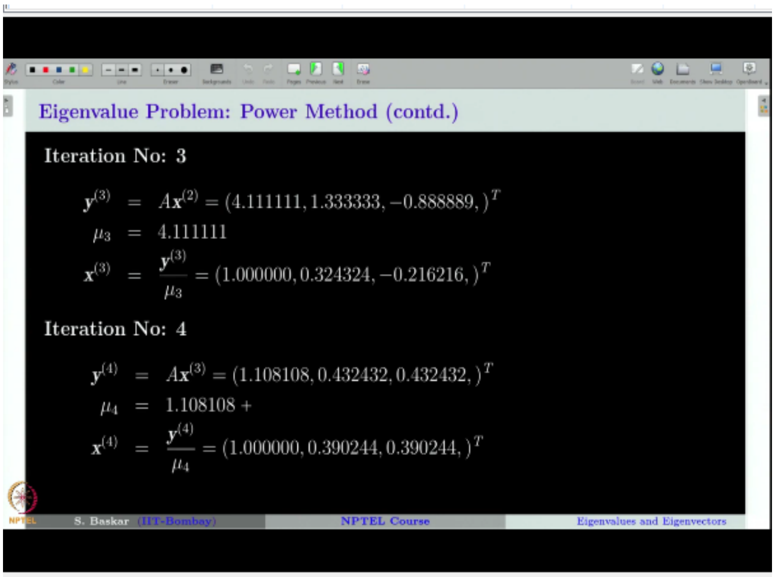

In the third iteration μ_3 comes out to be 4.111 and so on. And in the fourth iteration it is 1.11, you can observe that in the second iteration it is something like 1.1 and so on. In the third iteration it is 4.1 and so on. Again, in the fourth iteration it is 1.1 and so on.

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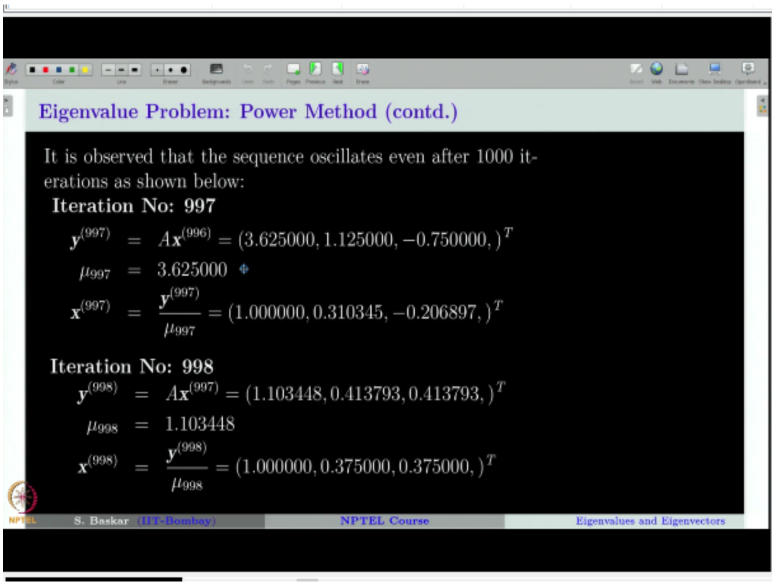

Like that it keeps on jumping between something around 1 to something around 4 and I kept on going till 997th iteration I went. I still see that μ is jumping between around 3.6 to 1.1, it never seems to be converging to the dominant eigenvalue either 2 or -2; it kept on jumping between these 2 numbers.

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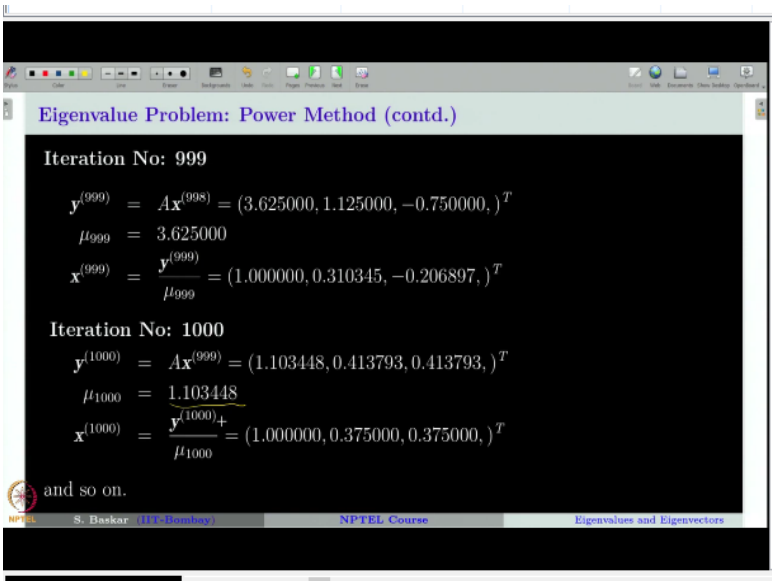

You can see in the 999th iteration again it jumped from 1.1032, again 3.6, again at the 1000th iteration it did like this. Then I went for quite some more iteration I observed that the sequence μ kept on jumping between 3.6 so on to 1.1 so on. So, that gives us a feeling that the power method for this example that is the matrix B given in this example seems to be not converging to the dominant eigenvalue.

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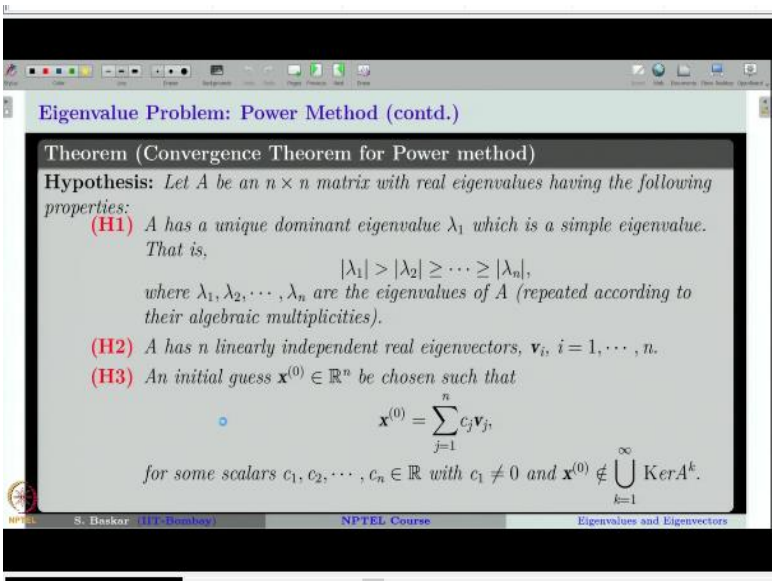

And therefore, not to a corresponding eigenvector also. That gives us an interesting question of when can we expect the convergence of these sequences? For that let us state this theorem; we need the following hypothesis, you are given a matrix A which is a $n \times n$ matrix. Suppose A satisfies the following hypothesis. Hypothesis 1 is that A has a unique dominant eigenvalue λ_1.

So, these are the conditions that we have already spoken in the last class. We now know why we are imposing this condition. This is very clear from the way we have constructed the sequences μ_k and $\boldsymbol{x}^{(k)}$. Therefore, you can understand why these hypotheses are part of the convergence theorem. The next hypothesis is that A has n linearly independent eigenvectors $\boldsymbol{v}_i$. That is a set of eigenvectors v_1, v_2 up to v_n should form a basis for $\mathbb{R}^n$.

Again, if you recall from the last class why this hypothesis is needed because when we choose an arbitrary vector as an initial guess we started with its representation in terms of the eigen bases and then from there we recursively pre multiplied A and that is how we constructed the sequences. Therefore, this condition is also important for us in the construction of the sequences in power method.

The third condition is that the initial guess that we choose should satisfy the condition that the initial guess should always be away from the kernel of A^k for every k = 1, 2 up to infinity and also when you write $\boldsymbol{x}^{(0)}$ as the scalar multiple of the eigenvectors then the first scalar that is c_1 should be not equal to 0. If you recall because the sequence that is converging to a scalar multiple of the eigenvector is actually multiplied with c_1.

If c_1 is 0 then the first term in our expression will vanish and therefore your sequence will never converge to the dominant eigenvalue and its corresponding eigenvector. Therefore c_1 should not be equal to 0 for the initial guess that we have chosen.

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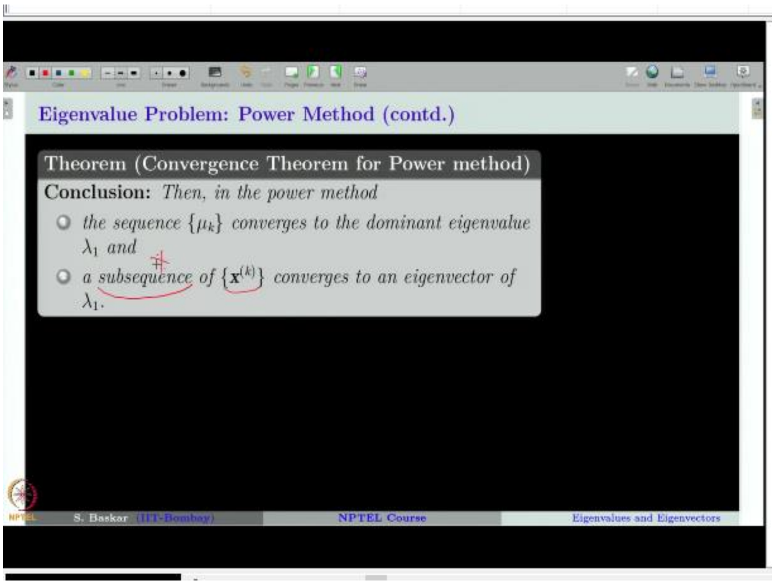

If all this hypothesis are satisfied then we can say that the sequence μ_k converges to the dominant eigenvalue λ_1 of the matrix A and the sequence $\boldsymbol{x}^{(k)}$ well it may not converge as a full sequence what we can conclude is we can find a subsequence of $\boldsymbol{x}^{(k)}$, t24hat is very important. You see you can find the subsequence of the sequence $\boldsymbol{x}^{(k)}$ that converges to an eigenvector of λ_1. So, that is very important here. Let us try to prove this theorem.

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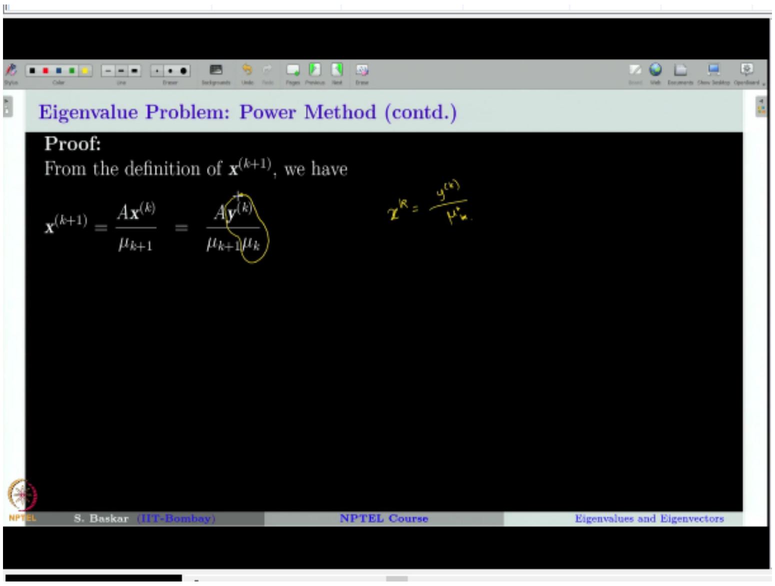

If you recall the definition of the sequence $\boldsymbol{x}^{(k)}$ is given by $\boldsymbol{x}^{(k+1)} = \frac{\boldsymbol{y}^{(k+1)}}{\mu_{k+1}}$. What is $\boldsymbol{y}^{(k+1)}$? That is nothing but $A\boldsymbol{x}^{(k)}$ that is the value of $\boldsymbol{x}$ in the previous iteration that is what I am writing here $\boldsymbol{x}^{(k+1)}$ is equal to instead of $\boldsymbol{y}^{(k+1)}$ I have written $A\boldsymbol{x}^{(k)}$. Now you see $\boldsymbol{x}^{(k)}$ is nothing but $\boldsymbol{y}^{(k)}$ divided by μ_k. That is what I am writing here instead of $\boldsymbol{x}^{(k)}$ I am just writing $A\boldsymbol{y}^{(k)}$.

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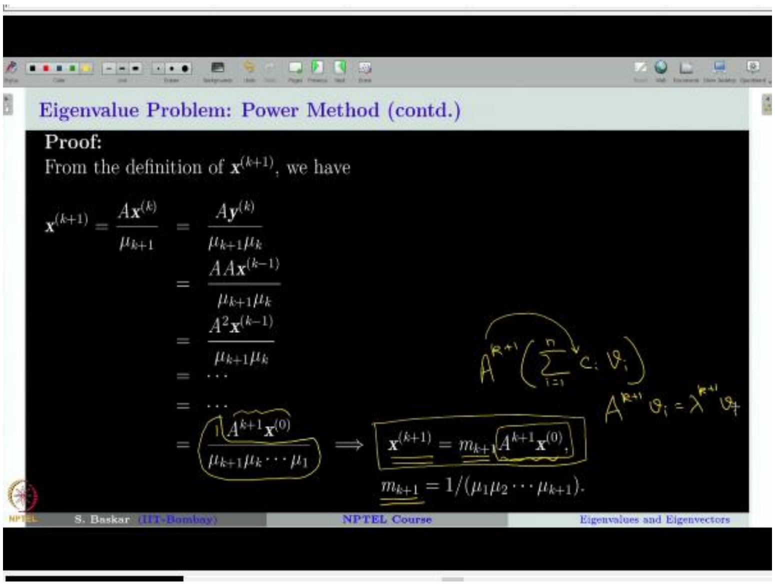

Now what is $\boldsymbol{y}^{(k)}$? $\boldsymbol{y}^{(k)}$ is nothing but $A\boldsymbol{x}^{(k-1)}$. So, therefore I am replacing $\boldsymbol{y}^{(k)}$ by $A\boldsymbol{x}^{(k-1)}$ and that can be simply written $\frac{A^2\boldsymbol{x}^{(k-1)}}{\mu_{k+1}\mu_k}$. Therefore, what we see that $\boldsymbol{x}^{(k+1)}$ can be written like this. Now once you understand this step, you can apply the same idea to this as $A^3\boldsymbol{x}^{(k-2)}$ and that will leave 1 more μ term here that is $\mu_{k+1}\mu_k\mu_{k-1}$.

Like that you keep on going how long you can go till you reach $\boldsymbol{x}^{(0)}$ because that was the starting point of our iteration. When you reach $\boldsymbol{x}^{(0)}$ you will have A^{k+1} here and in the denominator, you will have the products $\mu_{k+1}\mu_k \cdots \mu_1$. Therefore, $\boldsymbol{x}^{(k+1)}$ can be written as, I will just take 1 by this product as m_{k+1}. Therefore, $\boldsymbol{x}^{(k+1)} = m_{k+1}A^{k+1}\boldsymbol{x}^{(0)}$. That is very clear. So, this is what we obtained the sequence $\boldsymbol{x}^{(k+1)}$ is finally written like this in terms of $\boldsymbol{x}^{(0)}$.

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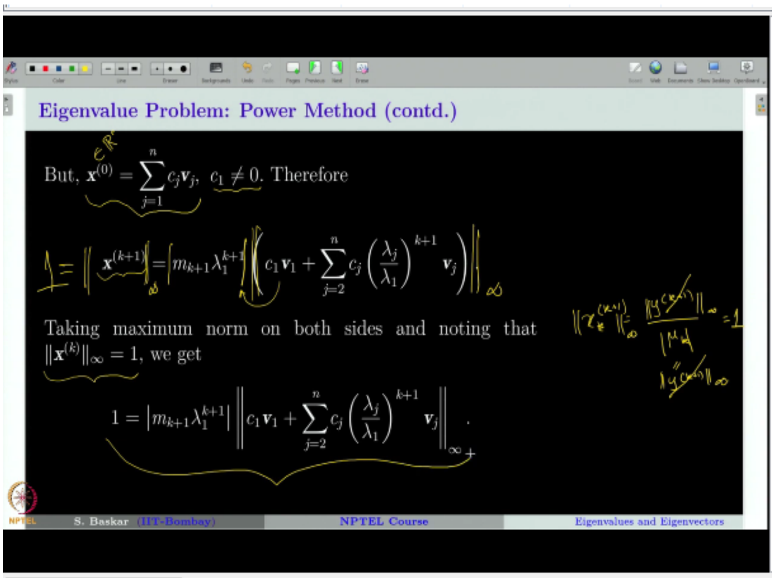

Let us recall $\boldsymbol{x}^{(0)}$ is something we chose arbitrarily and since our set of eigenvectors form a basis for $\mathbb{R}^n$ and $\boldsymbol{x}^{(0)}$ is a vector in $\mathbb{R}^n$, you can write $\boldsymbol{x}^{(0)}$ as a linear combination of the eigenvectors and our hypothesis also says that the $\boldsymbol{x}^{(0)}$ is such that c_1 is not equal to 0. Now what we will do we have this quantity and $A^{k+1}(\sum_{i=1}^{n} c_i v_i)$. Now you take this A^k inside.

And you can see that $A^{k+1}v_i$ is nothing but $\lambda^{k+1}v_i$. This kind of calculations are already done in our last class. Therefore, it should not be difficult for you to understand how we got this expression. Now what you got is from here, you have each term involving something like this $\lambda_i v_i$ and then you will remove λ_1 from the first term and you will divide λ_1 in all the other terms.

So, this is how in the last class we have seen that we have constructed the sequences for power method. That same idea I am just putting one more time as a proof of this theorem; we are not doing anything new here. Now what you do is if you observe the way $\boldsymbol{x}^{(k+1)}$ is defined you can see that it is a unit vector with respect to the maximum norm why because $\boldsymbol{x}^{(k+1)}$ is nothing but $\frac{\boldsymbol{y}^{(k+1)}}{\mu_k}$.

What is μ_k? μ_k is nothing but the value of that coordinate of $\boldsymbol{y}$ at which the maximum norm is achieved. Therefore, if you take modulus on both sides, you will have this and this mod and that is precisely norm of $\boldsymbol{y}^{(k+1)}$. Of course, everywhere it is an infinite norm that we are taking. Therefore, this gets canceled and you will have 1 here. So, $\boldsymbol{x}^{(k)}$ as we defined is a unit vector with respect to the maximum norm or l_∞ norm.

Therefore, if I take modulus that is norm on both sides, I will have this into norm of this vector everywhere we are taking the maximum norm. So, this is equal to 1 that is what we have written here.

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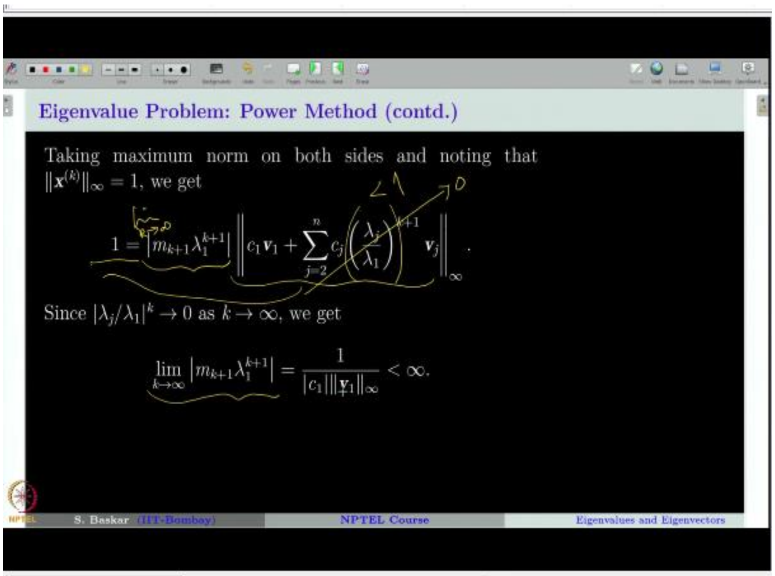

Therefore, you can see that this term is equal to 1 by this term, it will come in the denominator on the left hand side. Now take limit k tends to infinity in this term. So, that gives you $\lim_{k\to\infty} |m_{k+1}\lambda_1^{k+1}|$ is equal to when you take the limit you can see that all these terms are in the absolute sense less than 1. It means they lie between -1 and 1.

Therefore, if you take the power of that number $k + 1$ and then tend that k to infinity you will see that this term will go to 0 and you will have $|c_1|||\boldsymbol{v}||_\infty$ and that is taken on the other side therefore $1/|c_1|||\boldsymbol{v}||_\infty$ and that is a finite number.

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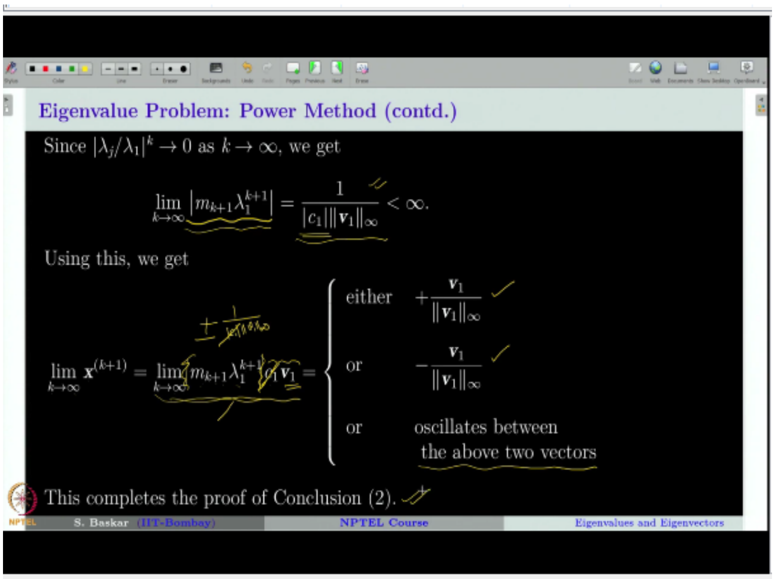

Therefore, if you take $\lim_{k\to\infty}$ of $\boldsymbol{x}^{(k+1)}$, if you recall how $\boldsymbol{x}^{(k+1)}$ was defined; $\boldsymbol{x}^{(k+1)}$ was defined like this. I am taking now $\lim_{k\to\infty}$ for $\boldsymbol{x}^{(k+1)}$ and that is nothing but $m_{k+1}\lambda_1^{k+1}c_1\boldsymbol{v}_1$, because this term goes to 0. That is what we are writing here $\lim_{k\to\infty} \boldsymbol{x}^{(k+1)} = \lim_{k\to\infty} m_{k+1}\lambda_1^{k+1}c_1\boldsymbol{v}_1$.

We know what is mod of this quantity, this is anyway independent of k. Therefore, you just can see what is this, this is nothing this. Now if I remove this mod what can I say this term will tend to either $+1/|c_1|||\boldsymbol{v}_1||_\infty$ or it will tend to minus of this or it may oscillate between these two.

Therefore, you can see that $\lim_{k\to\infty} \boldsymbol{x}^{(k+1)}$ will be either $\boldsymbol{v}_1$, remember c_1 is already there in the denominator therefore that and this c_1 will get canceled that is this c_1 and this c_1 will get canceled leaving either a plus sign or minus sign there. So, therefore this entire limit will either be + $\boldsymbol{v}_1$ $||\boldsymbol{v}_1||_\infty$ or $-\boldsymbol{v}_1$ $||\boldsymbol{v}_1||_\infty$ or it may simply oscillate between these two also, sometimes this may also happen that this sequence may oscillate because only in the modulus it is converging to this number.

Therefore, the sequence as such may even oscillate. So, we cannot ignore that case. That is why if you recall in our statement, we have only given guarantee that the sequence may not converge, only a subsequence can converge because this sequence may become an oscillating sequence as we have seen here, because it is coming as $\lim_{k\to\infty} m_{k+1}\lambda_1^{k+1}$.

We only know its absolute convergence. Therefore, we cannot conclude that it converges it may oscillate also. So, that completes the proof of the conclusion 2 because the conclusion 2 says that we can always find a subsequence that converges. If this happens then the subsequence is the full sequence itself. Similarly, if this happens then also the subsequence is the full sequence itself.

If this happens then it means that the sequence is oscillating between these two numbers. Therefore, you can choose those terms which involves this. So, that the sequence will converge to this one or you may choose those terms which has this value then that subsequence will converge to this one. That is why we can only give you guarantee up to the convergence of a subsequence here. So, we have proved conclusion 2.

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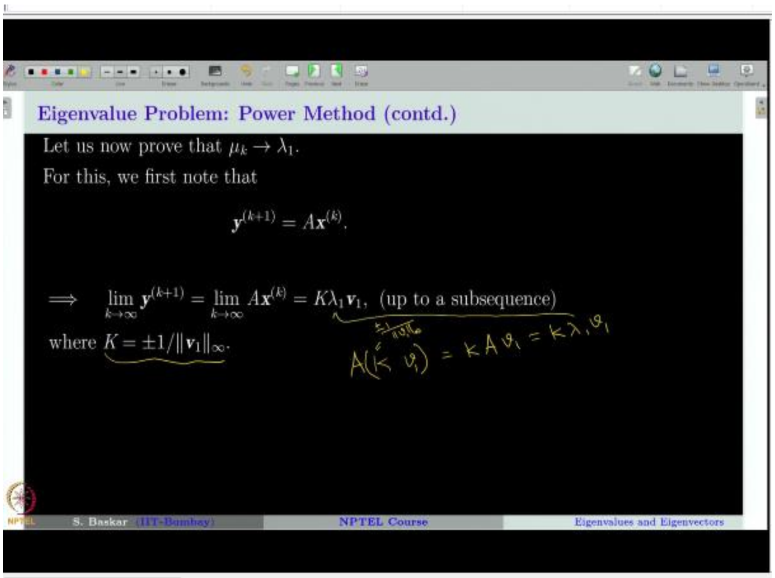

Now let us try to prove the conclusion 1 which is about the convergence of the sequence of real numbers μ_k. So, let us see how to prove that μ_k converges to λ_1. For that we first note that $\boldsymbol{y}^{(k+1)}$ is given by $A\boldsymbol{x}^{(k)}$. This is how we have defined our sequence $\boldsymbol{y}^{(k+1)}$. Therefore, taking limit on both sides will give us $\lim_{k\to\infty} A\boldsymbol{x}^{(k)} = A \lim_{k\to\infty} A\boldsymbol{x}^{(k)}$.

Now we know this how it looks like; we have already derived that, $\lim_{k\to\infty} A\boldsymbol{x}^{(k)}$, whether it is $\boldsymbol{x}^{(k+1)}$ or $\boldsymbol{x}^{(k)}$ does not matter because we are taking $\lim_{k\to\infty}$. Therefore, $\lim_{k\to\infty} A\boldsymbol{x}^{(k)}$ is actually behaving like this; one of these 3 cases will come. That is what we are having here. Therefore your $\lim_{k\to\infty} A\boldsymbol{x}^{(k)}$ can be written as $K\lambda_1\boldsymbol{v}_1$.

Why because it is nothing but $K\boldsymbol{v}_1$; what is *K*? *K* is nothing but this one, 1 by this that is what I am just denoting here by *K*, *K* is nothing but $1/||\boldsymbol{v}_1||_\infty$; it can be plus or minus. That is what we have written here $K\boldsymbol{v}_1$ and then you are applying *A* on it. So, *K* will come out and you will have $A\boldsymbol{v}_1$ and that is nothing but $K\lambda_1\boldsymbol{v}_1 1$. So, that is what we are writing here.

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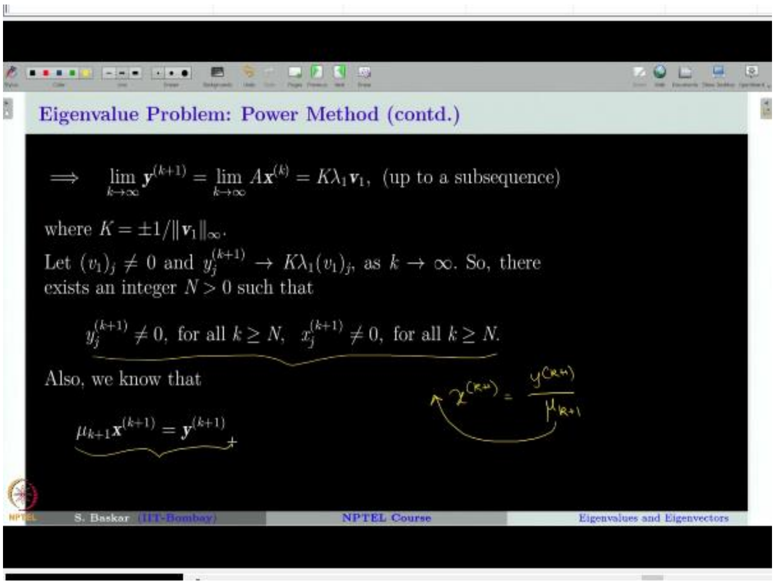

Now let us see what happens. You Know that this sequence is converging to this one up to a subsequence because it may be simply oscillating between plus and minus also. That *K* should not be ignored. Therefore, we will only say that this sequence is converging to this limit whether it is plus or minus up to a subsequence. Now what you do is you take one typical coordinate of this vector $\boldsymbol{v}_1$.

Let us choose that as the *j*th coordinate which is non zero that is more important. Then you choose the same coordinate here in each of the terms in the sequence

and you can see that that sequence will converge to k lambda 1 and this is the vector v 1 and we are choosing the jth component of that vector. Therefore $\boldsymbol{y}^{(k)}$ it will converge to $\boldsymbol{v}_1$ the *j*th component. That is what we are writing here and $\boldsymbol{y_j}^{(k+1)}$ converges to this real number as *k* tends to infinity, up to a subsequence.

That we should always keep in mind; I may just loosely say converges but it is up to a subsequence. So, there exists an integer $N > 0$, such that you can say that $\boldsymbol{y_j}^{(k+1)}$ is not equal to 0 for sufficiently large k, why because we have chosen that coordinate of $\boldsymbol{v}_1$ to be non zero, how can we do that because we surely know that $\boldsymbol{v}_1$ is a non-zero vector because it is nothing but an eigenvector of λ_1.

Therefore, it is a non zero vector. So, there is surely one coordinate of $\boldsymbol{v}_1$ which is non zero. Now you see that this sequence is converging to that non zero value. Therefore, for sufficiently k the sequence will have non zero values; therefore your x_j will also be non zero for sufficiently large k, why because $\boldsymbol{x}_j{}^{(k+1)}$ is nothing but $\frac{\boldsymbol{y}_j{}^{(k+1)}}{\mu_{k+1}}$. μ is the value of that coordinate of $\boldsymbol{y}$ at which the maximum is achieved and this is some coordinate of $\boldsymbol{y}$ which is non zero.

Therefore, the jth coordinate of $\boldsymbol{x}$ vector will also be non zero at least after sufficiently large terms. So, we obtained up to here. Now you see what we can do with that. If you recall $\boldsymbol{x}^{(k+1)}$ is nothing but $\frac{\boldsymbol{y}^{(k+1)}}{\mu_{k+1}}$. By this time, we have understood it very clearly. Now what I am doing is, I am simply bringing μ_{k+1} on the right hand side and writing this equation.

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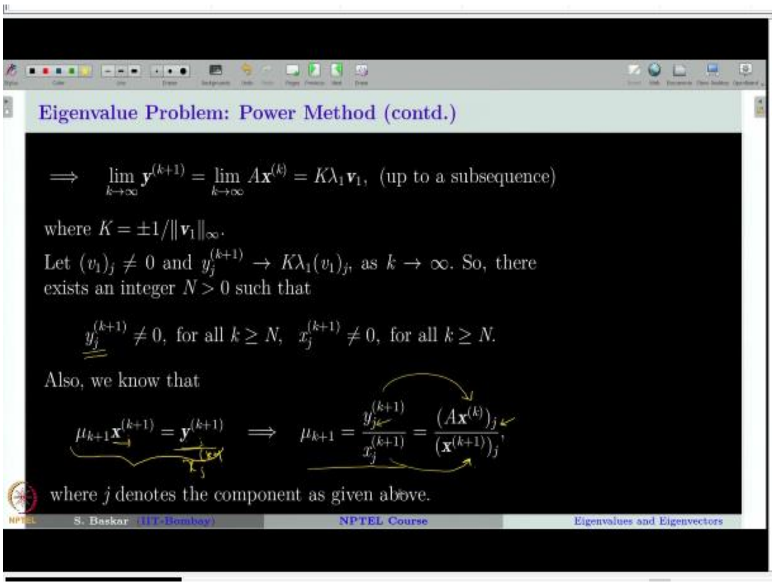

That gives me μ_{k+1} is equal to what I chose the jth coordinate because this is a vector this is a vector. I am just picking the jth coordinate of these 2 vectors then that jth coordinate will also satisfy the same equation. Therefore, I can write $\mu_{k+1} = \frac{\boldsymbol{y}_j{}^{(k+1)}}{\boldsymbol{x}_j{}^{(k+1)}}$. That is what I am writing here and this is just the definition of $\boldsymbol{y}^{(k+1)}$ and since I am only taking the jth coordinate I am putting here j and similarly this I am just retaining as it is.

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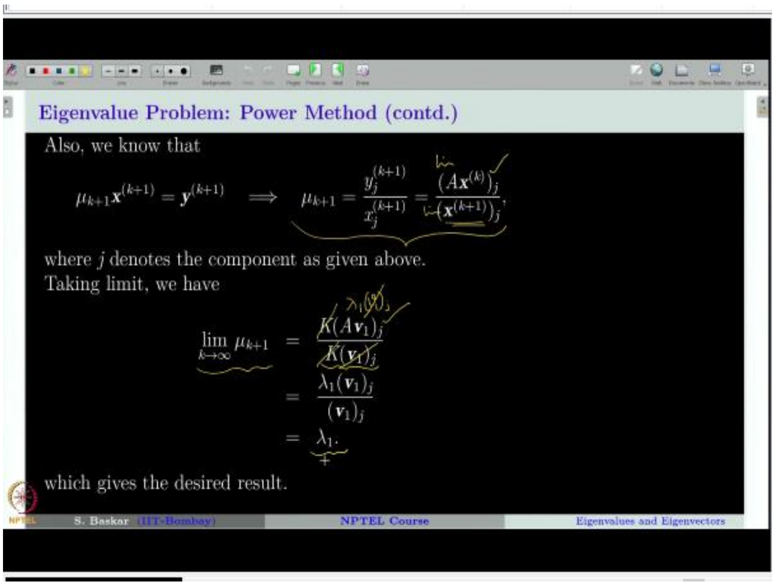

So, we obtained this expression for one typical coordinate of the vectors $\boldsymbol{y}$ and $\boldsymbol{x}$. With this we will take the limit as $\lim_{k\to\infty}$, we will get $\lim_{k\to\infty} \mu_{k+1}$ is equal to, if you recall I am taking limit both for numerator as well as for the denominator. For numerator we already know how this limit looks like. This limit looks like this. $A\boldsymbol{x}^{(k)}$ vector is $K\lambda_1\boldsymbol{v}_1$.

Therefore, its *j*th coordinate is the *j*th coordinate of the vector $\boldsymbol{v}_1$ and that is what I am writing here. The same without multiplying *A* will have $K\boldsymbol{v}_1$ that we have proved here. So, we have derived that part in this step. Therefore, combining those two information you can write $\lim_{k\to\infty} \mu_{k+1} = \frac{K(A\boldsymbol{v}_1)_j}{K(\boldsymbol{v}_1)_j}$.

Now *K* gets canceled, this is nothing but $\lambda_1\boldsymbol{v}_1$ and for $\boldsymbol{v}_1$ it is only the *j*th coordinate because I am only taking the *j*th coordinate of that vector and so again the *j*th coordinate of $\boldsymbol{v}_1$ will get canceled with the *j*th coordinate of $\boldsymbol{v}_1$ in the denominator that leaves us only with λ_1. Therefore, this sequence is converging to the dominant eigenvalue of the matrix . And that completes the proof of the convergence theorem for power method.

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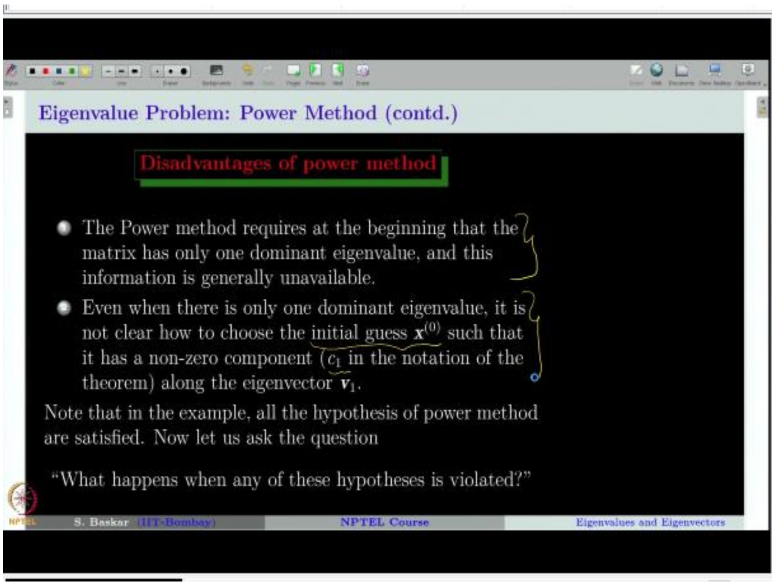

There are some serious disadvantages of the power method, what are they? The power method requires at the beginning that the matrix has only one dominant eigenvalue. That is the dominant eigenvalue should be unique, but when we have the matrix we do not any information about the eigenvalues and the eigenvectors of that matrix. In fact, that is why we are going for the numerical method.

Therefore, knowing this information is impossible and second disadvantage is that when you choose an initial guess, we see that in its representation the first scalar c_1 should be not equal to 0. This is also something which we do not know when we are choosing the initial guess simply because we do not know the eigenvectors. Therefore, we absolutely have no way to check the hypothesis of the power method.

Therefore, applying power method to any matrix is more or less a blind work computationally. So, this is something which we have to keep in mind and we have also seen in 2 examples where power method may not converge always. Now we can at least see when it converges, when it does not converge especially when we have more information about the matrix.

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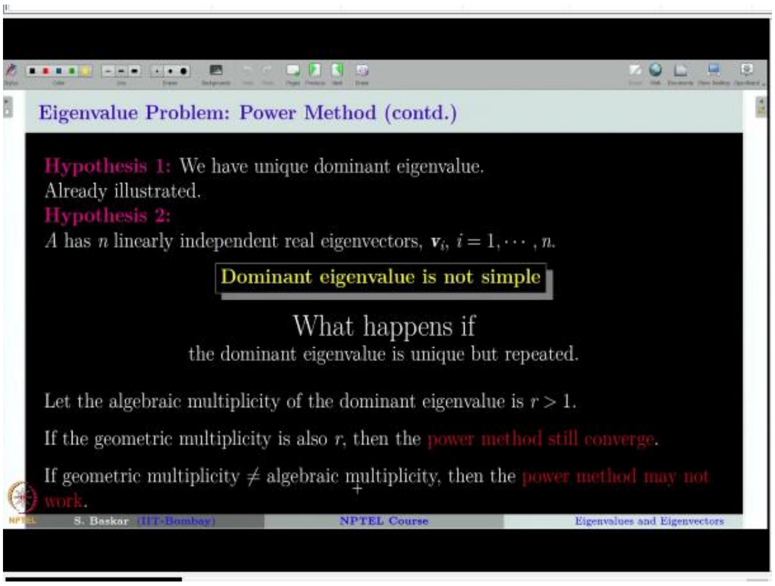

Let us try to see how power method behaves when it violates one or the other hypotheses that are given in the theorem. Let us take the first hypothesis. If you recall the first hypothesis of the theorem says that the matrix should have unique dominant eigenvalue. Now the question is what happens if the matrix does not have unique eigenvalue? We have already illustrated it at the beginning of this class.

We have taken a matrix B where the eigenvalues of B are (1, -2, 2). Therefore although the matrix B had distinct eigenvalues, its dominant eigenvalue is not unique and we have seen at least up to thousand iterations that the power method sequence was not going close to the eigenvalue or the corresponding eigenvector. It kept on oscillating between 2 numbers therefore that gives us a feeling that the power method was not converging in that particular example.

From there we can say that when the dominant eigenvalues is not unique for a matrix then power method may not converge. Let us take hypothesis 2. The hypothesis 2 says that a set of eigenvectors should form a basis for $\mathbb{R}^n$. Now if the dominant eigenvalue is not simple what happens if the dominant eigenvalue is unique but repeated? Let the algebraic multiplicity of the dominant eigenvalue is say $r > 1$.

If the geometric multiplicity is also r then the power method still converges. However, if the geometric multiplicity is not equal to the algebraic multiplicity then the power method may not work. This is just a remark.

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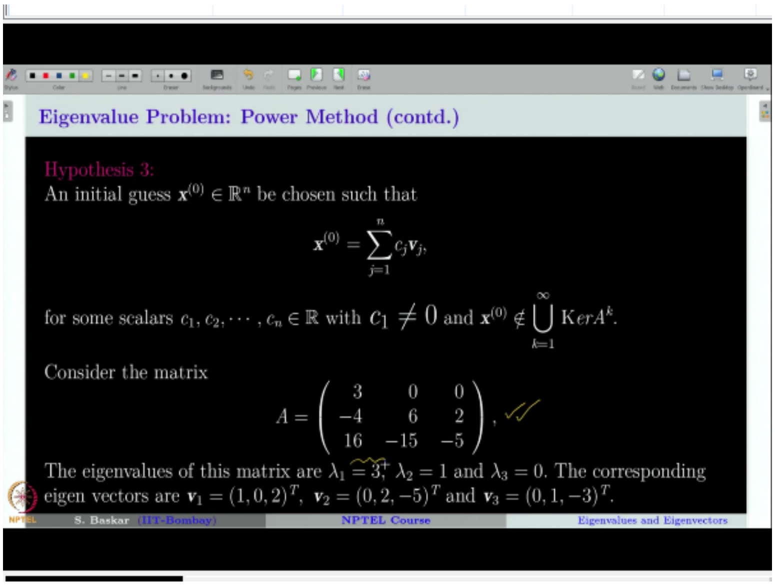

Now let us go to the hypothesis 3. The hypothesis says, importantly, that when you represent your initial guess as the scalar multiple of the eigenvectors then the scalar in the first term that is c_1 should not be equal to 0. That is an important condition that we have imposed which we have no way to check when we are applying power method to any matrix unless we know all the eigenvectors.

That is something which is too much to expect, because if we know all this information why are we going for power method. Therefore, practically these are not something which we have to or we can check, this is just for the understanding purpose we are now discussing what happens if c_1 for that initial guess $x^{(0)}$ that we have chosen. Let us see what happens. Let us take this nice matrix A. Why it is nice because it has unique dominant eigenvalue.

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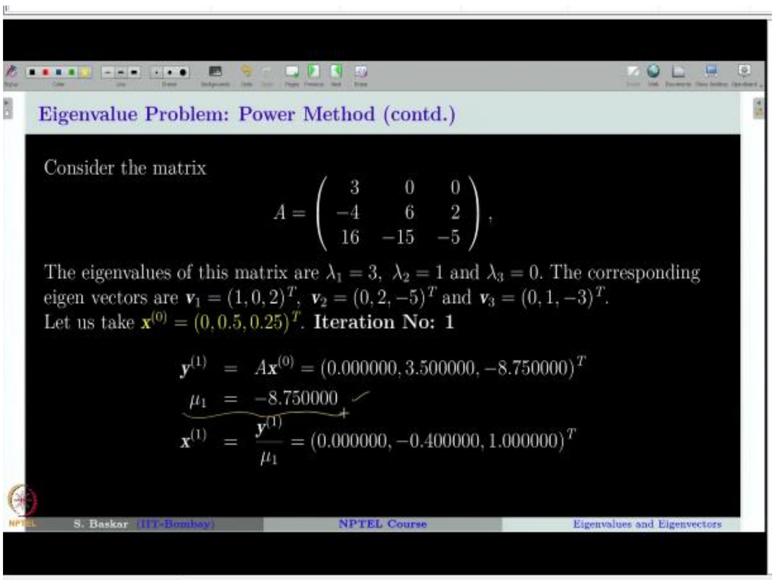

Now let us take this initial guess that is $x^{(0)} = (0,0.5,0.25)^T$. Now see what is going to happen. At the iteration 1 we have like this, just keep watching μ_1 that will give us a clear idea of where we are converged.

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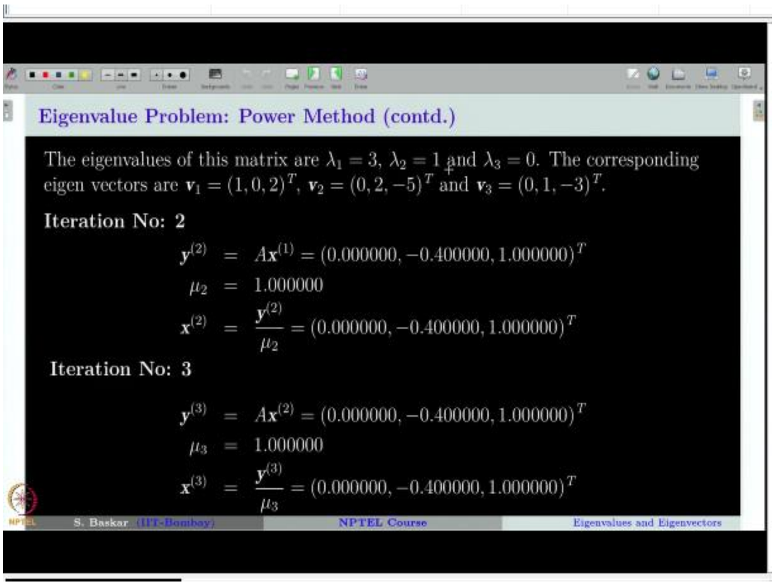

Then the second iteration gives immediately jump to 1 which is exactly the second dominant eigenvalue of the matrix A. Then iteration 3 is also the same.

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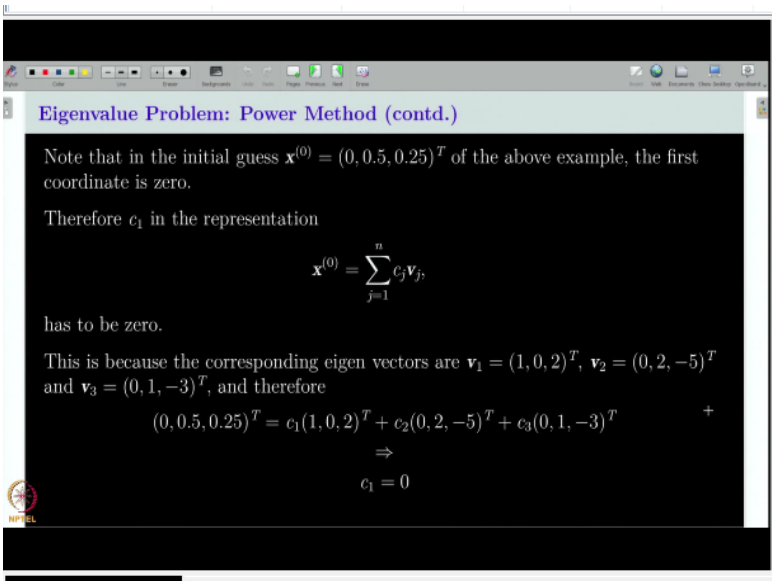

Iteration 4 and so on. Now you can see that once it gets stuck with this it is not going to move anywhere. Therefore, you can conclude that the power method in this particular case, that is for the matrix A with the initial guess as (0, 0.5, 0.25), it converges to this second dominant eigenvalue. Why it converges to second dominant eigenvalue because you can see that in this representation when you take $c_1 v_1 + c_2 v_2 + c_3 v_3 = x^{(0)}$.

That is what we are writing. So, what is this c_1, v_1 is $(1,0,2)^T + c_2(0,2,-5)^T + c_3(0,1,-3)^T$ = $(0,0.5,0.25)^T$. From here you can clearly see that c_1 is 0.

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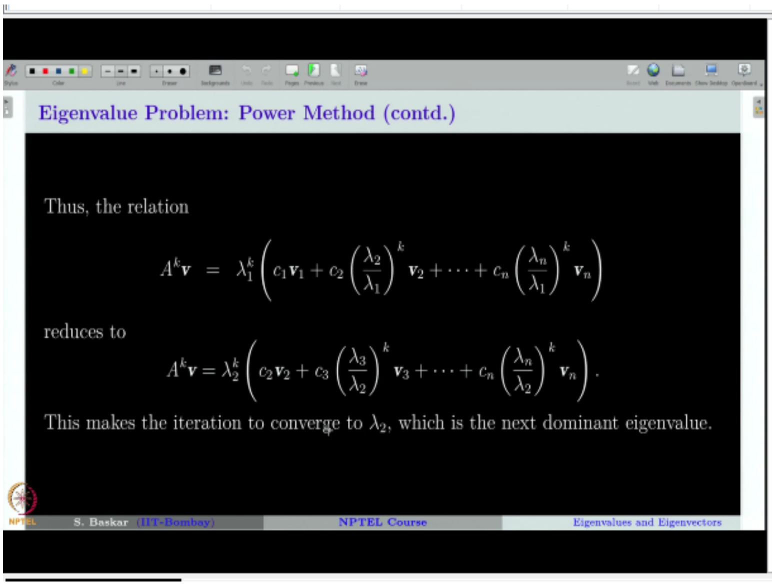

That means in all our derivation, if you go to our previous lecture and recall when we keep on multiplying A with the vector $x^{(0)}$ now, you will have this expression. In that now you see this happen to be 0. Therefore, who is dominating here the second term onwards is what is remaining in this expression. Therefore, the immediate second dominant eigenvalue is taking the position of your dominant eigenvalue.

And you can write that expression when c_1 is 0 you can write it as $\lambda_2^k(c_2 v_2 +)$ that is the role of $c_1 v_1$ is now replaced by $c_2 v_2 + c_3$ onwards you have. Now when you take k tending to infinity all these terms are going to 0 and you are left out with the term with λ_2. That is why the sequence is converging to this second dominant eigenvalue. It is not because the first coordinate of the $x^{(0)}$.

So, this is not going to decide where your sequence μ will converge. Remember it is because in that representation your c_1 was 0 that is why the sequence converges to the second dominant eigenvalue. So, often students make mistake; they see the first coordinate of the initial guess 0 immediately they will conclude that the sequence will converge to the second dominant eigenvalue.

No, you have to write this system and solve it for c_1, c_2, c_3 and check whether c_1 is 0. If c_1 is 0 then if the second dominant eigenvalue is unique then the sequence will converge to the second dominant eigenvalue. Similarly, if c_1 and c_2 both are 0 and if the third dominant eigenvalue is unique then the sequence will converge to the third dominant eigenvalue and so on. So, this is very important for us to remember.

So, in this case c_1 happened to be 0. That is why the sequence was converging to the second dominant eigenvalue λ_2. So, with this we will stop our discussion on power method in this class. We will continue our discussion in the next class. Thanks for your attention.