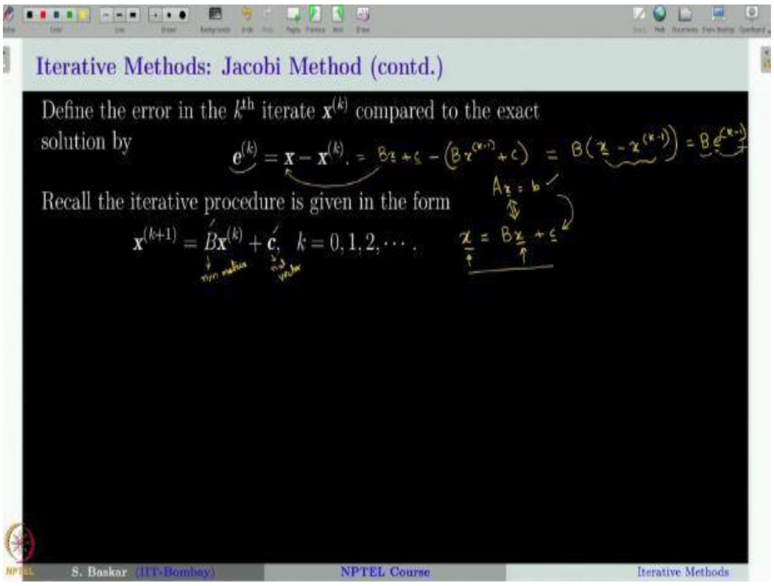
Numerical Analysis
Prof. S. Baskar
Department of Mathematics
Indian Institute of Technology-Bombay

Lecture-18
Linear System: Iterative Methods (Jacobi-Convergence)

Hi, we are discussing iterative methods for non-singular linear systems. In the last class we have discussed an iterative method called Jacobi method and we have also seen two examples. In the first example the Jacobi iterative sequence tend to converge to the exact solution and we have also given another example, where the Jacobi method tends to go away from the exact solution. That gives us an interesting question of when the Jacobi iterative sequence will converge. In this lecture we will try to answer this question.

(Refer Slide Time: 01:04)



We will start our discussion, with the definition of the error involved in the k^{th} iteration of anniterative method. It is not only for Jacobi method it can be any iterative method from where suppose you computed the term $x^{(k)}$ then the error involved in $x^{(k)}$ when compared to the exact solution is defined as $x - x^{(k)}$ and we use the notation $e^{(k)}$ to denote that quantity $x - x^{(k)}$.

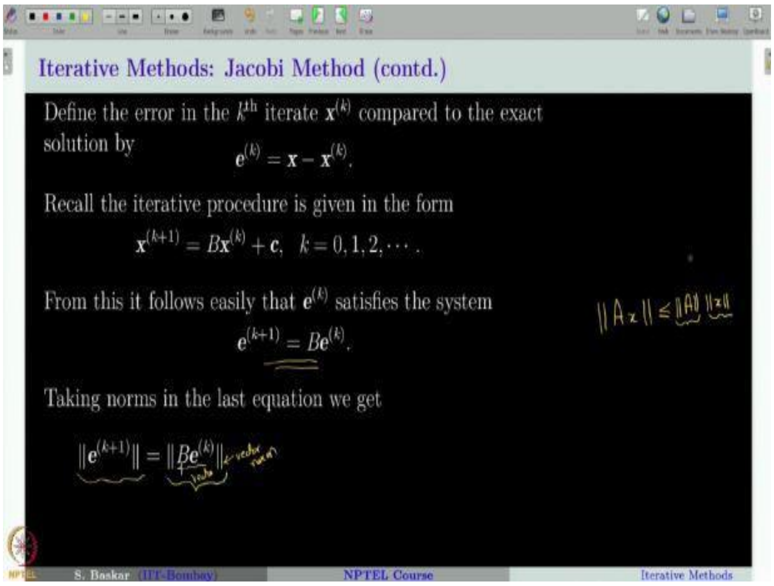
If you recall in the last class, we have derived the expression for the iterative sequence of the Jacobi method and it is given by $x^{(k+1)} = Bx^{(k)} + c$, where B is an $n \times n$ matrix and c is a n dimensional vector. So, we have derived the expression for this B and c in the last class. And if you recall the same equation will also be satisfied by the exact solution of the linear system

$Ax = b$ that is x also satisfies the equation $x = Bx + c$ why because, simply we have written $Ax = b$ in this form that is all.

We did not do anything, we just rewritten this system in this form and then we have plugged in a known quantity on the right-hand side and got the corresponding vector and called that as the terms of the iterative sequence**.** So, therefore we also have this equation satisfied by the exact solution. Therefore, the error involved in the k^{th} iteration can be written as $Bx + c$ which I am replacing with this x and $x^{(k)}$ is replaced by its formula, which is given by $Bx^{(k-1)} - c$**.**

Therefore, you can get this as B, which is a matrix therefore you can write it as $B\left(x - x^{(k-1)}\right)$**.** And this is again the definition of the error involved in the $(k-1)$ iteration**.** Therefore, you can write it as $Be^{(k-1)}$**.** So, therefore the error involved in kth iteration is nothing but B times error involved in the $(k-1)$ iteration.

(Refer Slide Time: 04:34)

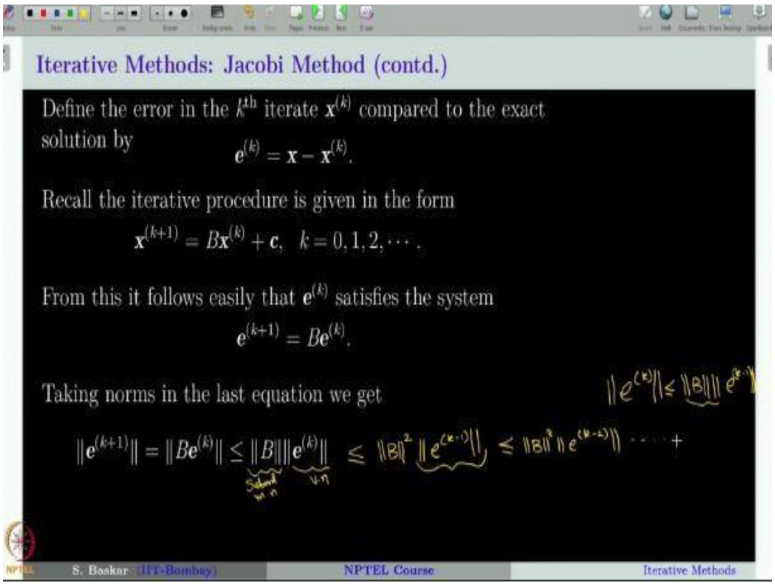


That is what I am writing $e^{(k+1)} = Be^{(k)}$. Now, take the vector norm on both sides of this equation to get $||e^{(k+1)}|| = ||Be^{(k)}||$. Remember $Be^{(k)}$ is a vector, therefore this is a vector norm**.** Similarly, this is also a vector norm that we have taken. Since, this is a general discussion we are not restricting ourselves to any particular vector norm like l_1 norm or l_2 norm or l_∞.

Now, this discussion will go on with any choice of the vector norm. Now, if you recall in one of the previous lectures, we have discussed subordinate matrix norm that is matrix norm subordinate to a vector norm**.** What it is? You give me a vector norm from there I will generate a matrix norm using that vector norm. Why we do that, because it satisfies three important properties that we have proved in one of the previous classes.

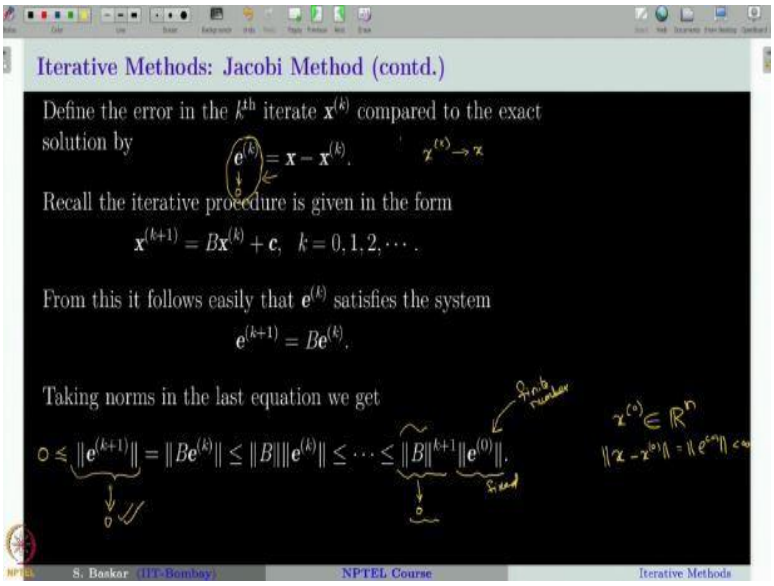
Whenever you see an expression like this Ax kind of expression you should immediately recall the property that this is less than or equal to $||A||$, this is the subordinate matrix norm into $||x||$, which is a vector norm**.** Here also you see a similar expression Be where B is a matrix e is a vector.

(Refer Slide Time: 06:32)



Therefore, this can be written as less than or equal to $||B||$, which is the subordinate matrix norm and this is the vector norm**.** And now you see this inequality holds for every k**.** In particular it also holds for $e^{(k)}$ that is you can write $e^{(k)}$ as $||B||\ ||e^{(k-1)}||$**.** Now, if I apply that here I will get this is less than or equal to already one norm B is there, now one more norm B will come from here and makes it $||B||^2||e^{(k-1)}||$. Again, you apply the same inequality for this term and that will give you $||B||^q||e^{(k-2)}||$**.** So, like that you can keep on going up to $e^{(0)}$**.**

(Refer Slide Time: 07:48)

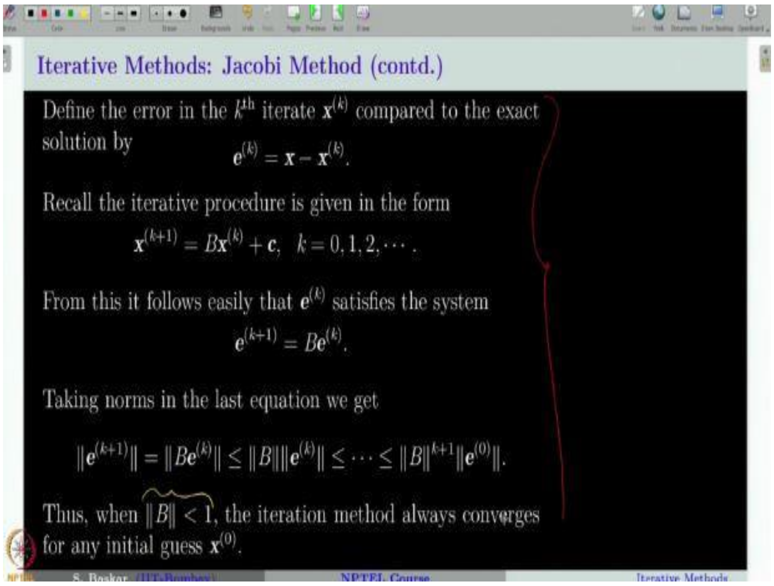


When you reach $e^{(0)}$ here correspondingly we will get $||B||^{k+1}$**.** Now, you see this term is some finite number why because, we have chosen $x^{(0)}$ arbitrarily**.** We will choose it in the space $\mathbb{R}^n$. Therefore, $||x - x^{(0)}||$ which is precisely your $e^{(0)}$ is going to be surely some finite number**.** Now, what is our interest? If you recall our interest in this discussion is to see when our Jacobi method that is the iterative sequence generated by the Jacobi method should converge**.**

When will it converges the question. Equivalently, the question is to see when this term goes to 0 in some sense**.** If this goes to 0 then $x^{(k)}$ goes to x**.** Therefore, our interest is to see when this term goes to 0 in some sense; for that we have taken a norm it means we are trying to see this condition in the sense of a given norm. That is, we want this term to go to 0**.** From this inequality we can see that this happens when this term goes to 0, because this is already a fixed number.

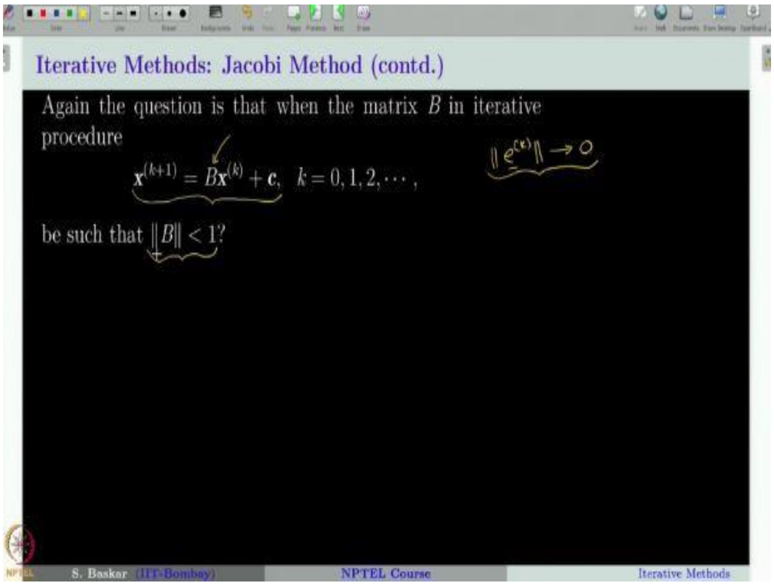
Therefore, this goes to 0, if this goes to 0. Why it is so we know that the norm is always greater than or equal to 0. Therefore, if this term goes to 0, then you can apply the Sandwich theorem for sequence to see that this is already 0 and if this goes to 0 then this will also go to zero. That is the idea. Now, the question is when this term goes to 0?

(Refer Slide Time: 10:20)



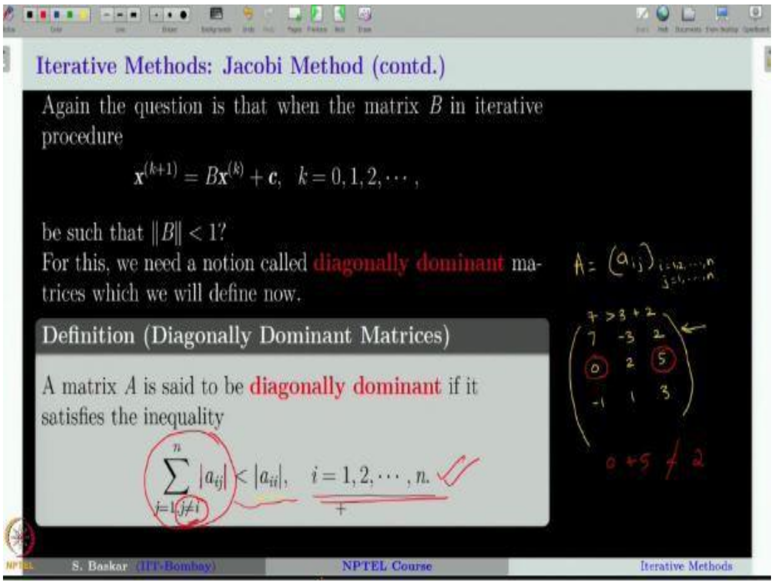
We can clearly see that this term goes to 0 as k tends to infinity, if $||B||$ is less than 1.

(Refer Slide Time: 10:29)



So, to summarize we want this sequence to converge**.** In order that this sequence should converge we need $e^{(k)}$ with respect to some given norm should converge to 0. We have just now seen that if $||B||$ is less than 1, then the error will converge to 0 with respect to the vector norm. Remember, you choose a vector norm and you want $e^{(k)}$ to converge to 0 with respect to that vector norm. For that this Jacobi iterations matrix should have it is corresponding subordinate matrix norm to be less than 1**.**

(Refer Slide Time: 11:20)

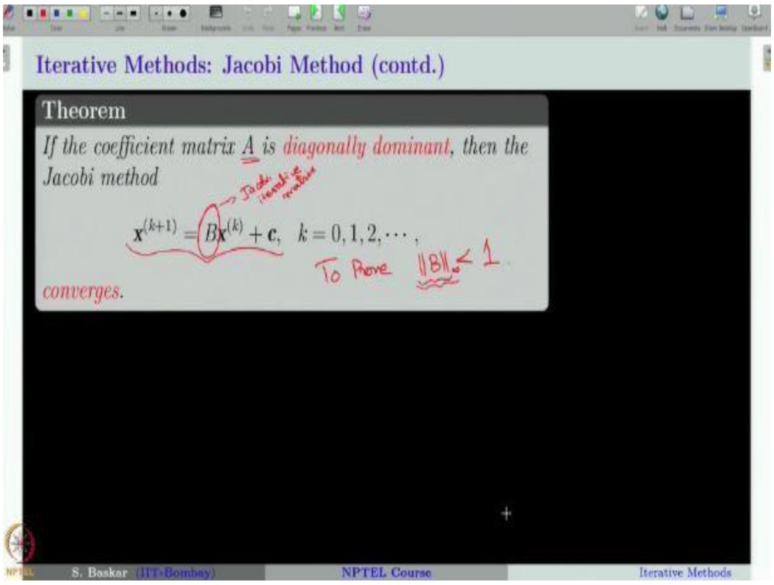


When can we say that a matrix is diagonally dominant? Suppose you have a matrix A, which is given by (a_{ij}), where i varies from 1 to i and j varies from 1 to n. Now, say for instance you take a matrix something like 7, -3 and 2, 0, 2 and 5 and -1, 1 and 3**.** So, to check whether this matrix is a diagonally dominant matrix or not what you have to do is you take all the terms other than the diagonal term that is what we meant by saying j not equal to i.

Remember you fix a_i for instance, you fix say first row and take all the non-diagonal elements take their absolute value and then sum them up. That is what the left-hand side says. In this example it is nothing but 3 + 2 and that should be less than the absolute value of the diagonal term. That is this should be less than the diagonal term that is 7 in the first row. Similarly, you take the second row you see that the non-diagonal elements are 0 and 5**.**

Therefore, 0 + 5 should be less than the diagonal element, which is 2. So, this is not happening therefore this matrix is not a diagonally dominant matrix, because this should happen for all the rows**.** So, you remove the diagonal element and sum all the non-diagonal elements of a row with their absolute value and then check whether this inequality is satisfied. If this happens for all the rows then we call the matrix A as diagonally dominant.

(Refer Slide Time: 13:40)

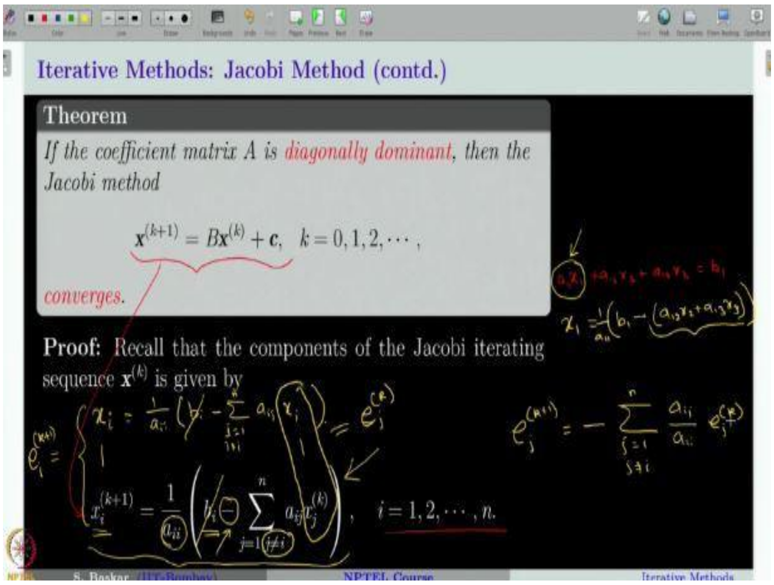


Now, our convergence theorem says that if the coefficient matrix A is diagonally dominant then the Jacobi method converges. It means the iterative sequence given by this expression converges. Now, the idea is to use the condition that the coefficient matrix is diagonally dominant and prove that $||B||$ is less than 1. That is the idea of the proof of this theorem. We will present this theorem by choosing some particular norm, which is the l_∞ norm, because in that way the proof goes more nicely and it is easy to understand.

Therefore in the proof of this theorem we will consider only l_∞ norm to compute the quantity $||B||$ and see how it goes. The theorem says that if A is diagonally dominant, then the Jacobi iterative matrix this is called the Jacobi iterative matrix will have it is subordinate matrix known to be less than 1. The proof of this theorem is more or less similar to what we have shown here, only thing is the same derivation we will do coordinate wise.

In order to see how the diagonally dominance of the matrix A is going to make this quantity $||B||$ to be less than 1, to very clearly see this we will do the same computation coordinate wise.

(Refer Slide Time: 15:43)



Remember this equation is given like this for each coordinate. How it is so? What we are doing is suppose you have $a_{11}x_1 + a_{12}x_2 + a_{13}x_3 = b_1$. If you take then what you are doing in the Jacobi method you are keeping the diagonal element since this is the first equation of your system, the diagonal element is the first term. You are keeping that on the left-hand side and then taking all the other elements that is $a_{12}x_2 + a_{13}x_3$ on the left hand side and finally dividing both sides by $\frac{1}{a_{11}}$.

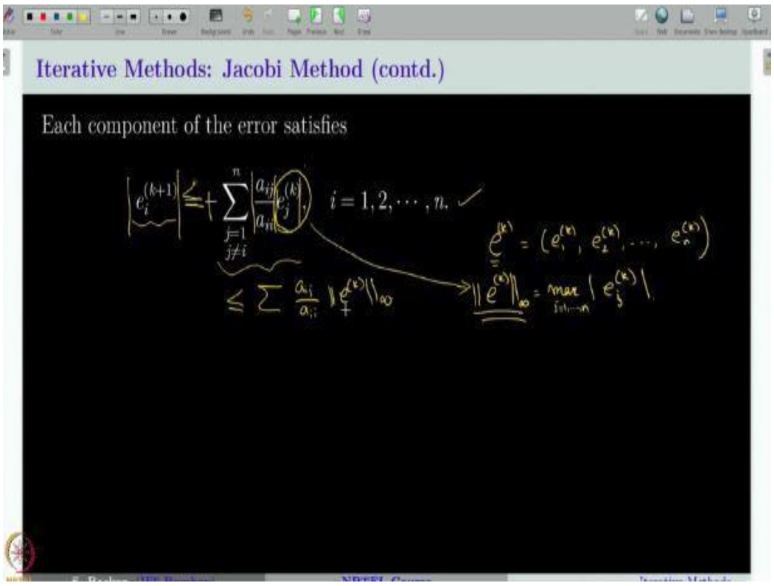
That is what I am writing. Here the *i*th equation can be written as x_i is equal to I am dividing by the diagonal element a_{ii} then keeping b_i as it is all the non-diagonal elements are taken on the right hand side. Therefore, you have a minus here and then the summation. This summation, remember, it will exclude the diagonal element because it was not taken to the right-hand side.

That is why we have written $j \neq i$ and this is for one typical equation that is the *i*th equation and that *i* will run from 1 to *n*. This is precisely what the Jacobi method is. Remember the same equation also holds for the exact solution. That is what we remarked that is x_i can be written as $\frac{1}{a_{ii}}\left(b_i - \sum_{j=1, j\neq i}^{n} a_{ij}\, x_j^{(k)}\right)$. Now, if you subtract these two $x_i - x_i^{(k+1)}$ that will give you the *i* th component of the vector error vector $e_i^{(k+1)}$.

When you subtract these two this gets canceled and similarly you will have here the coefficient a_{ij} will remain and then you will have $x^{(k)}$, there is no *k* here this is exact solution. Therefore,

$x_j - x_j^{(k)}$ will come and that will give us $e_j^{(k)}$**.** Therefore, you will have $e_j^{(k+1)}$ is equal to minus because this minus is here $\sum_{j=1, j\neq i}^{n} \frac{a_{ij}}{a_{ii}} e_j^{(k)}$**.** So, that is what we are having here.

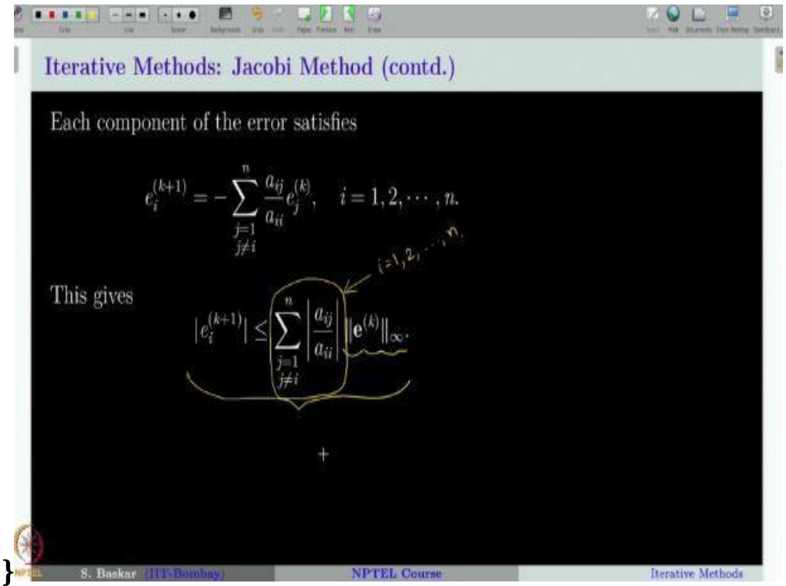
(Refer Slide Time: 19:14)



And this should hold for each equation that is for i = 1, 2 up to n. So, the corresponding error involved in the Jacobi iteration at the (k + 1)th iteration is given by this expression**.** Now, take the modulus on both sides and then you take the modulus inside the summation. That will give you this to be less than or equal to norm of this into norm of this and there is a plus sign here. Now, what happens is remember your $e^{(k)}$ vector is nothing but $e_1^{(k)}$, $e_2^{(k)}$ and so on up to $e_n^{(k)}$**.**

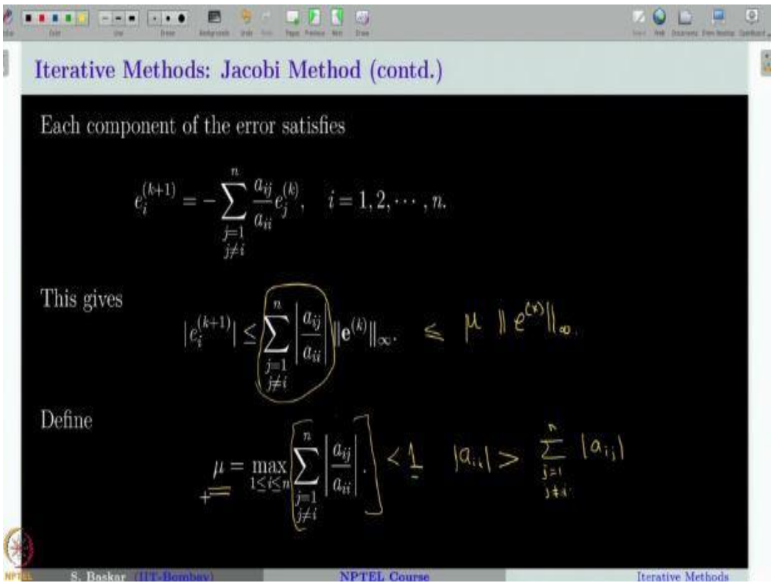
Now, what is the infinite norm of $e^{(k)}$ it is nothing but maximum over all $||e_i^{(k)}||$ or maybe e_j I will put here j varies from 1 to n**.** Now, you see you have e_j in each of these terms; what if I replace each of this term by it is infinite norm. Then I can say that this is further less than or equal to $\sum \frac{a_{ij}}{a_{ii}} ||e^{(k)}||_\infty$. Because I am replacing each term in the sum by it is maximum value therefore that will give me something value bigger than this**.**

(Refer Slide Time: 21:08)



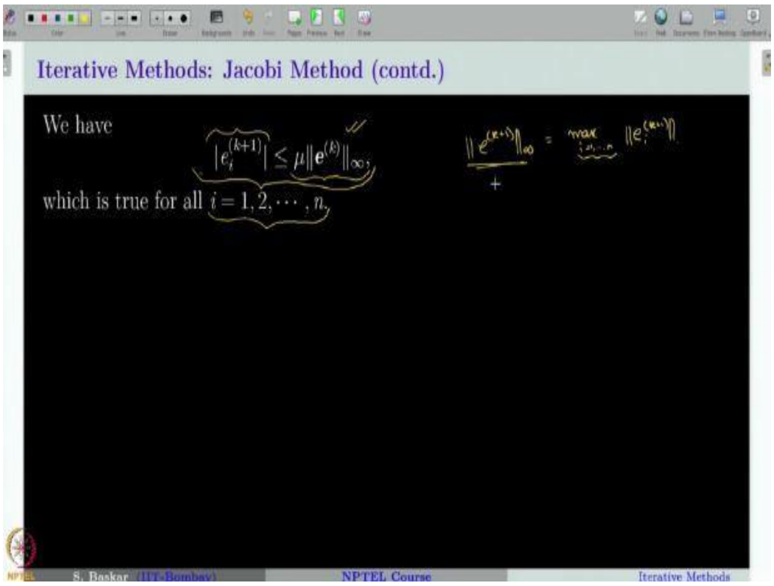
Therefore, I will have this inequality. Now, you see this inequality still involves this term. Let us see how to dominate this term. You see for each i you have this number i = 1 you have a number like this 2 and so on up to n and this is common for all these i's because it does not depend on i as well as it does not depend on j. Therefore, it will also come out of this summation and you have this.

(Refer Slide Time: 21:47)



Now, take the maximum over all these numbers. Take maximum over all these numbers and call this as μ. Then you can say that this is further less than or equal to $\mu||e_{(k)}||_\infty$.

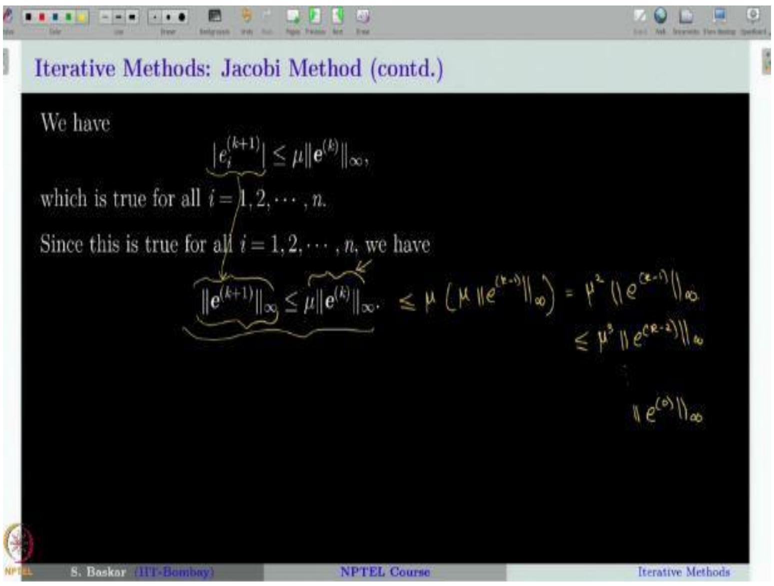
(Refer Slide Time: 22:10)



So, that is what we are getting here. $||e_i^{(k+1)} \leq \mu||e_k||_\infty$ and this happens for all i = 1, 2 up to n**.** Remember the right-hand side is independent of i, whereas the left hand side depends on i and this inequality holds for all i. Therefore, suppose you take this quantity, which is nothing but maximum over all i = 1 to n such that $||e_i^{(k+1)}$.

So, this maximum will achieve at some index between 1 and n**.** So, therefore this inequality will hold for that index also, because this holds for all i therefore it will also hold for that ind*e*x at which the maximum norm is achieved.

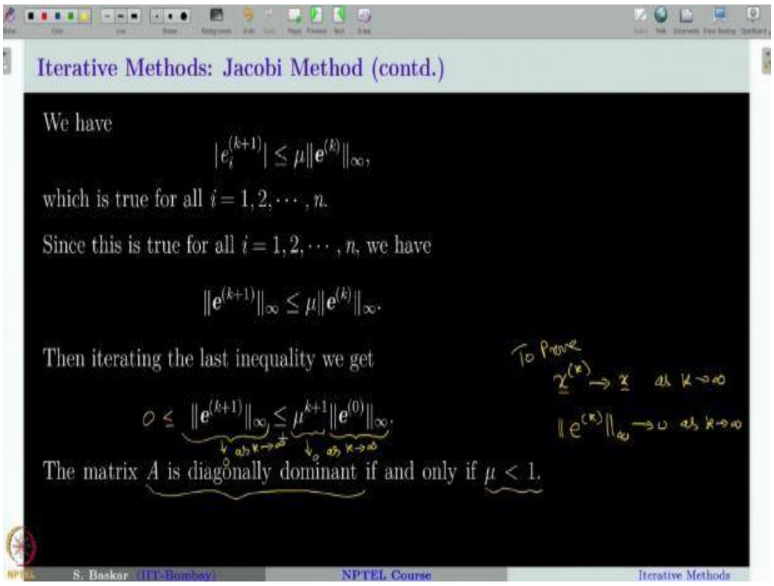
(Refer Slide Time: 23:16)



Therefore, you can write with respect to that maximum norm. So, that is you can replace this by the maximum norm, because it holds for all i therefore it will hold for that coordinate at which the maximum norm is achieved and that is less than or equal to $\mu||e^{(k)}||_\infty$. Now, you see finally we got this inequality and this is again a recursive inequality. In the sense, that you can apply this inequality to this term, just like what we did in the previous slide.

You can apply this inequality to this and that will give you μ into μ times $||e^{(k-1)}||$ and that will actually give us $\mu^2||e^{(k-1)}||_\infty$. Again, you do the same idea you can say that this is less than or equal to $\mu^3||e^{(k-2)}||_\infty$ and so on. You can go until where you can go up to $e^{(0)}$ here and correspondingly that will give us norm $||e^{(k+1)}||_\infty \leq \mu^{k+1}||e^{(0)}||_\infty$.

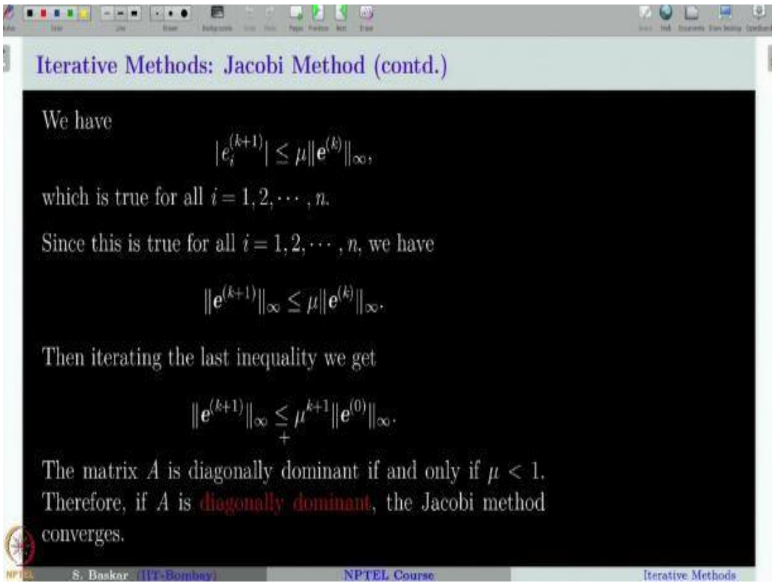
(Refer Slide Time: 24:48)



Now, if you recall in tutorial 1, we have solved the problem where we have seen that if your sequence $x_n$is such that $|x_n - L| \leq \mu^k|x_0 - L|$. If that is so then you can say that this sequence x_n converges to L as n tends to infinity if μ is less than 1 this is i here. Now, we have the same kind of inequality here and therefore you can see that if μ is less than 1 then of course this goes to 0 as k tends to infinity.

You can also see this result using the Sandwich theorem, because this is always greater than equal to 0 and so if this goes to 0 this is a fixed number therefore this will also goes to 0 as k tends to Infinity. So, that is what we get. Therefore, all we need is to see whether this μ is less than 1 or not.

(Refer Slide Time: 26:25)

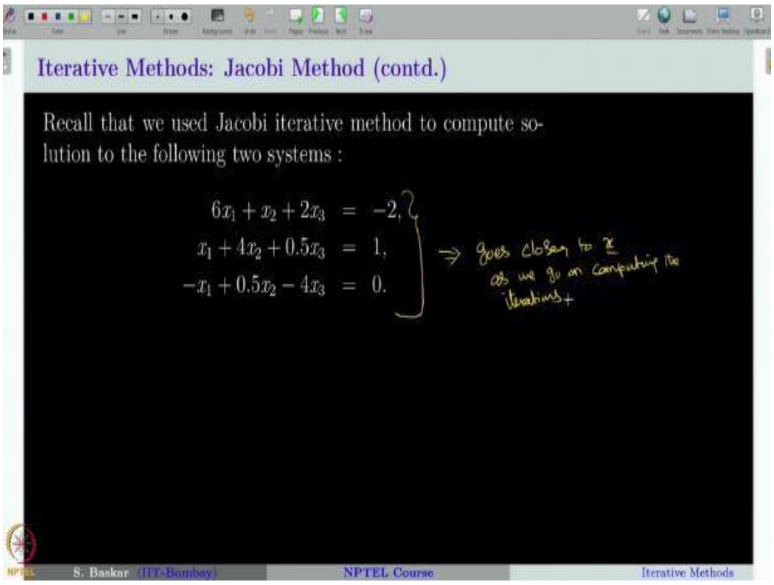


In fact, from the way μ is defined you can see that this is true if A is a diagonally dominant matrix. Remember if A is diagonally dominant then a_{ii} will always be greater than $\sum_{j=1, j\neq i}^{n} |a_{ij}|$. Therefore, all these terms are less than 1. And now you are taking maximum of all numbers which are less than one, therefore μ will also be less than 1.

So, this is what precisely we wanted to show that if A is diagonally dominant then the Jacobi sequence converges that is $x^{(k)}$ converges to x as k tends to infinity is what to prove. What we did is, we just took the infinite norm and tried to prove this. This is equivalent to saying that $e^{(k)}$ with respect to infinite norm tends to 0 as k tends to infinity. By going into coordinate wise we have seen that the required term that is infinite norm of $e^{(k)}$. Well here, we did with one index more $e^{(k+1)}$ is less than or equal to this quantity times the infinite norm of $e^{(0)}$.

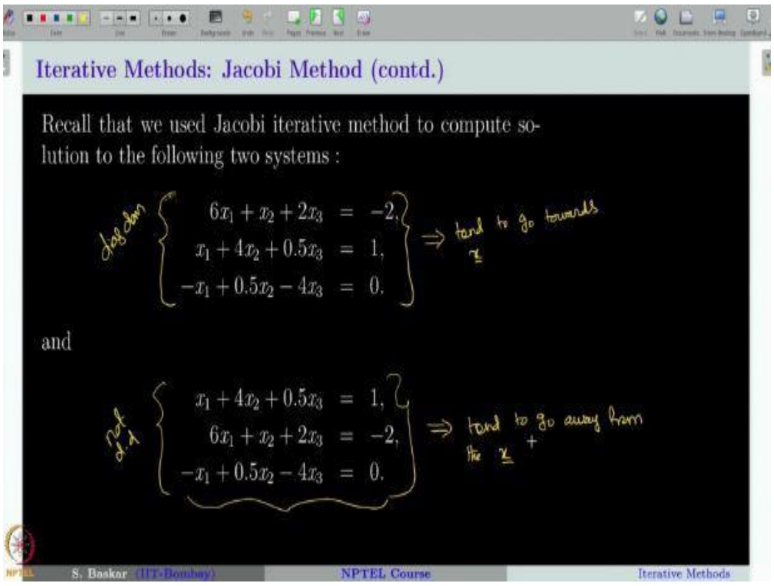
And by imposing the condition that A is diagonally dominant we saw that μ is less than one. Therefore, this goes to 0 as k tends to infinity and therefore by Sandwich theorem, we can see that this goes to 0 as k tends to infinity. So, that completes the convergence proof for Jacobi method. Remember that this theorem gives only the sufficient condition for the convergence of the Jacobi method. With this let us go to the examples that we have discussed in the last class.

(Refer Slide Time: 28:36)



If you remember we considered two examples. In the first example, we took this system and we have seen that the Jacobi iteration tends to go closer and closer to the exact solution as we go on computing the iterations.

(Refer Slide Time: 28:59)



On the other hand, the second example that we took where we took this system this tend to go away from the exact solution x that is what we have seen in the last class. Now, it is more clear for us, why this system was behaving well in terms of the Jacobi iteration, because this system is diagonally dominant**.** Whereas, this system is not diagonally dominant, but that does not mean that the Jacobi iteration should diverge**,** because our theorem says only that the diagonal dominance of A implies convergence.

If the matrix is not diagonally dominant our theorem simply does not say anything. In other words what we proved is only the sufficient condition for the convergence of the Jacobi method. Therefore our theorem is simply silent for this system, because this system is not diagonally dominant. Numerically we have seen at least to the terms that we have computed. Those terms of the sequence were going away from the exact solution**.**

In fact we can also get a necessary and sufficient condition for the convergence of the Jacobi iteration sequence. We will see all this in the coming lectures. Thank you for your attention.