

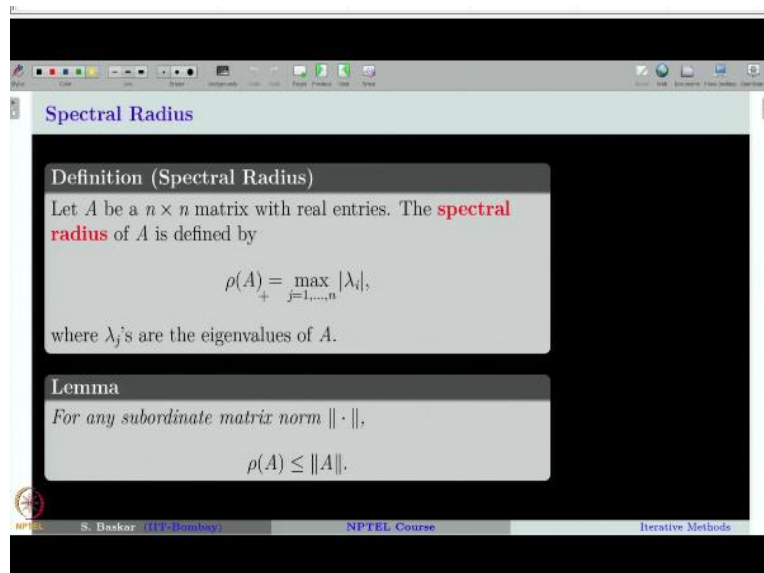
**Numerical Analysis**  
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**Lecture-20**  
**Linear Systems: Iterative Methods (General Convergence Theorem)**

Hi, we are discussing iterative methods for linear systems. In this we have completed Jacobi method and Gauss-Seidel method. We have also learnt sufficient condition under which the iterative sequences generated by these two methods converge. In the last class we have also seen through an example that an iterative sequence may converge rather slowly or it may not even converge.

In this lecture we will try to prove another theorem which gives us a necessary and sufficient condition under which an iterative method converges. This condition depends on the spectral radius of the iterative matrix of a method. In this we will consider a general form of the iterative method and we will prove this necessary and sufficient condition for a general iterative method.

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The image shows a presentation slide titled "Spectral Radius". The slide content is as follows:

**Spectral Radius**

**Definition (Spectral Radius)**  
Let  $A$  be a  $n \times n$  matrix with real entries. The **spectral radius** of  $A$  is defined by

$$\rho(A) = \max_{j=1, \dots, n} |\lambda_j|,$$

where  $\lambda_j$ 's are the eigenvalues of  $A$ .

**Lemma**  
For any subordinate matrix norm  $\|\cdot\|$ ,

$$\rho(A) \leq \|A\|.$$

At the bottom of the slide, there is a footer with the text: "S. Baskar (IIT-Bombay) NPTEL Course Iterative Methods".

For this we need a concept called spectral radius of a matrix. Let us first define this concept. Let  $A$  be a  $n \times n$  matrix. The spectral radius of the matrix  $A$  is defined as the maximum of the absolute values of the eigenvalues of the matrix  $A$  and it is denoted by  $\rho(A)$ .

Now that is an interesting result concerning this spectral radius which says that you give me any subordinate matrix norm, the norm of  $A$  with respect to that subordinate matrix norm will

surely be greater than or equal to this spectral radius of the matrix  $A$ . Not only that you can also see that the spectral radius will be the greatest lower bound of the set of all subordinate matrix norm of  $A$ . Means what? You collect all possible subordinate matrix norms. This is a mathematical concept you cannot do this practically.

Just mathematically speaking; take all subordinate matrix norm, find the value of this norm when it is applied to the matrix  $A$ , which is fixed for you. Then the spectral radius will be the greatest lower bound of that set. That is little difficult for us to prove. So, we will not get into the proof of that result. Anyway, we will not be using that; it is only an interesting result. Whereas, the result which is stated here is an interesting result and also it is easy to prove.

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The screenshot shows a presentation slide with the following content:

**Spectral Radius**

**Lemma**  
For any subordinate matrix norm  $\|\cdot\|$ ,

$$\rho(A) \leq \|A\|.$$

**Proof**

For any eigenvalue  $\lambda$  of  $A$  and a corresponding unit eigenvector  $v$ , we have

$$\begin{aligned} |\lambda| &= |\lambda| \cdot \|v\| \\ &= \|\lambda v\| \\ &= \|Av\| \\ &\leq \|A\| \|v\| = \|A\| \end{aligned}$$

Handwritten notes on the slide include:

$$|\lambda| \leq \|A\|$$

$$\rho(A) \leq \|A\|$$

At the bottom of the slide, it says: S. Daskar (IIT Bombay) NPTEL Course Iterative Methods

So, let us try to prove this lemma. You take any eigenvalue of  $A$  and a corresponding eigenvector of that eigenvalue. Let us make it unit vector and call it as  $v$ . Then you can write  $|\lambda| = |\lambda| \|v\|$ . Why we can write it because  $v$  we have taken as the unit vector. Therefore,  $\|v\|$  will be equal to 1. Therefore, we can write like this. Now use one of the properties of the vector norm and push this modulus of lambda into the norm.

And you can get  $\|\lambda v\|$ , you just go to the definition of vector norm, you can see that this is one of the properties of the vector norm. So, we are using that property. Now once you see  $\lambda v$ , you can immediately write it as  $Av$ , why? Because  $\lambda$  is the eigenvalue and  $v$  is the corresponding eigenvector of lambda. Therefore, you can write  $\lambda v$  as  $Av$ . Now whenever you see this expression immediately one important property of the subordinate matrix norm should come in our mind.

That is nothing but  $\|Av\|$  which is the vector norm is less than or equal to subordinate norm  $A$  into vector norm  $v$ . Again, you can see that vector norm  $v = 1$  because  $v$  is a unit vector with respect to this norm. Therefore, this is equal to norm  $A$ . Now you see you have chosen this eigenvalue arbitrarily. Therefore, this inequality that  $|\lambda| \leq \|A\|$  holds for any eigenvalue of  $A$ .

In particular, it will also hold for that eigenvalue at which the maximum is achieved. That is, it also holds for the spectral radius of  $A$ . Spectral radius of  $A$  is nothing but the maximum of modulus of all such  $\lambda$ s. Therefore, the spectral radius will also be less than or equal to norm  $A$ . So, that is what we understand and that is what we want to prove in this lemma also. This lemma gives us an encouraging message that if we use spectral radius in our analysis, we may get a better estimate for the error. So, that is what we are going to do.

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The slide is titled "Convergence of General Iterative Methods". It contains the following text and equations:

Let us consider an iterative method in the form

$$\mathbf{x}^{(k+1)} = B\mathbf{x}^{(k)} + \mathbf{c}, \quad k = 0, 1, 2, \dots \quad (*)$$

**Theorem (Necessary and Sufficient Conditions)**

For any  $\mathbf{x}^{(0)} \in \mathbb{R}^n$ , the sequence  $\{\mathbf{x}^{(k)}\}$  defined by (\*) converges to the solution of  $\mathbf{x} = B\mathbf{x} + \mathbf{c}$  if and only if

$$\rho(B) < 1.$$

Handwritten notes on the right side of the slide include:

- $\rho(A) < 1$  (crossed out)
- $A\mathbf{z} = \mathbf{b}$
- $\mathbf{z} = B\mathbf{z} + \mathbf{c}$

At the bottom of the slide, it says "S. Baskar (IIT-Rohrky)", "NPTEL Course", and "Iterative Methods".

Before that, what we will do is instead of considering Jacobi method or Gauss-Seidel method we will make our discussion a bit general and consider any iterative method which can be written in this form. Recall in the last class we have seen that Gauss-Seidel method can also be written in this matrix form as well as Jacobi method can also be written in this matrix form.

Now just to make our discussion little general; we will consider any iterative method which can be written in this form. Therefore, in our discussion Jacobi and Gauss-Seidel can be taken as a particular case. Now let us state the important theorem which is the necessary and sufficient condition for convergence of any iterative sequence that can be written in this form. What the theorem says? You take any initial condition that is more important.

You take any initial condition the sequence  $x^{(n)}$  generated using this formula converges to the solution  $x = Bx + c$ , why we are taking this as the system because if you recall we had  $Ax = b$ . Generally, we will write this system in the equivalent form as  $x = Bx + c$ . Therefore, any solution of our original system will also be the solution of this system. That is how we generally derive the iterative methods.

With that in mind, we say that this sequence converges to the solution of this system which in that sense is equivalent to the solution of our original system and the theorem says that this happens. That is, the convergence happens for any initial guess if and only if the spectral radius of the iterative matrix is less than 1. Remember the condition is imposed on the iterative matrix.

Recall, in our previous two theorems the condition was imposed on the coefficient matrix. That is the coefficient matrix should be diagonally dominant. That is what the condition we imposed in our previous theorems, they are sufficient conditions. Now here, we are imposing condition on the iterative matrix. Often students make mistake of checking this condition that is  $\rho$  of the coefficient matrix is less than 1.

That is not correct, you should not check this condition for the coefficient matrix but you should check this condition for the iterative matrix  $B$ . Now our interest is to prove this theorem, but to prove this theorem we need two important results from linear algebra. We will recall these results without proving them.

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The image shows a screenshot of a presentation slide. The title is "Convergence of General Iterative Methods (contd.)". The slide content is as follows:

We need the following two lemmas. Proofs are omitted.

**Lemma**  
For any  $n \times n$  matrix  $B$ ,  $\rho(B) < 1$  if and only if

$$\lim_{n \rightarrow \infty} B^n \mathbf{x} = \mathbf{0},$$

for every  $\mathbf{x} \in \mathbb{R}^n$ .

**Lemma**  
Let  $B$  be an  $n \times n$  matrix with  $\rho(B) < 1$ . Then  $(I - B)^{-1}$  exists and we have

$$(I - B)^{-1} = I + B + B^2 + \dots$$

At the bottom of the slide, there is a footer with the text: "S. Baskar (IIT-Bombay) NPTEL Course Iterative Methods".

The first result is for any  $n \times n$  matrix  $B$ , any matrix need not be an iterative matrix. This result is from linear algebra and it holds for any  $n \times n$  matrix. Of course, for us everything is with real entries. The spectral radius of  $B$  is less than 1 if and only if  $\lim_{n \rightarrow \infty} B^n x = 0$  and this should happen for every  $x \in \mathbb{R}^n$  and this is one result that we will be using in the proof of our theorem.

And the next result says that if  $B$  is a  $n \times n$  matrix and the spectral radius is less than 1 then  $(I - B)^{-1}$  exists and it can be written as  $I + B + B^2 \dots$ . This is a familiar result for us when  $B$  is a real number, but it also holds for matrices. That is what we are seeing here. You would have seen these results in linear algebra course. So, we will not prove these theorems here. But we will use them in our main theorem.

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**Convergence of General Iterative Methods (contd.)**

**Theorem (Necessary and Sufficient Conditions)**  
 For any  $\mathbf{x}^{(0)} \in \mathbb{R}^n$ , the sequence  $\{\mathbf{x}^{(k)}\}$  defined by (\*) converges to the solution of  $\mathbf{x} = B\mathbf{x} + \mathbf{c}$  if and only if

$$\rho(B) < 1.$$

**Proof**

To prove  $\rho(B) < 1 \Rightarrow$  convergence for any  $\mathbf{x}^{(0)}$ .

$$\begin{aligned} \mathbf{x}^{(k+1)} &= B\mathbf{x}^{(k)} + \mathbf{c} \\ &= B(B\mathbf{x}^{(k-1)} + \mathbf{c}) + \mathbf{c} \\ &= B^2\mathbf{x}^{(k-1)} + (B + I)\mathbf{c}. \end{aligned}$$

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With these two results in mind, let us prove our main theorem. Let us first assume that the spectral radius of  $B$  is less than 1 and let us prove that the sequence  $x^{(n)}$  converges for any given initial guess  $x^{(0)}$ . How to prove this? Let us start with the expression for the iterative sequence which is given by  $x^{(k+1)} = Bx^{(k)} + c$ . Now you see this expression was used in computing  $x^{(k)}$  also.

Therefore, the same expression can be applied to  $x^{(k)}$  by replacing  $x^{(k+1)}$  by  $x^{(k)}$ . Then you will get instead of  $x^{(k)}$ , you are just putting its expression from where it is computed and now with a simplification you can write this expression as  $B^2x^{(k-1)}$  plus, I am just clubbing these two terms and getting,  $(B + I)c$ . Now once you have this again you can apply the same expression for  $x^{(k-1)}$ .

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**Proof**

To prove  $\rho(B) < 1 \Rightarrow$  convergence for any  $x^{(0)}$ .

$$\begin{aligned} x^{(k+1)} &= Bx^{(k)} + c \\ &= B(Bx^{(k-1)} + c) + c \\ &= B^2x^{(k-1)} + (B + I)c \\ &\dots\dots\dots \\ &\dots\dots\dots \\ &\dots\dots\dots \\ &= B^{k+1}x^{(0)} + (B^k + \dots + B + I)c. \end{aligned}$$

$$\lim_{k \rightarrow \infty} x^{(k+1)} = \lim_{k \rightarrow \infty} B^{k+1}x^{(0)} + \left( \sum_{j=0}^{\infty} B^j \right) c$$

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In that way you can keep on going like this and till you hit the last step of  $x^{(0)}$ . At that stage you will have  $B^{k+1}x^{(0)}$  and the second term would have become the finite sum  $(B^k + B^{k-1} + \dots + B + I)c$ . Now you see we have this nice form  $x^{(k+1)}$  is now given like this. Now you see we have scope to use our two lemmas that we stated before. How can you use them?

For that you take  $\lim_{k \rightarrow \infty}$  on both sides. Once you take this you can recall the first lemma since you have assumed that the spectral radius of  $B$  is less than 1 that lemma says that  $\lim_{k \rightarrow \infty} B^{k+1}x = 0$  for any  $x$ . In particular whatever initial guess that you have chosen will also satisfy that condition. With that you can see that this term goes to 0.

And the second term of course when you take limit  $k$  tends to infinity this finite sum will become a series like this. Now again you can use the second lemma which says that if the spectral radius of matrix  $B$  is less than 1 then this series converges and the limit of this series is nothing but  $(I - B)^{-1}$ .

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Convergence of General Iterative Methods (contd.)

**Proof**

To prove  $\rho(B) < 1 \Rightarrow$  convergence for any  $\mathbf{x}^{(0)}$

$$\mathbf{x}^{(k+1)} = B^{k+1}\mathbf{x}^{(0)} + (B^k + \dots + B + I)\mathbf{c}.$$

$$\lim_{k \rightarrow \infty} \mathbf{x}^{(k+1)} = \lim_{k \rightarrow \infty} B^{k+1}\mathbf{x}^{(0)} + \left( \sum_{j=0}^{\infty} B^j \right) \mathbf{c}$$

Use the above two lemmas to get

$$\lim_{k \rightarrow \infty} \mathbf{x}^{(k+1)} = (I - B)^{-1}\mathbf{c}.$$

Let  $\mathbf{x} = (I - B)^{-1}\mathbf{c}$ .  $(I - B)\mathbf{x} = \mathbf{c}$   
 $\Rightarrow \mathbf{x} = B\mathbf{x} + \mathbf{c}$

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Now we will put these two lemmas into this expression. Remember this tends to 0 and this will be equal to  $(I - B)^{-1}$ . Therefore,  $\lim_{k \rightarrow \infty} \mathbf{x}^{(k+1)}$  is equal to this. This is the limit of this sequence. So, limit exists, that is what we have seen using these two lemmas. Now what is this limit? Let us call this limit as  $x$ , then you can see that  $(I - B)x = c$ . That is nothing but  $x = Bx + c$ .

This is precisely what we have demanded in the statement of our theorem. If you go back and see what we said is that the sequence should converge to the solution of this system and this is what now we are precisely seeing that the limit is nothing but the solution of this system.

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Convergence of General Iterative Methods (contd.)

**Proof**

To prove  $\rho(B) < 1 \Rightarrow$  convergence for any  $\mathbf{x}^{(0)}$

$$\mathbf{x}^{(k+1)} = B^{k+1}\mathbf{x}^{(0)} + (B^k + \dots + B + I)\mathbf{c}.$$

$$\lim_{k \rightarrow \infty} \mathbf{x}^{(k+1)} = \lim_{k \rightarrow \infty} B^{k+1}\mathbf{x}^{(0)} + \left( \sum_{j=0}^{\infty} B^j \right) \mathbf{c}$$

Use the above two lemmas to get

$$\lim_{k \rightarrow \infty} \mathbf{x}^{(k+1)} = (I - B)^{-1}\mathbf{c}.$$

Let  $\mathbf{x} = (I - B)^{-1}\mathbf{c}$ .  
 Then  $\mathbf{x}^{(k)} \rightarrow \mathbf{x}$  as  $k \rightarrow \infty$ .  $\mathbf{x} = B\mathbf{x} + \mathbf{c}$

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So, that completes the proof that if the spectral radius of the iteration matrix  $B$  is less than 1 then the corresponding iteration sequence will converge to a vector  $x$  which is the solution of the system  $x = Bx + c$ .

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Convergence of General Iterative Methods (contd.)

**Proof**

To prove convergence for any  $x^{(0)} \Rightarrow \rho(B) < 1$ .

Assume the contrary that  $\rho(B) \geq 1$ .

Choose the initial guess  $x^{(0)}$  such that  $x^{(0)} + x$  is an eigenvector corresponding to  $\lambda_j$ , where  $\rho(B) = |\lambda_j|$ .

We give the proof only for a particular case when all eigenvalues of  $B$  are real.

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So, we finish the proof of one side. Let us now assume that the iterative sequence converges for any initial guess  $x^{(0)}$ . That is very important. The statement is that the sequence converges for any initial guess  $x^{(0)}$  and this is the assumption and we have to prove that the spectral radius of  $B$  is less than 1. How to prove this? There are many ways to prove it. Let us take a simple approach.

Assume the contrary that the spectral radius of  $B$  is greater than or equal to 1. That is, we are given that the iterative sequence is converging for any initial guess, but we are also assuming that the spectral radius is greater than or equal to 1. With this assumption we will now see that we can arrive at a contradiction. What is that contradiction? I will pick up one initial guess for which the corresponding iterative sequence is diverging.

That is what I am going to prove. For that I have to choose a particular initial guess conveniently to make the sequence diverge. How am I going to choose this initial guess? I will choose my initial guess in such a way that  $x^{(0)}$  that is my initial guess  $-x$ . Remember  $x$  is the exact solution of the system,  $x = Bx + c$ . This is a vector now. Now I will choose my initial guess in such a way that this vector is an eigenvector corresponding to which eigenvalue?

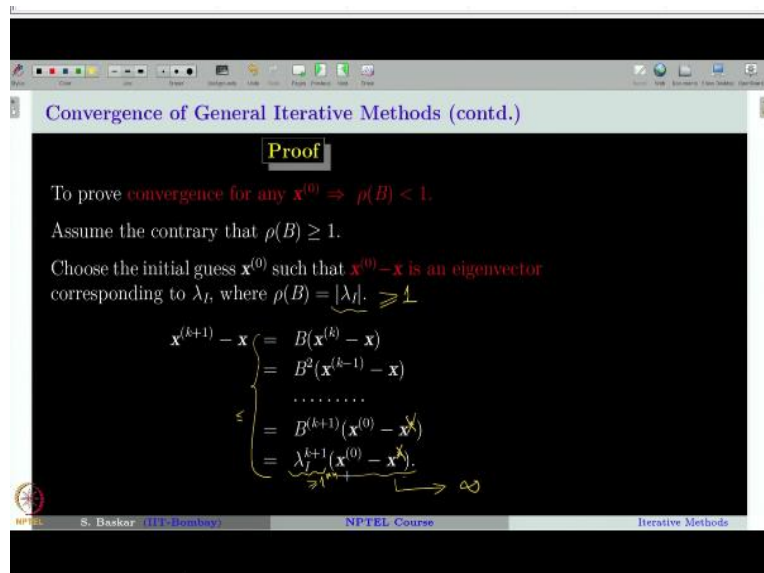
The eigenvalue at which the maximum is achieved, that is nothing but that eigenvalue which came as the spectral radius of my iterative matrix. So, this can always be done, you are given eigenvalue. Then you can just take any eigenvector and then you can just write it in this form



by conveniently choosing your vector  $x^{(0)}$ . You have freedom to choose  $x^{(0)}$ , but you have no freedom on  $x$  because it comes as the solution of your system.

But you have freedom on  $x^{(0)}$ , so you can always choose such a vector. Now we will see how we are going to arrive at a contradiction.

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Let us take this term; if you recall  $x^{(k+1)}$  is nothing but  $Bx^{(k)} + c$ . That is the way we have defined the iterative sequence. The same equation also holds for  $x$ . Therefore,  $x$  should be equal to  $Bx + c$ . Now we are subtracting these two. Therefore, this will go off and you will have  $Bx^{(k)} - Bx$  that I can write as  $B(x^{(k)} - x)$ . Now the idea is something similar to what we did in the previous case.

You apply the same expression for  $x^{(k)}$  and write this as  $B^2(x^{(k-1)} - x)$ . Now, once you do that you keep on continuing this recursively until you hit the dead end where you have  $B^{(k+1)}(x^{(0)} - x)$ . Now what is this? This is nothing but the eigenvector. Therefore, you can write it as  $\lambda_l^{k+1}$  into the eigenvector. Again, this star should not be there it is a typo  $x^{(0)} - x$  and you have this because this is a eigenvector.

Now what we assumed as a contrary that  $\lambda_l$  is greater than or equal to 1. That is what we have assumed. That is this is something greater than or equal to 1 rise to  $k + 1$ . If  $\lambda_l = 1$  then you can see that the right-hand side will never go to 0 as  $k$  tends to infinity, it will be a fixed number. On the other hand, if  $\lambda_l$  is strictly greater than 1 then the right-hand side will go to infinity.

And therefore, the left-hand side will also go to infinity. Remember in this derivation we only have equality, there is no inequality. Therefore, whatever happens on the right-hand side will surely happen to the left-hand side also. If it was less than or equal to then you cannot say that the left-hand side will go to infinity if right-hand side goes to infinity. But in this case, you can say because it is an equality.

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**Proof**

To prove convergence for any  $\mathbf{x}^{(0)} \Rightarrow \rho(B) < 1$ .

Assume the contrary that  $\rho(B) \geq 1$ .

Choose the initial guess  $\mathbf{x}^{(0)}$  such that  $\mathbf{x}^{(0)} - \mathbf{x}^*$  is an eigenvector corresponding to  $\lambda_l$ , where  $\rho(B) = |\lambda_l|$ .

$$\begin{aligned} \mathbf{x}^{(k+1)} - \mathbf{x}^* &= B(\mathbf{x}^{(k)} - \mathbf{x}^*) \\ &= B^2(\mathbf{x}^{(k-1)} - \mathbf{x}^*) \\ &\dots\dots\dots \\ &= B^{k+1}(\mathbf{x}^{(0)} - \mathbf{x}^*) \\ &= \lambda_l^{k+1}(\mathbf{x}^{(0)} - \mathbf{x}^*). \end{aligned}$$

Assumption  $\rho(B) \geq 1 \Rightarrow |\lambda_l| \geq 1$

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That means that our assumption that  $\lambda_l$  is greater than or equal to 1 made us to choose an initial guess for which the corresponding iterative sequence is not converging which is not correct as per our assumption that our iterative sequence always converges for any initial guess. Therefore, our assumption that the spectral radius of  $B$  is greater than or equal to 1 is not correct.

That means the spectral radius of  $B$  should be always less than 1 if the iterative sequence is converging for any  $x^{(0)}$ . That is more important. For any  $x^{(0)}$  if it converges that will surely imply that the spectral radius is less than 1 and this completes the proof of this important theorem.

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Convergence of General Iterative Methods (contd.)

To prove convergence for any  $\mathbf{x}^{(0)} \Rightarrow \rho(B) < 1$ .

Assume the contrary that  $\rho(B) \geq 1$ .

Choose the initial guess  $\mathbf{x}^{(0)}$  such that  $\mathbf{x}^{(0)} - \mathbf{x}$  is an eigenvector corresponding to  $\lambda_l$ , where  $\rho(B) = |\lambda_l|$ .

$$\mathbf{x}^{(k+1)} - \mathbf{x} = \lambda_l^k (\mathbf{x}^{(0)} - \mathbf{x})$$

Assumption  $\rho(B) \geq 1 \Rightarrow \lambda_l \geq 1 \Rightarrow \text{RHS} \rightarrow \infty$  as  $k \rightarrow \infty$ .

**Important Observation**

If we start with  $\mathbf{x}^{(0)}$  such that  $\mathbf{x}^{(0)} - \mathbf{x}$  is an eigenvector, then

- $\rho(B)$  is closer to 0  $\Rightarrow$  faster the convergence is;
- $\rho(B)$  is closer to 1  $\Rightarrow$  slower the convergence is.

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Let us also have an important observation which we have already made in the case of the previous theorem on Jacobi and Gauss-Seidel method. You can see that if you are starting with the  $\mathbf{x}^{(0)}$  such that  $\mathbf{x}^{(0)} - \mathbf{x}$  is an eigenvector, again there is a typo here. Then this equality shows that if  $\lambda_l$  is very close to 0 then the convergence is going to be faster. On the other hand, if the spectral radius that is in this case it is  $\lambda_l$  is very close to 1 then the convergence is going to be very slow.

You can see directly from here. As  $k$  tends to infinity the convergence is happening because this term is going to 0 as  $k$  tends to infinity if  $\lambda_l$  which is nothing but your spectral radius is less than 1. If it is very close to 1 the convergence will be very slow. If it is very close to 0 then it goes much faster. So, that is what is a clear observation from here and in fact it holds for any choice of  $\mathbf{x}^{(0)}$  also.

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Convergence of Jacobi and Gauss-Seidel Method

Example:  
Consider

$$\begin{cases} x_1 + \quad + x_3 = 0 \\ -x_1 + x_2 = 0 \\ x_1 + 2x_2 - 3x_3 = 0. \end{cases}$$

$$B_J = D^{-1}C = \begin{pmatrix} 0 & 0 & -1 \\ 1 & 0 & 0 \\ 1/3 & 2/3 & 0 \end{pmatrix} \text{ and } B_G = -(L+D)^{-1}U = \begin{pmatrix} 0 & 0 & -1 \\ 0 & 0 & -1 \\ 0 & 0 & -1 \end{pmatrix}$$

Jacobi iteration sequence:  $\mathbf{x}^{(k+1)} = B_J \mathbf{x}^{(k)}$

Gauss-Seidel iteration sequence:  $\mathbf{x}^{(k+1)} = B_G \mathbf{x}^{(k)}$

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With that note in mind, let us now see the example that we discussed in the last class and also at the beginning of this class. If you recall, we have considered this system which is not diagonally dominant, and not only that, you cannot exchange any of the rows to make it diagonally dominant. Then we have formulated the Jacobi method and Gauss-Seidel method and we have seen that Jacobi method was converging with the initial guess was taken as  $\mathbf{x}^{(0)} = (1, 1, 1)$ . With that we have observed that Jacobi method was converging, but very slowly why it was so?

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Convergence of Jacobi and Gauss-Seidel Method

Example:  
Consider

$$\begin{cases} x_1 + \quad + x_3 = 0 \\ -x_1 + x_2 = 0 \\ x_1 + 2x_2 - 3x_3 = 0. \end{cases}$$

$$B_J = D^{-1}C = \begin{pmatrix} 0 & 0 & -1 \\ 1 & 0 & 0 \\ 1/3 & 2/3 & 0 \end{pmatrix} \text{ and } B_G = -(L+D)^{-1}U = \begin{pmatrix} 0 & 0 & -1 \\ 0 & 0 & -1 \\ 0 & 0 & -1 \end{pmatrix}$$

$\rho(B_J) \approx 0.94444$

Jacobi iteration sequence:  $\mathbf{x}^{(k+1)} = B_J \mathbf{x}^{(k)}$

Gauss-Seidel iteration sequence:  $\mathbf{x}^{(k+1)} = B_G \mathbf{x}^{(k)}$

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Now we can try to answer that. For that you have to take this matrix  $B_J$  and find all its eigenvalues and then take the modulus of the eigenvalues. Let us say it is  $\lambda_1, \lambda_2, \lambda_3$ . Then you take the modulus of all these eigenvalues and then take the maximum of all these numbers. In this case it happens to be 0.9444. Now go back to our previous observation.

Your spectral radius of the Jacobi iteration sequence was pretty close to 1. So, that is why we witnessed that the convergence was pretty slow, also we have observed that the iterative sequence of Gauss-Seidel method was oscillating and never converged. Now by computing the spectral radius of  $B_G$  you can also understand why it was happening like that. It was because the spectral radius of  $B_G$ , remember  $B_G$  is the iterative sequence of the Gauss-Seidel method.

And its spectral radius is 1. That may be the reason for why it was not converging but it was oscillating. From here you can see more clearly the behaviour of these two methods and also we can see that for a given system you may not always expect these methods to converge even convergence of one method may not imply the convergence of the other and also the sequences may converge very slowly.

These are some of the disadvantages of iterative methods. Therefore, before applying these methods into your applications, you have to have an idea of whether these methods are going to converge; if so, how fast they will converge? For this all this analysis that we did in this section are very important. Like this in numerical analysis, doing analysis on methods will give us lot of ideas about how to use these methods more efficiently and what are all the limitations of these methods. With this node let us finish this lecture. Thank you for your attention.