

Numerical Analysis
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Lecture-19
Linear Systems: Iterative Methods (Gauss-Seidel Method)

Hi, we are learning iterative methods for linear systems. In this topic in the last class, we have learned Jacobi method and a sufficient condition for the convergence of Jacobi iteration sequence. In this class we will study Gauss-Seidel method.

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This method is named after 2 German mathematicians called Friedrich Gauss and Philipp Ludwig von Seidel. Von Seidel was an assistant to Jacobi and it is widely believed that Gauss knew these 2 methods that is what we now learn as Jacobi method and Gauss-Seidel method previously itself, but many of Gauss work were not communicated. Therefore, it was not known to the mathematical community until Jacobi and Gauss-Seidel discovered these methods independently.

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Iterative Methods: Gauss-Seidel Method

Gauss-Seidel method is a modified version of Jacobi method.
Consider the 3×3 system

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + a_{13}x_3 &= b_1, \\ a_{21}x_1 + a_{22}x_2 + a_{23}x_3 &= b_2, \\ a_{31}x_1 + a_{32}x_2 + a_{33}x_3 &= b_3. \end{aligned}$$

For $a_{ii} \neq 0, i = 1, 2, \dots, n$, we can write

$$\left. \begin{aligned} x_1 &= \frac{1}{a_{11}}(b_1 - a_{12}x_2 - a_{13}x_3), \\ x_2 &= \frac{1}{a_{22}}(b_2 - a_{21}x_1 - a_{23}x_3), \\ x_3 &= \frac{1}{a_{33}}(b_3 - a_{31}x_1 - a_{32}x_2). \end{aligned} \right\}$$

With these short historical notes, let us now start learning the Gauss-Seidel method. Gauss-Seidel method is a modified version of the Jacobi method. Let us quickly recall the Jacobi method and see how we modify this Jacobi method to get the Gauss-Seidel method. Let us again consider only the 3×3 system, because once we understand this generalizing it to any $n \times n$ system is not difficult.

So, if you recall we are given a linear system $Ax = b$, what we do is we keep the diagonal terms on the left-hand side and push all the non-diagonal terms on the right-hand side and then we divide both sides by the diagonal element a_{ii} . Of course, we have to assume that a_{ii} 's should be non-zero. Then we get an equivalent system like this. You can immediately note that any solution to this system will also be a solution to this system. It means these 2 systems are equivalent.

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Iterative Methods: Gauss-Seidel Method (contd.)

Let

$$\mathbf{x}^{(0)} = (x_1^{(0)}, x_2^{(0)}, x_3^{(0)})^T$$

be an initial guess to the true solution \mathbf{x} .

The iteration sequence (for $k = 0, 1, 2, \dots$) is given by

$$\begin{aligned} x_1^{(k+1)} &= \frac{1}{a_{11}} (b_1 - a_{12}x_2^{(k)} - a_{13}x_3^{(k)}) \\ x_2^{(k+1)} &= \frac{1}{a_{22}} (b_2 - a_{21}x_1^{(k+1)} - a_{23}x_3^{(k)}) \\ x_3^{(k+1)} &= \frac{1}{a_{33}} (b_3 - a_{31}x_1^{(k+1)} - a_{32}x_2^{(k+1)}) \end{aligned}$$

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With this observation, now what we will do is we will choose a vector arbitrarily and name it as $\mathbf{x}^{(0)}$ and plug in that on the right-hand side of the equivalent system and get a new vector which we will name as x_1 . Again, you plug in x_1 on the right-hand side to get x_2 and the iteration process goes on like this. In general Jacobi iteration is given like this. One can also write it in the matrix notation which we have done in the last class.

Now once we understood the Jacobi method let us come to the Gauss-Seidel method and see what is the modification that is done in order to get this new method. The idea is rather very nice; you look at the first equation. When you are computing x_1 , you only know x_2 and x_3 from the previous iteration. Therefore, you are substituting the values from the previous iteration. Now when you go to calculate x_2 you can see that the value of x_1 now is known as a recent one.

We do not know whether it is going to be better than the previous iteration as far as the exact solution is concerned or not. However, what we see is that we have a latest computed version of x_1 at the time of computing x_2 . Therefore, the idea is why not we will put $x_1^{(k+1)}$ here instead of the old information. That is the idea. Of course, x_3 is not known to us because we have not yet computed it.

Therefore, we will again borrow this value from the previous iteration. However, this value that is the value of x_1 will be put from the present iteration. Similarly, when you go to compute x_3 at that time both x_1 as well as x_2 are holding a rather latest information that is the value than

what is coming from your previous iteration. Therefore, why not we put $x_1^{(k+1)}$ and $x_2^{(k+1)}$ instead of the values from the previous iteration. So, that is the idea of Gauss-Seidel method.

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Therefore Gauss-Seidel method is defined as x_1 is equal to all the terms where these unknowns are replaced by the values from the previous iteration. When you come to x_2 this term is taken from the present iteration and x_3 is taken from the previous iteration and for the coordinate x_3 you take both the terms from the present iteration. In a sense what we are doing is you have $x_i^{(k+1)} = a_{i1}x_1 + a_{i2}x_2$ and so on $a_{ii}x_i + a_{i,i+1}x_{i+1}$ up to $a_{in}x_n$.

Here, when you are computing the i th coordinate all the terms below the diagonal term or known from the present iteration. Whereas, all the terms which are on the other side of the diagonal term is going to be known from the previous iteration only. Whereas the diagonal term is not appearing on the right-hand side, because it is kept here and then you are dividing everything by $\frac{1}{a_{ii}}$.

And there is one more term which is actually coming as $-b_i$, minus of these things. So, that is how we are getting. What is important here is we are now splitting the terms which are in the lower side of the diagonal element and the upper side of the diagonal element. The lower sides of the diagonal element are substituted with the value of the coordinate from the present iteration.

Whereas the terms which are on the right side that is upper side of the diagonal element are substituted with the values from the previous iteration. So, that is the main idea of the Gauss-Seidel method.

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Iterative Methods: Gauss-Seidel Method (contd.)

Let $\mathbf{x}^{(0)} = (x_1^{(0)}, x_2^{(0)}, x_3^{(0)})^T$ be an initial guess to the true solution \mathbf{x} .

Define a sequence of iterates (for $k = 0, 1, 2, \dots$) by

$$x_1^{(k+1)} = \frac{1}{a_{11}}(b_1 - a_{12}x_2^{(k)} - a_{13}x_3^{(k)})$$

$$x_2^{(k+1)} = \frac{1}{a_{22}}(b_2 - a_{21}x_1^{(k+1)} - a_{23}x_3^{(k)})$$

$$x_3^{(k+1)} = \frac{1}{a_{33}}(b_3 - a_{31}x_1^{(k+1)} - a_{32}x_2^{(k+1)}).$$

This method is called the **Gauss-Seidel Iteration method.**

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I hope you have understood the Gauss-Seidel method.

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Iterative Methods: Gauss-Seidel Method (contd.)

Theorem
If the coefficient matrix is diagonally dominant, then the Gauss-Seidel method converges.

Proof:
The Gauss-Seidel method reads

$$x_i^{(k+1)} = \frac{1}{a_{ii}} \left\{ b_i - \sum_{j=1}^{i-1} a_{ij}x_j^{(k+1)} - \sum_{j=i+1}^n a_{ij}x_j^{(k)} \right\}, \quad i = 1, 2, \dots, n.$$

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Let us now ask the question when does the sequence generated by the Gauss-Seidel method converge? Again, the idea is more or less similar to the Jacobi method. We can see that if the coefficient matrix A is diagonally dominant then the iteration sequence generated by the Gauss-Seidel method will always converge with respect to whatever may be the initial guess that we take.

Let us try to prove this theorem now. Again, we will follow the ideas that were introduced in the convergence theorem of the Jacobi method; what we do is we will try to write the error component-wise and then try to estimate it. For that we will first write the Gauss-Seidel method component wise as I explained the i th term is written with a split of the terms which are on the lower side of the diagonal element and the terms on the upper side of the diagonal element.

The lower side we are putting $k + 1$ and the other side of the terms are taken from the previous iteration. So, this is precisely what we have defined as Gauss-Seidel method. Now you can see that the same will also be satisfied by the exact solution. That is, if you take the i th component of the exact solution that can also be written as $x_i = \frac{1}{a_{ii}}$ into b_i minus this term with x_j minus the same term with x_j again.

Only thing is, j runs from 1 to $i - 1$ here and it will run from $j = i + 1$ to n . Now in order to get the i th component of the error involved in the $k + 1$ iteration what you have to do? You have to subtract these 2 equations; you have to subtract this and this. Again, as we did in the Jacobi method you can see that b_i will get canceled and $x_j - x_j^{(k+1)}$ will give you $e_j^{(k+1)}$ and similarly here it will give $e_j^{(k)}$ and the coefficient here. So, that is what we will get.

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Iterative Methods: Gauss-Seidel Method (contd.)

Theorem
 If the coefficient matrix is diagonally dominant, then the Gauss-Seidel method converges.

Proof:
 The Gauss-Seidel method reads

$$x_i^{(k+1)} = \frac{1}{a_{ii}} \left\{ b_i - \sum_{j=1}^{i-1} a_{ij} x_j^{(k+1)} - \sum_{j=i+1}^n a_{ij} x_j^{(k)} \right\}, \quad i = 1, 2, \dots, n.$$

Therefore, the error in each component is given by

$$e_i^{(k+1)} = - \sum_{j=1}^{i-1} \frac{a_{ij}}{a_{ii}} e_j^{(k+1)} - \sum_{j=i+1}^n \frac{a_{ij}}{a_{ii}} e_j^{(k)}, \quad i = 1, 2, \dots, n.$$

At the bottom of the slide, there is a small diagram showing the error term $x_i - x_i^{(k+1)}$ and the slide footer includes 'S. Dinkar (IIT Bombay)', 'NPTEL Course', and 'Iterative Methods'.

When we subtract these two $e_i^{(k+1)}$, b went off. Therefore, we are left out with this terms where $x - x_j^{(k+1)}$ is $e_j^{(k+1)}$ and $x_j^{(k)}$ is now $e_j^{(k)}$ because we are subtracting $x_i - x_i^{(k+1)}$. Now we have this expression for the error. Note that in Jacobi method, the only difference is these terms are

combined together with the superscript k , whereas here these terms are split into 2 parts. The left part is with $k + 1$ and the right part is with k . That is the only difference in the Gauss-Seidel method.

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Iterative Methods: Gauss-Seidel Method (contd.)

Proof:
 The Gauss-Seidel method reads

$$x_i^{(k+1)} = \frac{1}{a_{ii}} \left\{ b_i - \sum_{j=1}^{i-1} a_{ij} x_j^{(k+1)} - \sum_{j=i+1}^n a_{ij} x_j^{(k)} \right\}, \quad i = 1, 2, \dots, n.$$

Therefore, the error in each component is given by

$$|e_i^{(k+1)}| \leq \left| - \sum_{j=1}^{i-1} \frac{a_{ij}}{a_{ii}} e_j^{(k+1)} - \sum_{j=i+1}^n \frac{a_{ij}}{a_{ii}} e_j^{(k)} \right|, \quad i = 1, 2, \dots, n.$$

Define

$$\alpha_i = \sum_{j=1}^{i-1} \left| \frac{a_{ij}}{a_{ii}} \right|, \quad \beta_i = \sum_{j=i+1}^n \left| \frac{a_{ij}}{a_{ii}} \right|, \quad i = 1, 2, \dots, n,$$

with the convention that $\alpha_1 = \beta_n = 0$.

Now what we do? We will define this term as α , of course we are taking modulus because we are going to take modulus on the both sides in the next step. Therefore, we give a notation for the modulus of this term as α_i and modulus of this term as β_i . Here you can see that when $i = 1$ the sum goes from $j = 1$ to $i - 1$. It means when $i = 1$ it goes from 1 to 0 which is not looking nice.

Therefore, what we will do is, we will just define the case $i = 1$ for α as 0 separately. Similarly, the same problem will come for β when $i = n$. In that case j will run from $n + 1$ to n which does not look nice. Therefore, we will separately define β_n as 0. So, this is just a notational convention that we will keep in mind when we are defining these terms. So, these are some notations we have taken.

Now what we will do? We will take modulus on both sides of this equation and then when we push the modulus into this sum. We will get a less than or equal to sign and then what I will do as we did in the previous theorem on Jacobi method we will dominate each of these errors by their infinite norm.

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Iterative Methods: Gauss-Seidel Method (contd.)

Therefore, the error in each component is given by

$$|e_i^{(k+1)}| = \left| -\sum_{j=1}^{i-1} \frac{a_{ij}}{a_{ii}} e_j^{(k+1)} - \sum_{j=i+1}^n \frac{a_{ij}}{a_{ii}} e_j^{(k)} \right|, \quad i = 1, 2, \dots, n.$$

Define

$$\alpha_i = \sum_{j=1}^{i-1} \left| \frac{a_{ij}}{a_{ii}} \right|, \quad \beta_i = \sum_{j=i+1}^n \left| \frac{a_{ij}}{a_{ii}} \right|, \quad i = 1, 2, \dots, n, \quad \|\mathbf{e}^{(k)}\|_\infty = \max_j |e_j^{(k)}|$$

with the convention that $\alpha_1 = \beta_n = 0$.

$$\Rightarrow |e_i^{(k+1)}| \leq \alpha_i \|\mathbf{e}^{(k+1)}\|_\infty + \beta_i \|\mathbf{e}^{(k)}\|_\infty, \quad i = 1, 2, \dots, n. \quad (1)$$

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Then what we will get? We will get, remember we are taking modulus of this equation and that gives me $\text{mod } e_i^{(k+1)}$ less than or equal to; when I take modulus here and take that modulus inside this sum, I have α_i for this term that I have taken here. Similarly, β_i for this term I have taken and I have $\text{mod } e_j^{(k+1)}$. What I will do is, I will replace each of this term by their maximum. That is what we call as $\|\mathbf{e}^{(k+1)}\|_\infty$ and that is what is sitting here.

Similarly, the same thing we will do for the second term also and that is what is sitting here as the infinite norm; therefore, this less than or equal to is coming from two steps. One is when you take this modulus inside this sum and another one is when you replace $|e_{j+1}|$ and $|e_j^{(k)}|$ by their respective infinite norms. You get this less than or equal to sign. Remember this inequality is satisfied by e_i for each $i = 1, 2$ up to n , it means all the coordinates of the error vector $\mathbf{e}^{(k+1)}$ will satisfy this inequality.

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Iterative Methods: Gauss-Seidel Method (contd.)

Define

$$\alpha_i = \sum_{j=1}^{i-1} \left| \frac{a_{ij}}{a_{ii}} \right|, \quad \beta_i = \sum_{j=i+1}^n \left| \frac{a_{ij}}{a_{ii}} \right|, \quad i = 1, 2, \dots, n,$$

with the convention that $\alpha_1 = \beta_n = 0$.

$$\Rightarrow |e_i^{(k+1)}| \leq \alpha_i \|e^{(k+1)}\|_\infty + \beta_i \|e^{(k)}\|_\infty, \quad i = 1, 2, \dots, n. \quad (2)$$

Let l be such that

$$\|e^{(k+1)}\|_\infty = |e_l^{(k+1)}|.$$

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Therefore, it will also be satisfied by that coordinate where the infinite norm of $e^{(k+1)}$ is achieved. Say the infinite norm of $e^{(k+1)}$ is achieved at say some l th coordinate where l is something lying between 1 to n . Then since this inequality is satisfied by all such i 's it will also satisfy for that l at which this maximum norm is achieved. Therefore, for that coordinate you will have $e_l^{(k+1)}$ is less than equal to $\alpha_l \|e^{(k+1)}\|_\infty + \beta_l \|e^{(k)}\|_\infty$.

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Iterative Methods: Gauss-Seidel Method (contd.)

Define

$$\alpha_i = \sum_{j=1}^{i-1} \left| \frac{a_{ij}}{a_{ii}} \right|, \quad \beta_i = \sum_{j=i+1}^n \left| \frac{a_{ij}}{a_{ii}} \right|, \quad i = 1, 2, \dots, n,$$

with the convention that $\alpha_1 = \beta_n = 0$.

$$\Rightarrow |e_i^{(k+1)}| \leq \alpha_i \|e^{(k+1)}\|_\infty + \beta_i \|e^{(k)}\|_\infty, \quad i = 1, 2, \dots, n. \quad (2)$$

Let l be such that

$$\|e^{(k+1)}\|_\infty = |e_l^{(k+1)}|.$$

Then with $i = l$ in (3),

$$\|e^{(k+1)}\|_\infty \leq \alpha_l \|e^{(k+1)}\|_\infty + \beta_l \|e^{(k)}\|_\infty.$$

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So, that is what I will write here $e_l^{(k+1)}$ that is nothing but the infinite norm of the vector $e^{(k+1)}$ is less than or equal to $\alpha_l \|e^{(k+1)}\|_\infty + \beta_l \|e^{(k)}\|_\infty$. So, let us keep this inequality in hand and just have a small observation.

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Iterative Methods: Gauss-Seidel Method (contd.)

$$\Rightarrow |e_i^{(k+1)}| \leq \alpha_i \|e^{(k+1)}\|_\infty + \beta_i \|e^{(k)}\|_\infty, \quad i = 1, 2, \dots, n. \quad (3)$$

Let l be such that $\|e^{(k+1)}\|_\infty = |e_l^{(k+1)}|$.

Then with $i = l$ in (3),

$$\|e^{(k+1)}\|_\infty \leq \alpha_l \|e^{(k+1)}\|_\infty + \beta_l \|e^{(k)}\|_\infty.$$

Observe

$$\mu = \max_{1 \leq i \leq n} \sum_{\substack{j=1 \\ j \neq i}}^n \frac{|a_{ij}|}{|a_{ii}|} = \max_{1 \leq i \leq n} (\alpha_i + \beta_i)$$

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The observation is the following. We have so far derived this inequality and now let us have this observation. If you recall the μ which is given here is familiar to us from the convergence theorem of Jacobi method also. If you recall when we were proving the convergence theorem of Jacobi method we have defined this μ . Now you can see that the same μ is nothing but maximum of $\alpha_i + \beta_i$, why?

Because in the definition of μ we are removing the diagonal element and we have j going from 1 to $i - 1$ then i is removed then you are going from $i + 1$ to n . That is precisely how we defined α_i and β_i 's. Therefore, taking maximum on this entire sum is equivalent to taking maximum on the sum of α_i 's and β_i 's. So, at this level we have not done anything new, it is just similar to what we did with the Jacobi method.

Only thing is we are now considering the terms at the left of the diagonal and those terms on the right of the diagonal are treated separately. That is the only difference here.

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Iterative Methods: Gauss-Seidel Method (contd.)

Let l be such that $\|e^{(k+1)}\|_\infty = |e_l^{(k+1)}|$.

Then with $i = l$ in (3),

$$(1-\alpha_l)\|e^{(k+1)}\|_\infty \leq \alpha_l\|e^{(k+1)}\|_\infty + \beta_l\|e^{(k)}\|_\infty.$$

Observe

$$\mu = \max_{1 \leq i \leq n} \sum_{\substack{j=1 \\ j \neq i}}^n \left| \frac{a_{ij}}{a_{ii}} \right| = \max_{1 \leq i \leq n} (\alpha_i + \beta_i)$$

We know A is a diagonally dominant matrix
 $\Rightarrow \mu < 1 \Rightarrow \alpha_l < 1$.

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Now you see we have assumed that A is diagonally dominant and from the direct observation you can see that diagonal dominance of A implies that μ is less than 1. This we have explained in the last class itself, why diagonal dominance of A implies μ less than 1? It comes directly from the definition; if you carefully look into this expression and see the definition of diagonal dominance you will immediately see that μ should be less than 1.

Now μ is less than 1 and μ is nothing but the sum of α 's and β 's. Therefore, just α_i 's alone you take they all will be less than 1. In particular this α_l will also be less than 1. That is the important observation that we are getting. Here from the hypothesis that A is diagonally dominant. Now what we will do is we will take this term to the left-hand side what you will get is $1 - \alpha_l$ you will get.

This goes off and then you have this. Then you bring this term to the right-hand side again and write $1 - \alpha_l$. Now I can do that because $1 - \alpha_l$ is positive, why it is because just now we saw that α is less than 1. Therefore $1 - \alpha$ is positive.

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Iterative Methods: Gauss-Seidel Method (contd.)

Then with $i = l$ in (3),

$$\|e^{(k+1)}\|_\infty \leq \alpha_l \|e^{(k+1)}\|_\infty + \beta_l \|e^{(k)}\|_\infty.$$

Observe

$$\mu = \max_{1 \leq i \leq n} \sum_{\substack{j=1 \\ j \neq i}}^n \frac{a_{ij}}{a_{ii}} = \max_{1 \leq i \leq n} (\alpha_i + \beta_i)$$

We know A is a diagonally dominant matrix
 $\Rightarrow \mu < 1 \Rightarrow \alpha_l < 1.$

$$\Rightarrow \|e^{(k+1)}\|_\infty \leq \frac{\beta_l}{1 - \alpha_l} \|e^{(k)}\|_\infty.$$

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Therefore, you can divide both sides by $1 - \alpha$ to get $\|e^{(k+1)}\|_\infty \leq \frac{\beta_l}{1 - \alpha_l} \|e^{(k)}\|_\infty$. So, this is much more nice for us because all the $e^{(k+1)}$'s are kept on the left hand side and we have all $e^{(k)}$'s on the right hand side. Now let us try to understand how this number behaves that is what is our aim now.

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Iterative Methods: Gauss-Seidel Method (contd.)

We know A is a diagonally dominant matrix
 $\Rightarrow \mu < 1 \Rightarrow \alpha_l < 1.$

$$\Rightarrow \|e^{(k+1)}\|_\infty \leq \frac{\beta_l}{1 - \alpha_l} \|e^{(k)}\|_\infty.$$

Define

$$\eta = \max_{1 \leq i \leq n} \frac{\beta_i}{1 - \alpha_i}$$

Then the above inequality yields

$$\|e^{(k+1)}\|_\infty \leq \eta \|e^{(k)}\|_\infty \leq \eta^2 \|e^{(k-1)}\|_\infty \leq \dots \leq \eta^k \|e^{(0)}\|_\infty$$

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For that first we will remove the dependency of this number on a particular coordinate because in any estimate we want some fixed number it should not depend on anything which may vary; here l may be anything between $i = 1$ to n . So, let us dominate it by something very specific. For that we will take maximum on all the coordinates of this expression. Because such an expression is possible for each i , β_i , $1 - \alpha_i$, all are positive numbers because β is positive, all α_i 's are less than 1.

Therefore, $1 - \alpha_i$ is also positive for each $i = 1, 2$ up to n . Therefore, when we take maximum of all that that number will also be a positive number and what we will do is, we will now dominate this term by η and therefore we will have $\|e^{(k+1)}\|_\infty \leq \eta \|e^{(k)}\|_\infty$. That is what we finally got. So, this is much more nice than this because we have a nice estimate on this.

Now you see the question is what about this η ? Because we can now recursively write it like this $\eta^2 \|e^{(k-1)}\|_\infty$, the same idea that we always put to get a nice estimate finally is $k + 1$ times $\|e^{(0)}\|_\infty$. Why this is nice because we already chosen $x^{(0)}$ and therefore this term is something which is fixed. We have chosen this and x is something which we may not know but it is already given to us and this is fixed quantity that is more important.

Now we have to only see that η^{k+1} should go to 0 as k tends to infinity. For that we have to see whether η is less than 1 or not, it is not very clear from the way it is defined and the condition that A is a diagonally dominant matrix. But it is not very difficult to see.

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Iterative Methods: Gauss-Seidel Method (contd.)

$$\|e^{(k+1)}\|_\infty \leq \alpha_i \|e^{(k+1)}\|_\infty + \beta_i \|e^{(k)}\|_\infty$$

$$\Rightarrow \|e^{(k+1)}\|_\infty \leq \frac{\beta_i}{1 - \alpha_i} \|e^{(k)}\|_\infty.$$

Define

$$\eta = \max_{1 \leq i \leq n} \frac{\beta_i}{1 - \alpha_i}.$$

Then the above inequality yields

$$\|e^{(k+1)}\|_\infty \leq \eta \|e^{(k)}\|_\infty.$$

For each i ,

$$\frac{\beta_i}{1 - \alpha_i} = \frac{\alpha_i [1 - (\alpha_i + \beta_i)]}{1 - \alpha_i} \geq \frac{\alpha_i}{1 - \alpha_i} [1 - \mu] \geq 0,$$

Let us take this expression. If you recall this expression is what finally led to μ after taking maximum and this is the expression which gave us η after taking maximum? So, let us take this as μ_i, η_i and you see how this is going to be. For that we can just simplify this expression.

After simplification you can get this expression to be $\frac{\alpha_i(1 - (\alpha_i + \beta_i))}{1 - \alpha_i}$.

Now, this I can dominate by μ from the lower side because there is a negative sign here. Therefore, this term will be greater than or equal to the same expression with this replaced by μ . Now you can see that this is positive and also this is positive because of the diagonal dominance property of A which says that μ is strictly less than 1. Therefore, this is also positive. Therefore, both are going to be positive and that shows that μ_i which is this expression, η_i This minus this is greater than or equal to 0, it means this is greater than or equal to this.

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Iterative Methods: Gauss-Seidel Method (contd.)

Define

$$\eta = \max_{1 \leq i \leq n} \frac{\beta_i}{1 - \alpha_i}$$

Then the above inequality yields

$$0 \leq \|e^{(k+1)}\|_\infty \leq \eta \|e^{(k)}\|_\infty \leq \dots \leq \frac{\eta^{k+1}}{2^i} \|e^{(0)}\|_\infty$$

For each i ,

$$\frac{(\alpha_i + \beta_i) - \beta_i}{1 - \alpha_i} = \frac{\alpha_i [1 - (\alpha_i + \beta_i)]}{1 - \alpha_i} \geq \frac{\alpha_i}{1 - \alpha_i} [1 - \mu] \geq 0$$

$$\Rightarrow \eta \leq \mu < 1$$

Thus, when the coefficient matrix A is diagonally dominant, Gauss-Seidel method converges. \square

Now what we have proved is we have shown that η_i is less than or equal to μ_i , from here is greater than or equal to 0. Now this is true for all $i = 1$ to n . In particular for that component where the maximum was achieved also will satisfy this. Therefore, I can as well write it as η_i 's are less than or equal to μ . Remember μ is nothing but the maximum of all these expressions.

Now again this is true for all i . Therefore, for that i for which η attains its maximum also will satisfy this equation. Therefore, we can write η is less than equal to μ . That is what I have written and we already know that μ is less than 1. Therefore, η is less than 1. Therefore, this is less than or equal to up to $\eta^{k+1} \|e^{(0)}\|_\infty$. Now you know how to conclude that the sequence x_n converges to x because we know that this is less than 1 and therefore this term goes to 0 as k tends to infinity.

This is a fixed term and this is always greater than equal to 0. Therefore, you use Sandwich theorem to show that this goes to 0 as k tends to infinity. Therefore, the Gauss-Seidel iteration sequence will also converge if the coefficient matrix A is diagonally dominant.

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Gauss-Seidel Method in Matrix Notation

Recall the Jacobi method can be written as

$$\mathbf{x}^{(k+1)} = B_J \mathbf{x}^{(k)} + \mathbf{c}_J,$$

where

$$B_J = D^{-1}C, \quad \mathbf{c}_J = D^{-1}\mathbf{b},$$

with $D = \text{diag}(a_{11}, \dots, a_{nn})$ and $C = D - A$.

Can we write Gauss-Seidel method in matrix form?

Answer: Yes.

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Now our next question is can we write Gauss-Seidel method in the matrix notation? Recall we have written Jacobi method in the matrix form that is we have written the Jacobi sequence as $\mathbf{x}^{(k+1)}$ is equal to some matrix $B\mathbf{x}^{(k)} + \mathbf{c}$. Let us denote this matrix by B_J and it is called the Jacobi iteration matrix and similarly we will call that \mathbf{c} as \mathbf{c}_J in order to see that it is the vector and the matrix coming from the Jacobi method.

If you recall B_J is given by $D^{-1}C$ and \mathbf{c}_J is given by $D^{-1}\mathbf{b}$, where D is the diagonal matrix, whose elements are precisely the diagonal elements of the matrix A . Similarly, C is given by $D - A$. Now the question is, can we write Gauss-Seidel method also in this form? That is the question. The answer is yes. We can write Gauss-Seidel method also in this form.

How to write it? If you recall instead of keeping all the non-diagonal elements as one matrix what we are doing? We are further decomposing that non-diagonal element as lower part of the diagonal element and upper part of the diagonal element. That is what we are doing in the Gauss-Seidel method.

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Gauss-Seidel Method in Matrix Notation

Write $A = L + D + U$

$$Ax = b \Rightarrow (L + D)x = b - Ux.$$

$$x = (L + D)^{-1} b - (L + D)^{-1} Ux$$

Gauss-Seidel method can be written as

$$x^{(k+1)} = B_G x^{(k)} + c_G, \quad k = 0, 1, 2, \dots$$

where

$$B_G = -(D + L)^{-1} U, \quad c_G = +(D + L)^{-1} b$$

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Therefore, we will now write the matrix A as L which contains all the lower part of the diagonal elements plus diagonal element plus all the upper part of the diagonal elements as separate matrix. Once you do that then the given system $Ax = b$ can be written as $(L + D + U)x = b$. That can be written as $(L + D)x = b - Ux$. Now from there we can easily write $x = (L + D)^{-1}b - (L + D)^{-1}Ux$.

And thereby we can define the Gauss-Seidel iteration sequence in this form. That is x is equal to some $Bx + c$. That is the form which we would like to have for the Gauss-Seidel method also which was already there for the Jacobi method and now we know how to write this iterative matrix. This is called the Gauss-Seidel iterative matrix. Let us give the notation B_G for it and similarly the vector is denoted by c_G , where B_G is given by $-(D + L)^{-1}U$ and c_G is given by $+(D + L)^{-1}b$.

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Convergence of Jacobi and Gauss-Seidel Method

Example:
Consider

$$\begin{aligned} x_1 + \quad + x_3 &= 0 \\ -x_1 + x_2 &= 0 \\ x_1 + 2x_2 - 3x_3 &= 0. \end{aligned}$$

$$A = \begin{pmatrix} 1 & 0 & 1 \\ -1 & 1 & 0 \\ 1 & 2 & -3 \end{pmatrix}$$

For Jacobi iteration matrix,

$$\Rightarrow D = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -3 \end{pmatrix} \text{ and } C = \begin{pmatrix} 0 & 0 & -1 \\ 1 & 0 & 0 \\ -1 & -2 & 0 \end{pmatrix}$$

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Let us take an example. You can clearly observe that in this example the coefficient matrix is not a diagonally dominant matrix not only that any interchange of rows will not make this system diagonally dominant. Therefore, none of our previous theorems whether it is Jacobi method or Gauss-Seidel method will give us any idea of whether the sequences are going to converge or not.

Let us see what is going to happen with this system when we apply Jacobi and Gauss-Seidel methods. Remember the coefficient matrix is given by $A = 1, 0, 1, -1, 1, 0$ and $1, 2$ and -3 . First to write Jacobi iteration in the matrix form you take D which is precisely the diagonal matrix with all the diagonal elements from A and rest of the elements are arranged to get C .

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Convergence of Jacobi and Gauss-Seidel Method

Example:

$$A = \begin{pmatrix} 1 & 0 & 1 \\ -1 & 1 & 0 \\ 1 & 2 & -3 \end{pmatrix}$$

For Jacobi iteration matrix,

$$\Rightarrow D = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -3 \end{pmatrix} \text{ and } C = \begin{pmatrix} 0 & 0 & -1 \\ 1 & 0 & 0 \\ -1 & -2 & 0 \end{pmatrix}$$

$$\Rightarrow \underline{B_J} = D^{-1}C = \begin{pmatrix} 0 & 0 & -1 \\ 1 & 0 & 0 \\ 1/3 & 2/3 & 0 \end{pmatrix}$$

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Therefore, your Jacobi iteration matrix is given like this.

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Convergence of Jacobi and Gauss-Seidel Method

Example:

$$A = \begin{pmatrix} 1 & 0 & 1 \\ -1 & 1 & 0 \\ 1 & 2 & -3 \end{pmatrix}$$

For Gauss-Seidel iteration matrix,

$$\Rightarrow L = \begin{pmatrix} 0 & 0 & 0 \\ -1 & 0 & 0 \\ 1 & 2 & 0 \end{pmatrix} \quad D = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -3 \end{pmatrix} \quad \text{and} \quad U = \begin{pmatrix} 0 & 0 & -1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$\Rightarrow B_G = -(L+D)^{-1}U = \begin{pmatrix} 0 & 0 & -1 \\ 0 & 0 & -1 \\ 0 & 0 & -1 \end{pmatrix}$$

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Similarly for Gauss-Seidel iteration what you do? You collect all these terms that is the lower part of the diagonal terms as L then keep all the diagonal terms with D and take all the upper part of the diagonal elements and arrange them in this form to get U . Once you get that you can write the iterative matrix for Gauss-Seidel method and it is given like this.

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Convergence of Jacobi and Gauss-Seidel Method

Example:

Consider

$$\begin{aligned} x_1 + \quad + x_3 &= 0 \\ -x_1 + x_2 &= 0 \\ x_1 + 2x_2 - 3x_3 &= 0. \end{aligned}$$

$$B_J = D^{-1}C = \begin{pmatrix} 0 & 0 & -1 \\ 1 & 0 & 0 \\ 1/3 & 2/3 & 0 \end{pmatrix} \quad \text{and} \quad B_G = -(L+D)^{-1}U = \begin{pmatrix} 0 & 0 & -1 \\ 0 & 0 & -1 \\ 0 & 0 & -1 \end{pmatrix}$$

Jacobi iteration sequence: $\mathbf{x}^{(k+1)} = B_J \mathbf{x}^{(k)}$

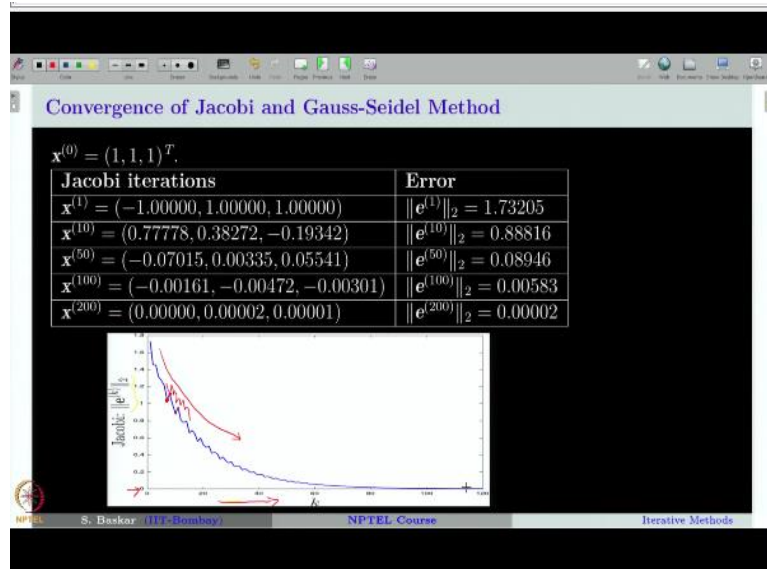
Gauss-Seidel iteration sequence: $\mathbf{x}^{(k+1)} = B_G \mathbf{x}^{(k)}$

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Therefore, the Jacobi iteration sequence in the matrix notation is given like this and the Gauss-Seidel iterative sequence in the matrix notation is given like this. You can also write it in the component form and also you can implement it as a computer code. That may be much more simple and efficient rather than writing in the matrix form because you need to invert this matrix which may not be efficient computationally.

This is just for our understanding I am giving you this. You will understand why I am giving this form probably in the next class. But for now, we will do the iteration using Jacobi method and Gauss-Seidel method.

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For that we will take our initial guess as 1, 1, 1. I am not now going to choose the 0 vector because 0 vector is the solution of our system. Therefore, that may not be a good choice for us as an initial guess. So, I am taking this vector; there is a reason for this also but that I will tell you later. Let us do the computation using Jacobi method. You can see that the Jacobi method started with this iteration $x^{(1)}$ and the l_2 error is given in this column.

I am not showing you all the iterations but I am only showing you some iterations. After 10 iterations the Jacobi method gave this vector, whereas the error it is improved but not so fast. You see even after 10 iterations we have not even captured one significant digit of the exact solution. That is really too much, because generally 10 iteration itself is very big. Not only that I went up to 50 iterations.

Hardly perhaps one significant digit it might have captured or even not that. And after 10 iterations you can see that only 2 digits after decimal points are captured as 0. That is really very slow and then further I went up to 200 iterations even there hardly four digits of 0 are captured. So, this shows that well Jacobi method seems to be converging but it is very, very slow; 200 iterations are generally computationally not affordable.

Because you can count the number of operations involved in each iteration and now if you are going say 500 iteration, 600 iteration, 1000 iterations then it may not be meaningful for us to go for iterative methods rather than going for Gaussian elimination method. So, we should get the convergence within 10, 20 or something like that depending on the dimension of the system.

Such convergence are really not affordable computationally. Let us try to see what is going to happen with Gauss-Seidel. Because we may expect something better in Gauss-Seidel. Why? Because we have seen that η is less than or equal to μ , what it means in the error bound for Jacobi method we had this $\mu^{(k+1)} \|e^{(0)}\|_\infty$. Of course, we computed it with infinite norm but that does not matter.

And similarly, this is for Jacobi method. For Gauss-Seidel method we had $\|e^{(k+1)}\|_\infty \leq \eta^{k+1} \|e^{(0)}\|_\infty$, therefore at least for diagonally dominant matrices this coefficient which we generally call as rate of convergence sometime. The rate of convergence seems to be better for Gauss-Seidel method than for the Jacobi method. Therefore at least in the diagonally dominant case Gauss-Seidel method is expected to perform better than the Jacobi method because of this factor.

But now remember we are working with non-diagonally dominant system, will Gauss-Seidel method perform better than Jacobi method? Let us see; we will compute the iteration sequence of Gauss-Seidel method and see what is happening. Before that I will just show you the graph of the error that is the 2 norm of the error is shown in this graph. You can see that as you go on increasing the iteration the x -axis is the index of the iteration.

That is k and y -axis is the l_2 norm of the error. You can see that the error is gradually decreasing. You also have some oscillations here; this is quite common in the iterative methods. It means the iteration here was better than the iteration here. So, the iteration error just increased. Again, it fell, again it increased like that it goes. But as you go on the error is tending to 0.

So, this is the 0 line, so it is tending to 0, but it is quite slow. In the Jacobi case let us see what is happening to Gauss-Seidel case.

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Convergence of Jacobi and Gauss-Seidel Method

$\mathbf{x}^{(0)} = (1, 1, 1)^T$.

Jacobi iterations	Error
$\mathbf{x}^{(1)} = (-1.00000, 1.00000, 1.00000)$	$\ e^{(1)}\ _2 = 1.73205$
$\mathbf{x}^{(10)} = (0.77778, 0.38272, -0.19342)$	$\ e^{(10)}\ _2 = 0.88816$
$\mathbf{x}^{(50)} = (-0.07015, 0.00335, 0.05541)$	$\ e^{(50)}\ _2 = 0.08946$
$\mathbf{x}^{(100)} = (-0.00161, -0.00472, -0.00301)$	$\ e^{(100)}\ _2 = 0.00583$
$\mathbf{x}^{(200)} = (0.00000, 0.00002, 0.00001)$	$\ e^{(200)}\ _2 = 0.00002$

Gauss-Seidel iteration	Error
$\mathbf{x}^{(1)} = (-1.0, -1.0, -1.0)$	$\ e^{(1)}\ _2 = 1.7320508$
$\mathbf{x}^{(2)} = (1.0, 1.0, 1.0)$	$\ e^{(2)}\ _2 = 1.7320508$
$\mathbf{x}^{(3)} = (-1.0, -1.0, -1.0)$	$\ e^{(3)}\ _2 = 1.7320508$
$\mathbf{x}^{(100)} = (1.0, 1.0, 1.0)$	$\ e^{(100)}\ _2 = 1.7320508$
$\mathbf{x}^{(199)} = (-1.0, -1.0, -1.0)$	$\ e^{(199)}\ _2 = 1.7320508$

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At the first iteration we get $\mathbf{x}^{(1)} = -1, -1$ and -1 and the l_2 error is given like this. Remember, we have the initial guess as 1, 1 and 1. If you go back to the iterative matrix of the Gauss-Seidel method you can see that B_G is given like this. Therefore, when you multiply it with 1, 1, 1 clearly you will get $-1, -1$ and -1 as $\mathbf{x}^{(1)}$. So, that is why you see this vector as the first term of the iterative sequence in the Gauss-Seidel method.

Now you can also see what happens with the next iteration by just looking at the iterative matrix. It will be simply $\mathbf{x}^{(0)}$. Now once you have this again $\mathbf{x}^{(3)}$ will be $\mathbf{x}^{(1)}$ and so on. So, in this way we get an oscillating sequence in the Gauss-Seidel method and therefore the Gauss-Seidel method never converges. In fact, I have gone up to 200 iterations and you can see that the sequence simply oscillates and keeps the error at a constant level.

Therefore, in this example we can see that the Jacobi method is converging but rather very slowly whereas Gauss-Seidel method does not converge, it simply oscillates. So, what we understood in today's class is that when that is the coefficient matrix is diagonally dominant then Gauss-Seidel method will perform better than Jacobi method; will both of them or order wise they are linear order.

However, Gauss-Seidel accuracy will be little better than Jacobi's accuracy if the coefficient matrix is diagonally dominant. If the coefficient matrix is not diagonally dominant then we cannot say anything. Jacobi method may converge for which Gauss-Seidel may not convert. The same can happen other way also. Gauss-Seidel may converge but Jacobi may not converge

in some other examples. There are examples like that where Gauss-Seidel converges, but Jacobi never converges.

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Convergence of Jacobi and Gauss-Seidel Method

$\mathbf{x}^{(0)} = (1, 1, 1)^T$.

Jacobi iterations	Error
$\mathbf{x}^{(1)} = (-1.00000, 1.00000, 1.00000)$	$\ e^{(1)}\ _2 = 1.73205$
$\mathbf{x}^{(10)} = (0.77778, 0.38272, -0.19342)$	$\ e^{(10)}\ _2 = 0.88816$
$\mathbf{x}^{(50)} = (-0.07015, 0.00335, 0.05541)$	$\ e^{(50)}\ _2 = 0.08946$
$\mathbf{x}^{(100)} = (-0.00161, -0.00472, -0.00301)$	$\ e^{(100)}\ _2 = 0.00583$
$\mathbf{x}^{(200)} = (0.00000, 0.00002, 0.00001)$	$\ e^{(200)}\ _2 = 0.00002$

Gauss-Seidel iteration	Error
$\mathbf{x}^{(1)} = (-0.0000113, -0.0000113, -0.0000113)$	$\ e^{(1)}\ _2 = 0.0000196$
$\mathbf{x}^{(2)} = (0.0000113, 0.0000113, 0.0000113)$	$\ e^{(2)}\ _2 = 0.0000196$
$\mathbf{x}^{(3)} = (-0.0000113, -0.0000113, -0.0000113)$	$\ e^{(3)}\ _2 = 0.0000196$
$\mathbf{x}^{(100)} = (0.0000113, 0.0000113, 0.0000113)$	$\ e^{(100)}\ _2 = 0.0000196$
$\mathbf{x}^{(199)} = (-0.0000113, -0.0000113, -0.0000113)$	$\ e^{(199)}\ _2 = 0.0000196$

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Here we saw as instant where the sequence is not converging but it is just oscillating. The sequence can also diverge, it means it can just go to infinity also; anything can happen if the coefficient matrix is not a diagonally dominant matrix. Therefore, our understanding on these methods is still not complete. Something interesting is happening in these methods which is when the system is not diagonally dominant.

Therefore, there is still some scope for us to analyze and understand what is going on with these methods when the coefficient matrix is not diagonally dominant. This is what precisely we are going to do in the next class. Thank you for your attention.