Numerical Analysis
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Lecture-10
Linear System: LU Factorization (Doolittle and Crout)

Well, in this lecture we will study some LU factorization methods. If you recall in the last lecture we have seen that the Gaussian elimination process itself can lead to a LU factorization for a given coefficient matrix. In this lecture we will further see 3 methods to do the LU factorization; we will also see how to solve a linear system once the LU factorization is done.

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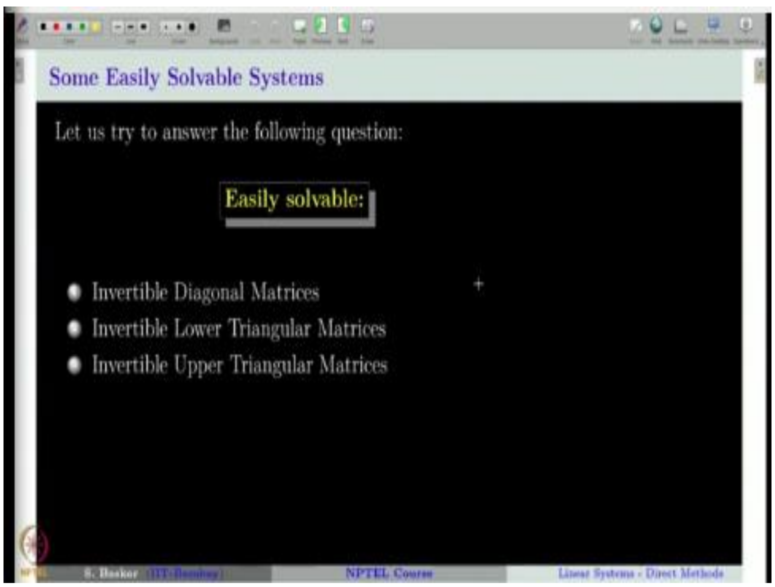

Let us start our class with a very simple question, what are all the easily solvable systems? We can at least think of 3 types of systems that can be readily solvable; one is the invertible diagonal system and next is the invertible lower triangular system and similarly invertible upper triangular systems.

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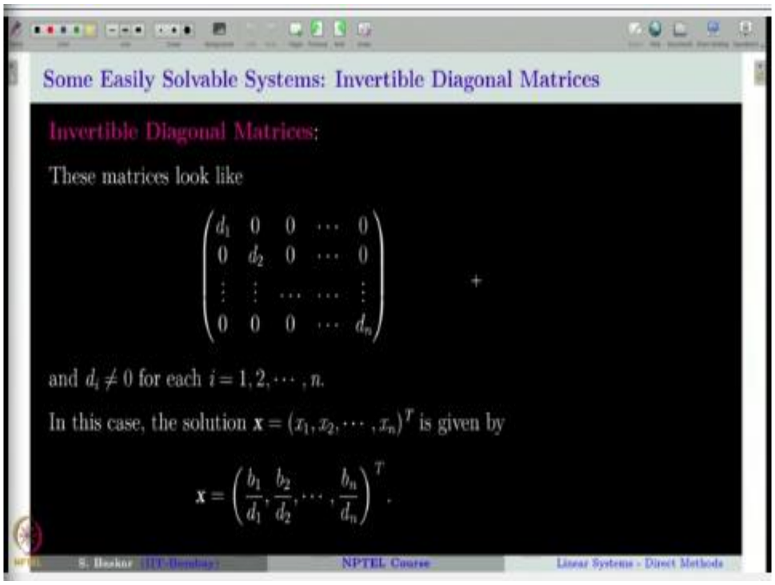

Let us take up each of this and see how to solve a system when the coefficient matrix is one of these 3. Let us first take invertible diagonal matrix, so these are the matrices for which the diagonal elements are non zero**.** And one can easily see that the system $Ax = b$ can be solved immediately to get the solution x.

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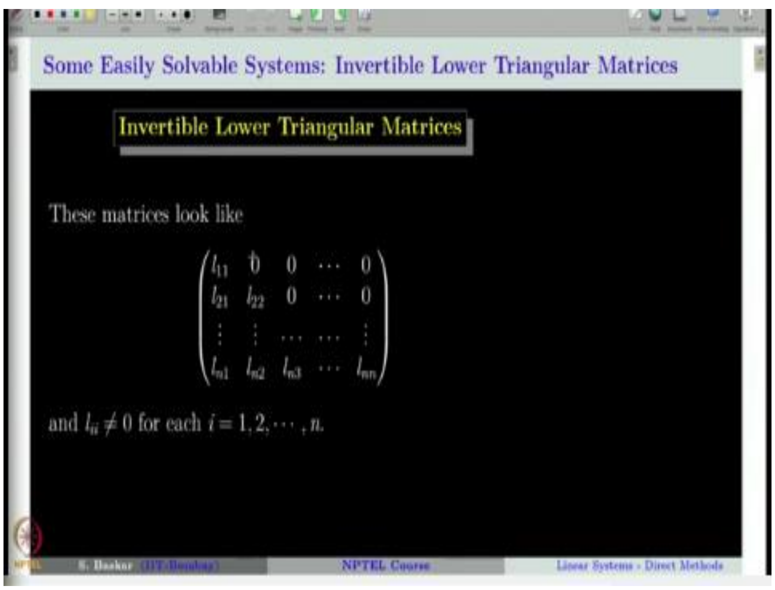

Now if our coefficient matrix is an invertible lower triangular matrix, that is if the coefficient matrix looks like this where all the entries above the diagonal elements are zeros and all the diagonal elements are non zero**.**

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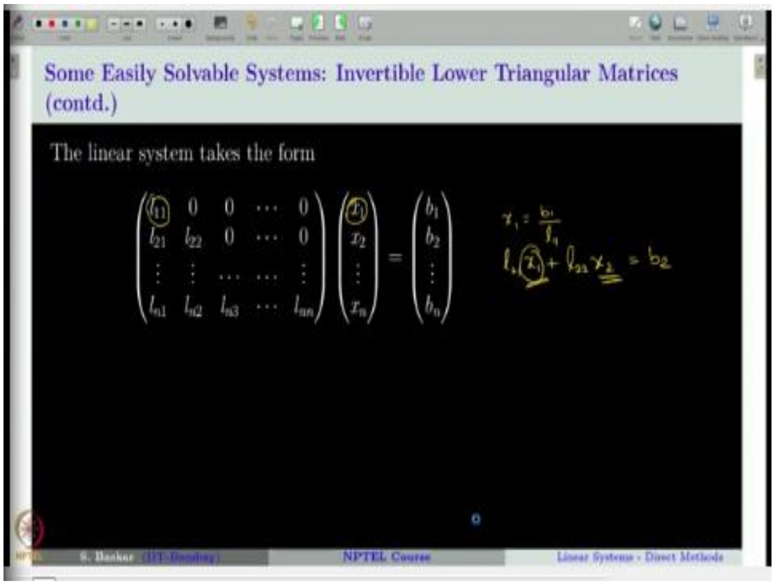

The corresponding linear system will look like this. Here you can see that x_1 can be obtained directly by taking $x_1 = \frac{b_1}{l_{11}}$. Now once you have x_1 you can put it in the second equation which will look like $l_{21}x_1 + l_{22}x_2 = b_2$. We have already obtained x_1 therefore you can obtain x_2 from this equation similarly you can go ahead and find other components of the unknowns in a forward substitution process.

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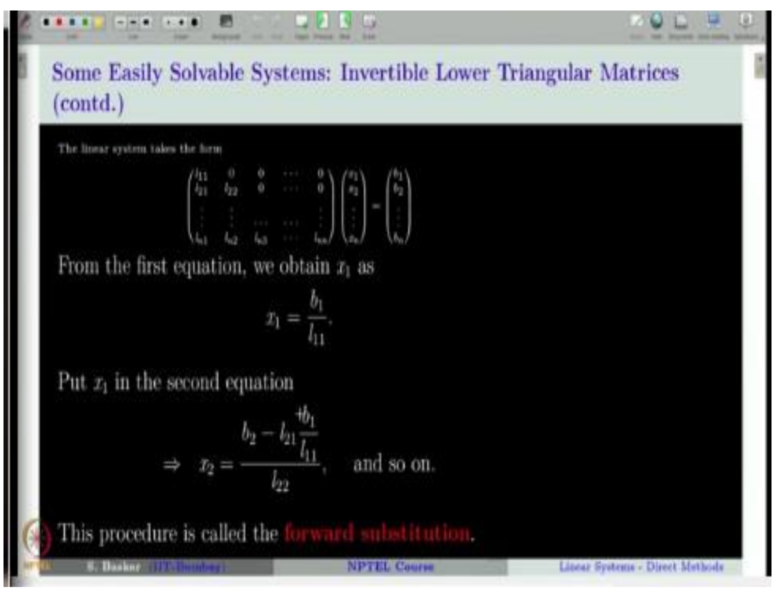

So, this is what I have told, x_1 is obtained as $\frac{b_1}{l_{11}}$. And once you have x_1 you can substitute that in the second equation to get x_2 as this expression and so on. So, this is called the forward substitution process. So, if you are given coefficient matrix is an invertible lower triangular matrix then you can do a forward substitution and get the solution of the linear system, you need not go for the elimination process as we have done in the Gaussian elimination method.

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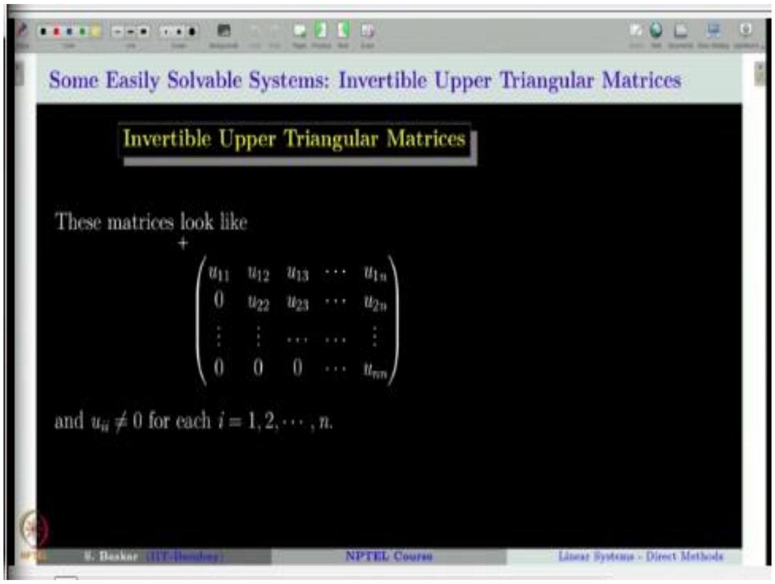

Now let us take the invertible upper triangular matrices. These are the matrices which will look like this, here all the elements under the diagonal elements are zeros and all the diagonal elements are non zeros.

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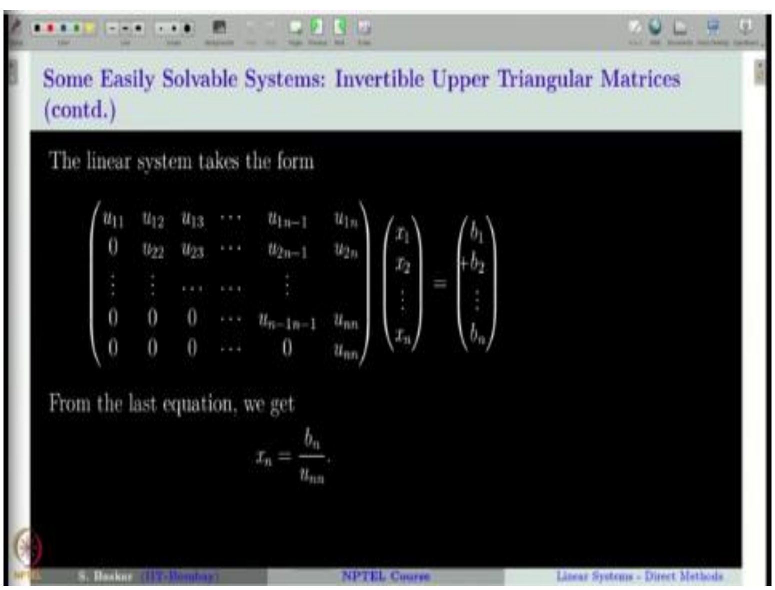

The corresponding linear system will look like this, where this is the matrix *A* and this is the vector *x* equal to *b* vector. This is very familiar to us because in the Gaussian elimination method finally we get an upper triangular matrix, so we know how to solve this system to get the unknown vector

x, we just have to do the backward substitution process where x_n is now obtained as $\frac{b_n}{u_{nn}}$. And once you have x_n you can substitute that in the previous equation that is the n minus first equation.

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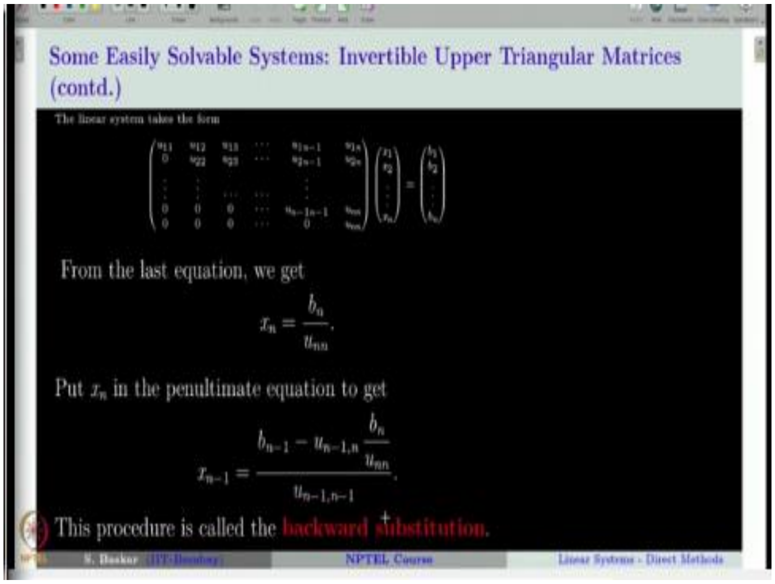

And get x_{n-1} as this expression and so on and that is the backward substitution process.

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Now our given matrix A may not be one of these 3 matrices in which case it is not possible for us to do just a substitution process rather one may have to go for the elimination process as we discussed in the Gaussian elimination method. At the end of this section I will show you that the

Gaussian elimination process is very costly from the computational point of view. That is why often Gaussian elimination method is not preferred; one needs to go for iterative methods.

However if we are very particular that we have to have the direct methods, then we have no choice other than going for Gaussian elimination method. In certain situations, for instance in the residual character method which we will see later, we come across a situation where your coefficient matrix is fixed and we have to solve the system $Ax = b$ with various right hand side vectors.

In that case what one generally does is, they factorize the matrix A as L and U and keep it aside and for every b now they just have to do a forward substitution and a backward substitution to get the answer. In such situations LU factorization is preferred. Now we will see how a general linear system can be solved by doing a LU factorization. Remember a LU factorization means you should first write your matrix A in the form of the product of a lower triangular matrix and an upper triangular matrix.

Generally, we denote the lower triangular matrix by L and the upper triangular matrix by U. Therefore we should find L and U such that $A = LU$. Now once you have such a factorization what is the advantage? Well, you can write now $Ax = b$ as $LUx = b$. Now take this Ux alone and name it as a vector z. Obviously z is unknown because x is not known to us therefore as a first step what you do is consider the system $Lz = b$.

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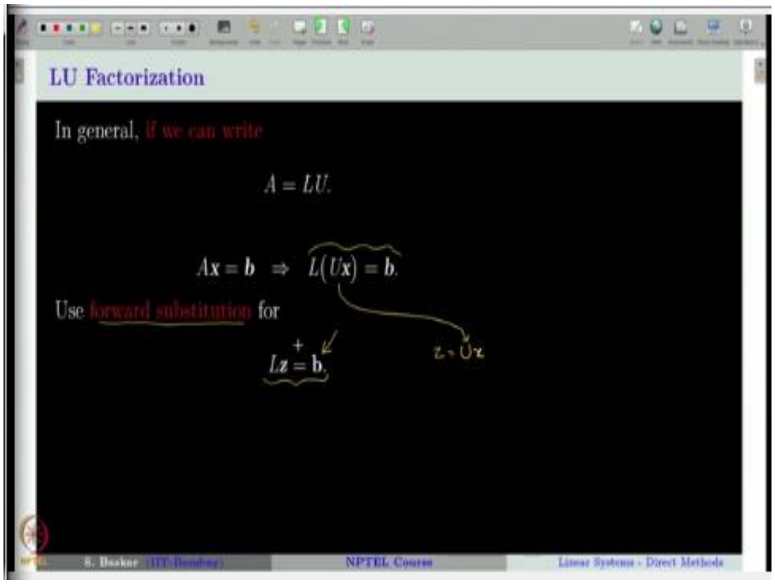

Remember z is nothing but Ux that is this vector is what we denote by z. Therefore this is now equivalent to $Lz = b$, b is the right hand side vector that is given to us therefore a forward substitution will give you this solution z for this system because it is a lower triangular system.

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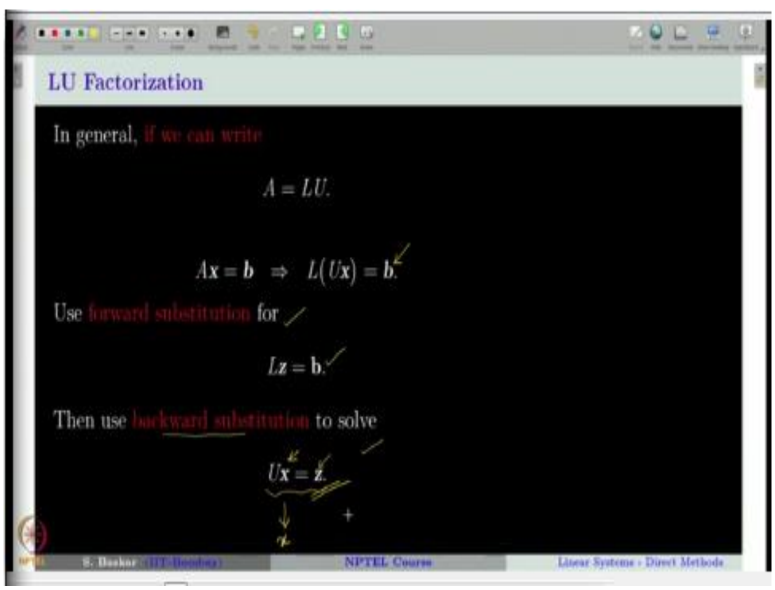

Once you get z you plug in that into $Ux = z$, this is what we have taken. Now this is again a linear system because z is already obtained here and therefore the right hand side vector for this system is known to as. So, you get a upper triangular system $Ux = z$ therefore you can do a back substitution to get the required solution x, so that is the idea. So, once you factorize your matrix A as the product of the lower triangular and the upper triangular matrix.

Then you give me any number of right hand sides I can just simply do 1 forward substitution and 1 backward substitution and give you *x*. The costliest step of the Gaussian elimination method is the elimination process and that elimination process is only done once. And with that one process you can now solve many systems where *A* is fixed but *b* varies. Such situation as I told will occur in certain methods like residual character methods. Now the question is how to factorize a given matrix *A* into lower and upper triangular matrices?

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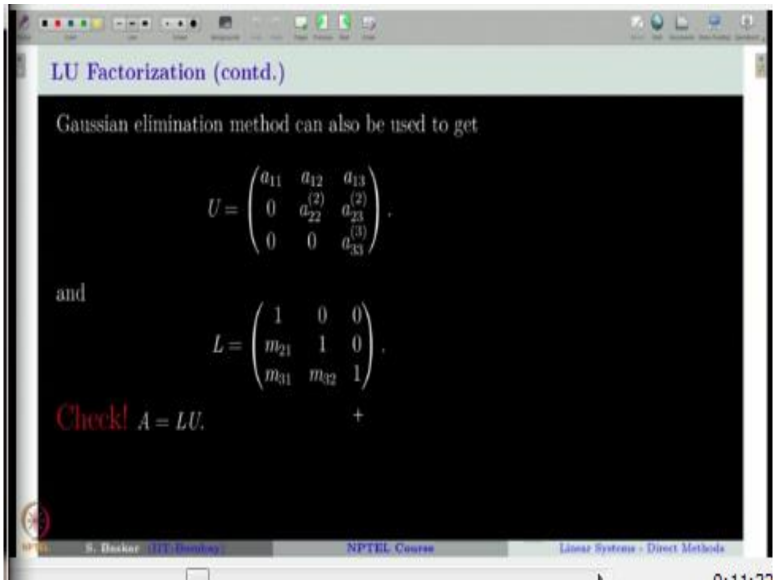

As we saw Gaussian elimination method itself gives us this upper triangular matrix which is the final matrix that you get out of the elimination process in the Gaussian elimination method and when you collect all the multiplications m_{ij}'s in each step in this form. Then we have seen in the last class that we get $A = LU$, I asked you to check this, I hope you would have checked and seen that this is true. Now the question is, is this LU factorization unique?

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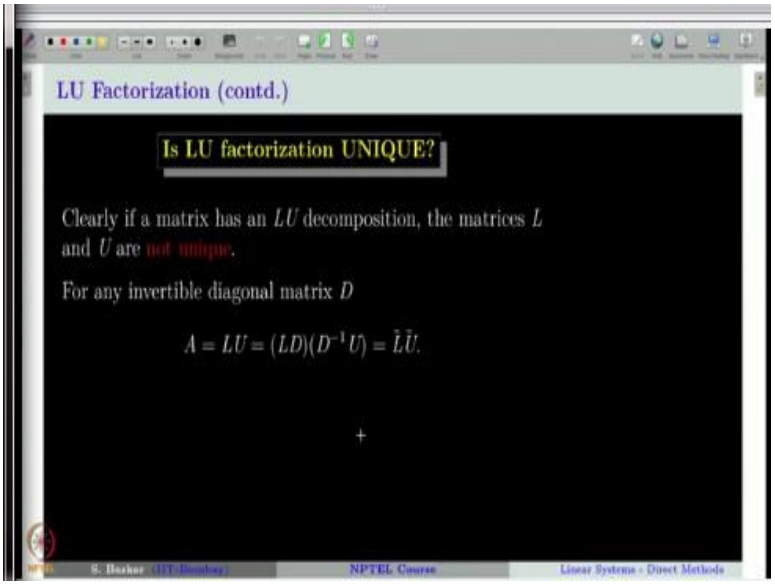

Well, the answer for this question is that the LU factorization is not unique. Why? It is very simple to see, once you have one factorization L into U then you take any invertible diagonal matrix D and just you can write L into U as $(LD)(D^{-1}U)$. I am just multiplying D and D^{-1} that will not change the system in any way. And now just (LD) that will again be a lower triangular matrix and then $D^{-1}U$ that will again be a upper triangular matrix.

And these lower triangular and upper triangular matrices will be surely different from the one which you have obtained if you choose this invertible diagonal matrix suitably. So, therefore once you have one factorization you can generate infinitely many LU factorizations.

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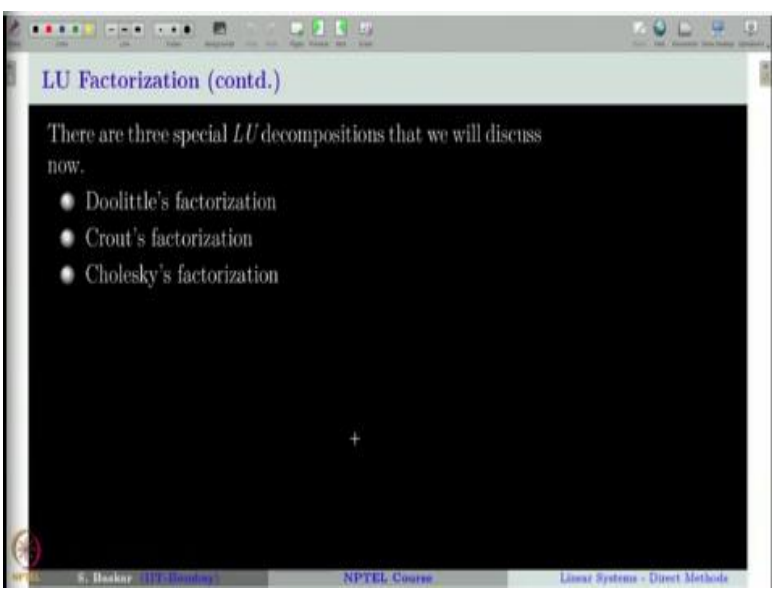

Now the next question is, Is Gaussian elimination method is the only way to obtain *LU* factorization? The answer is no, there are at least 3 methods that we will learn in this course. One is the Doolittle's factorization, another one is Crout's factorization and finally we will also learn Cholesky's factorization. Out of these 3 Doolittle and Crout factorizations are computationally as costly as Gaussian elimination method.

However Cholesky's factorization is little efficient than all these methods but it works only for symmetric and positive definite matrices. We will learn to construct *LU* factorization for a given invertible matrix using each of these 3 methods. Let us first consider the Doolittle factorization.

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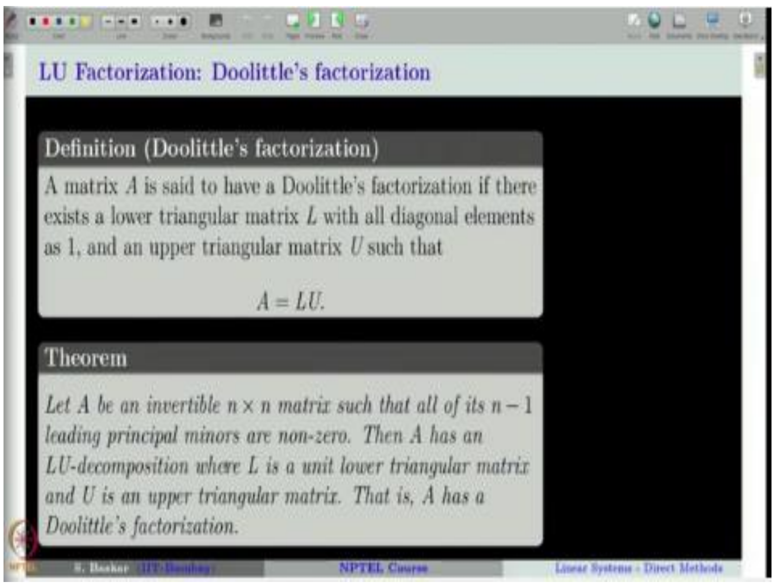

What is mean by Doolittle factorization? Well, it is a *LU* factorization of a given matrix where the lower triangular matrix has all it is diagonal elements as 1. If you recall in the Gaussian elimination method the *LU* factorization that we got is precisely the Doolittle factorization. You can see that all the diagonal elements of the lower triangular matrix have value 1. Therefore the Gaussian elimination method in fact gives us a Doolittle factorization.

Now the next question is, can we always get a Doolittle factorization for a given invertible matrix? Well, we have a condition under which Doolittle factorization surely exist, the condition is that all the n - 1 leading principal minors are non-zero, then the matrix A will surely have the Doolittle factorization. Well, I hope you know what is mean by principal minors and leading principal minors, you would have studied this in your linear algebra course.

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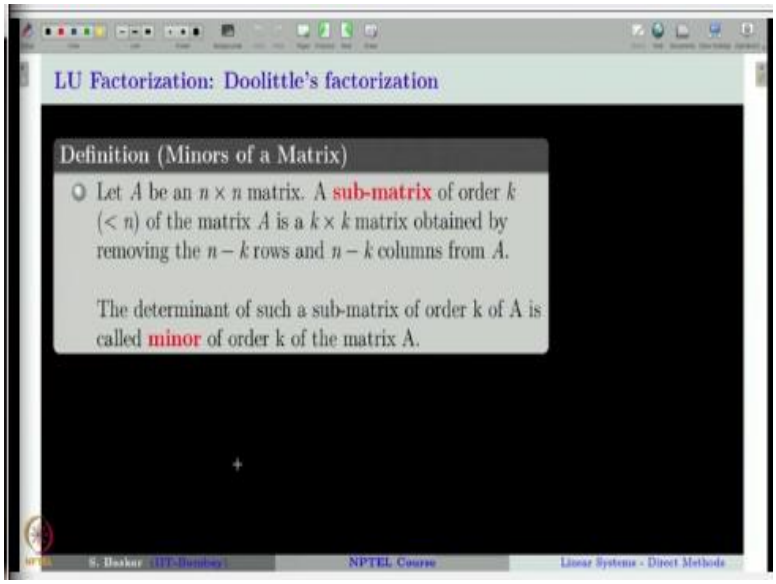

However we will quickly recall here let A be an $n \times n$ matrix. A sub matrix of order k, where $k < n$ of the matrix A is a $k \times k$ matrix obtained by removing $n - k$ rows and $n - k$ columns of A. The determinant of such a sub matrix of order k of A is called minor of order k of that matrix. Remember, just for the minor you can remove any set of rows and the same number of columns you can remove and get the sub matrix.

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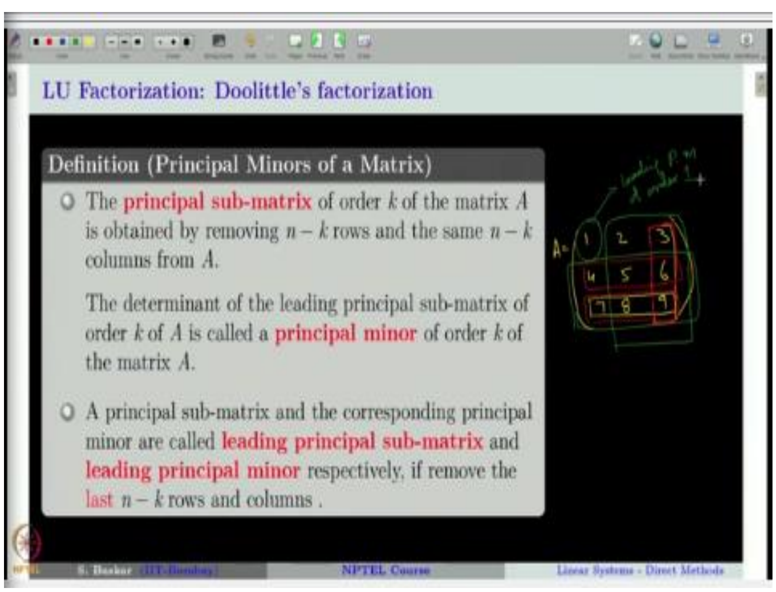

Whereas principal sub matrix means, well, you have to remove $n-k$ rows and whatever index of the rows are removed the same index of the columns has also be removed. Say for instance, you have 1, 2, 3, 4, 5, 6, 7, 8, 9 be the matrix A. Then if I remove say for instance third row and similarly the third column is removed, the remaining one that is 1, 2, 4, 5 is a principal sub matrix and it is determinant is the principal minor.

Whereas if I remove the second row and say third column then it is just a sub matrix and its determinant is just a minor because you are removing this second row but you are not removing the second column. If you are removing the same index then it is principal minor otherwise it just minor. Now what is mean by leading principal minor? Well, the leading principal minor or the principal minors where the sub matrix is obtained by removing the last $n-k$ rows and columns.

Therefore if I remove third row and third column, it is not only a principal minor, it is also a leading principal minor of order 2. Similarly if I remove second row and third row and similarly if I remove second column and third column then I am removing the last 2 rows and the same index last 2 columns also therefore the remaining one is the leading principal minor of order 1, I hope you understood the definition.

Now if you go back what the theorem says if all the $n-1$, that is the leading principal minor of order 1, 2, 3, up to $n-1$, if all of them are non-zero then we will surely have Doolittle factorization, that is what the theorem is. Well, we will not go to prove this theorem because the proof of the theorem is more or less similar to the existence proof of the Cholesky's factorization, we will spend time on understanding the existence theorem for Cholesky's factorization.

Interested students can first understand the Cholesky's factorization existence theorem and then come back and read the existence theorem of Doolittle factorization, you will surely understand it because the proof is almost similar.

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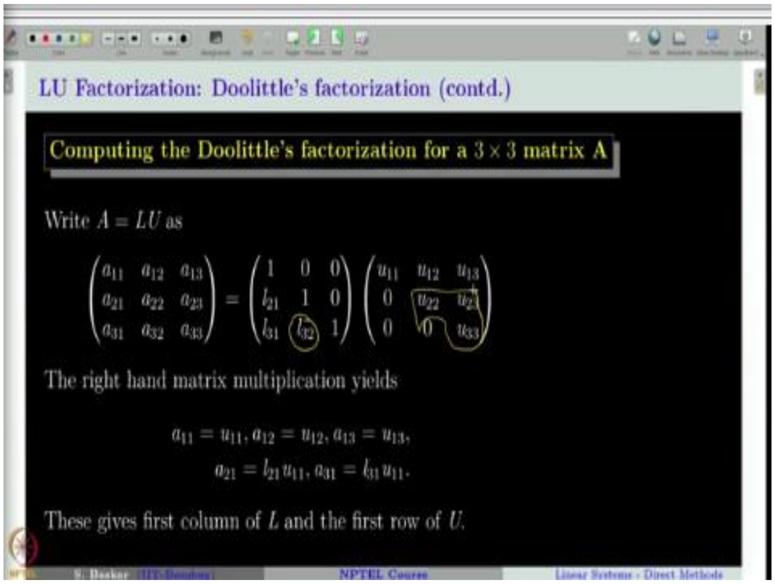

Rather here we will just see how to compute Doolittle factorization for a given matrix. Again to keep our discussion simple we will only explain the computational procedure for Doolittle factorization in the case of a 3×3 matrix. And the generalization to $n \times n$ matrix can be done in a similar way. Let us see how to obtain a Doolittle factorization, well, it is pretty straightforward.

You write the matrix A as the lower triangular matrix with all it is diagonal elements as 1. Remember, that is the definition of Doolittle factorization into the upper triangular matrix, there is no restriction for the upper triangular matrix. Only restriction for the lower triangular matrix is that all the diagonal elements are 1. Now once you write this what you do? You multiply these 2 matrices and compare the elements of the right hand side matrix with the elements of the matrix A.

That is how you will get all the unknowns on the right hand side; remember all the l's and u's are unknowns. You can multiply the first row of L and the first column of U; you can see that the left hand side will have a_{11}, whereas the right hand side you will get u_{11}. Therefore you directly get the value of u_{11} as a_{11}. Similarly you multiply the first row with the second column of U you will immediately get the value of u_{12} as a_{12}.

Similarly the first row of L with the third column of U will give u u_{13}. Again second row of L and the first column of U will give you $l_{21} = \frac{a_{21}}{u_{11}}$. Remember, for this you need this to be non-zero, that is nothing but a_{11} should be non-zero. If you recall we have already assumed that all the principal minors up to order $n-1$ are non-zeros. In particular, a_{11} is nothing but the principal minor of A of order 1, so that should also be non-zero which is assumed in the theorem.

Therefore this condition as per the theorem should be satisfied. If this is not so then you cannot go ahead with the Doolittle factorization. Well, we have obtained all the elements in the first column of L and also we obtained all the elements of the first row of U. Now let us go ahead with this idea and see how to get the other elements like we still have to get l_{32} and we have to get all these elements also.

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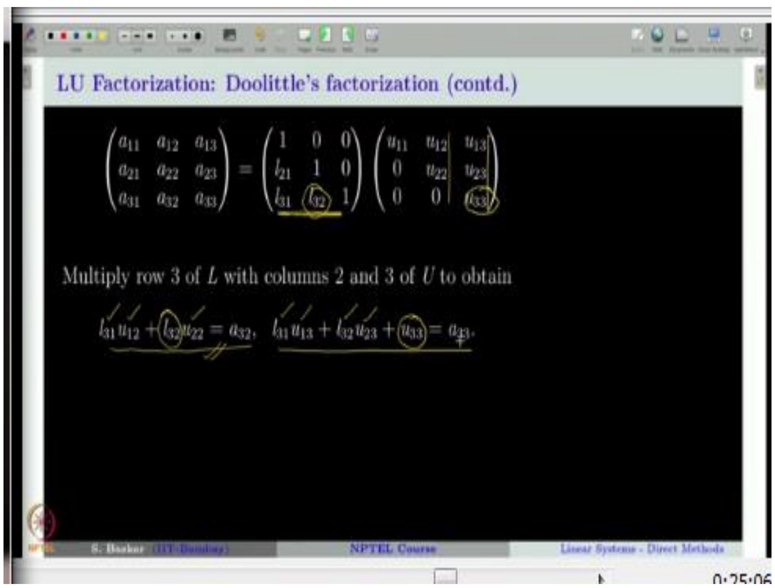

Well, you multiply the row 2 of L by column 2 of U that will give you this equation similarly row 2 and column 3 of U will give this equation. Here you can see that l_{21} is known to us, u_{21} is known to us therefore u_{22} can be obtained immediately. Similarly l_{21} is known to us, u_{13} is also known to us therefore u_{23} can be obtained. So, we have also obtained these elements, these elements are already obtained and we also know these elements.

So, we have to know this one and this one, for that we will multiply the third row of L with the second column of U to get this equation. Remember, this is known to us, u_{12} is known to us, u_{22} also is known to us in the previous step. Therefore this unknown that is this one is now obtained explicitly. And similarly you now take the row 3 of L and multiply it with column 3 of U to get this equation.

Here also you can see that this is known to us, this is known to us, l_3 is now known to us from just this equation, u_{23} is also known to us. Therefore this unknown is obtained in terms of known quantities.

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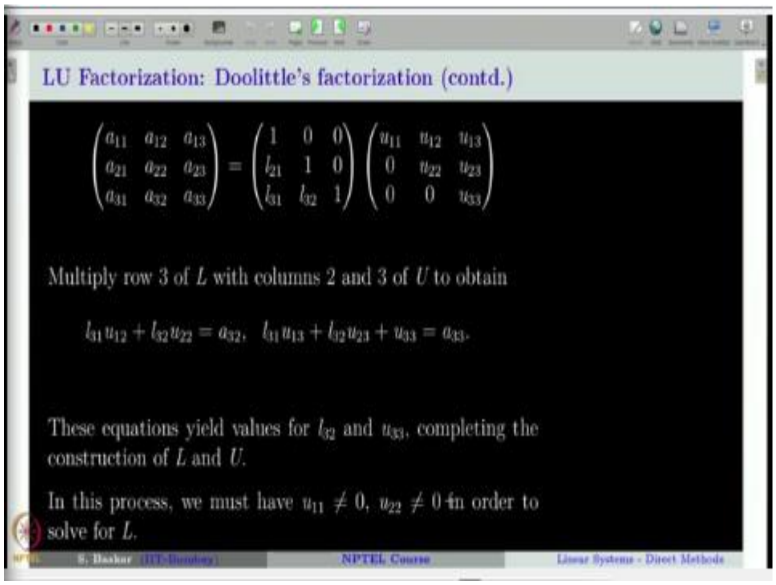

And in this way we have obtained all the elements of the lower triangular matrix and all the elements of the upper triangular matrix also. So, in that way we have obtained the Doolittle factorization, it is a very simple idea. Just write $A = LU$, where L alone you have to write with lower triangular matrix having all it is diagonals elements as 1 and then just multiply the right hand side, compare the coefficients with the coefficients of the matrix A and you will get all the unknowns on the right hand side.

In that process you only have to assume that u_{11} and u_{22} are non-zeros. That will be equivalent to assuming that the principal minor of order 1 and 2 are non-zeros. This is what our theorem was also demanding from us that all the principal minors of order $n-1$ should be non-zero. Here we have taken $n = 3$ therefore the principal minor of order 1 and 2 both should be non-zero, that is what is demanded from this construction also, well, this is just a construction.

One needs to prove that theorem for that we will just give the proof in the case of Cholesky's factorization and one can easily understand the Doolittle also once you know the proof of the Cholesky' factorization existence theorem. Well, let us take an example.

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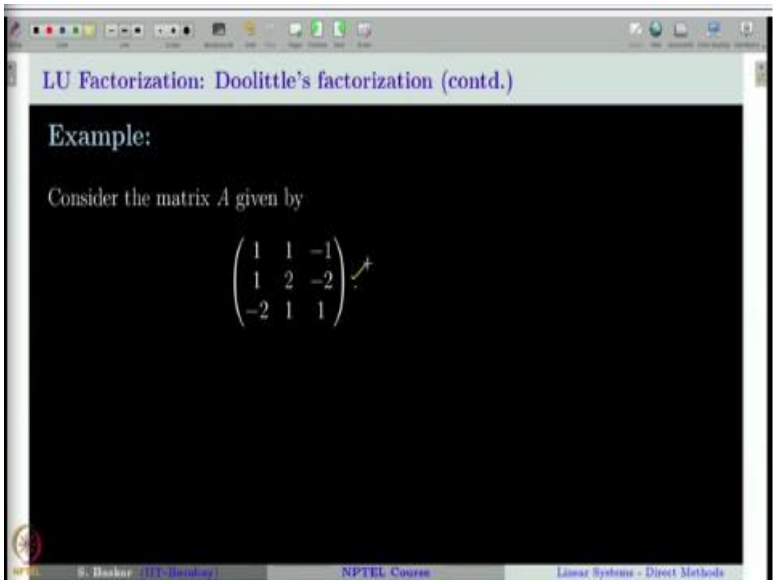

Let us consider the matrix A given like this.

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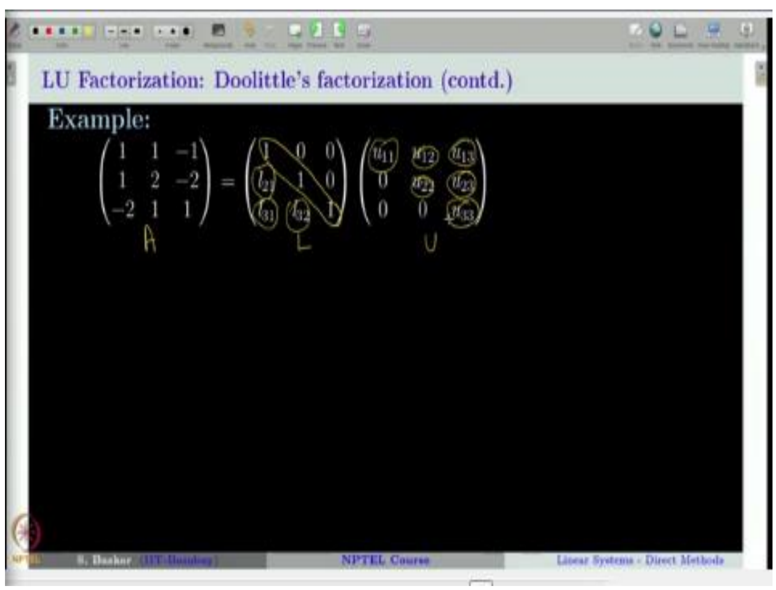

Now we are looking for a LU factorization of the matrix A, remember you have to write $A = LU$ with L having all it is diagonal elements as 1, this is the only key idea in Doolittle factorization. Once you have this then you know how to obtain each of these unknowns, all these unknowns.

Now you have the expressions for all these unknowns or you can simply multiply them and compare the coefficients also.

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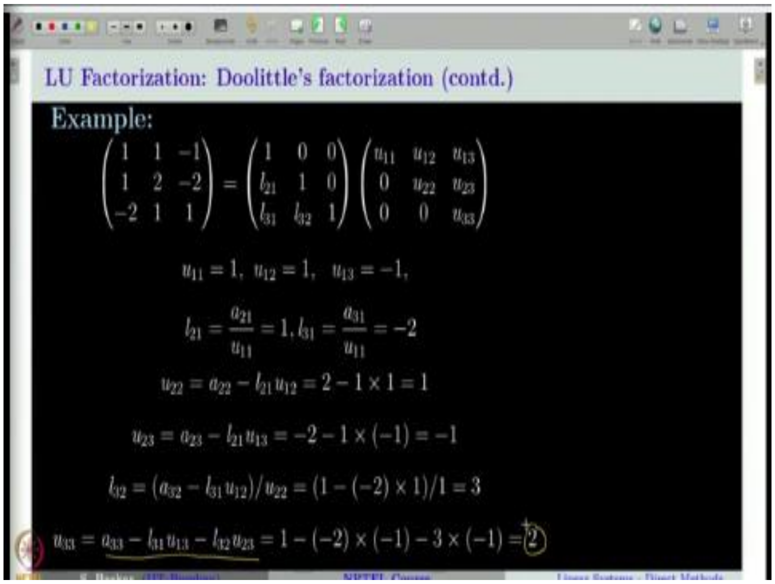

So, you can readily see that u_{11} is just 1, u_{12} is also 1 and u_{13} is –1 and similarly you can get these 2 unknowns also that is l_{21} and l_{31}, they are given by a_{21} divided by u_{11} which is 1 and l_{31} is a_{31} divided by u_{11} and that is given by –2. And then go on with finding u_{22} and that is given by 1 and u_{23} can be obtained using this formula and that is given by –1. And finally we have to get l_{32} and its expression is given like this and its value can be obtained as 3. And also u_{33} which is given by this expression and it is value is 2. Therefore we have obtained all the unknowns.

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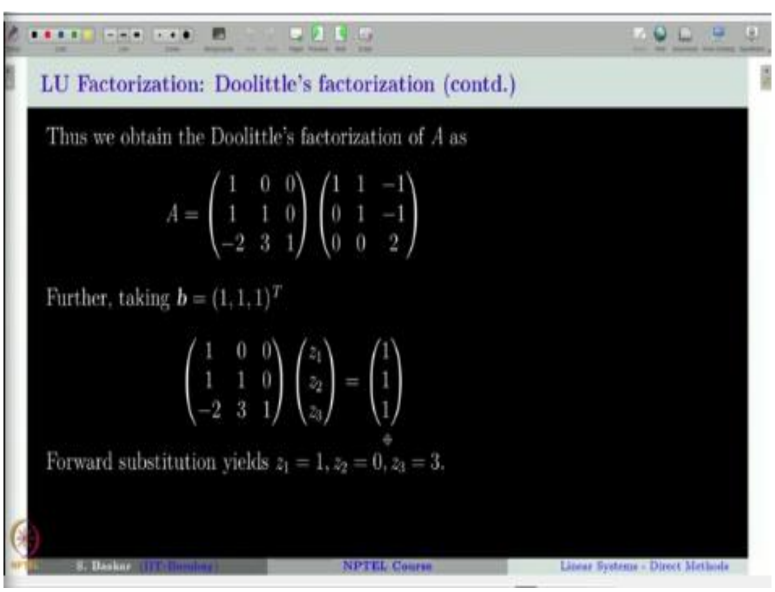

And hence we obtained the LU factorization of the matrix A as L given like this and U given like this. Now if I give you a right hand side vector b, it is very simple for you to obtain the solution of the system $Ax = b$, how will you do that? First write $Lz = b$, remember this is L and this is z and b and do a forward substitution that will give you the unknown vector z, which is in our case it is 1, 0 and 3.

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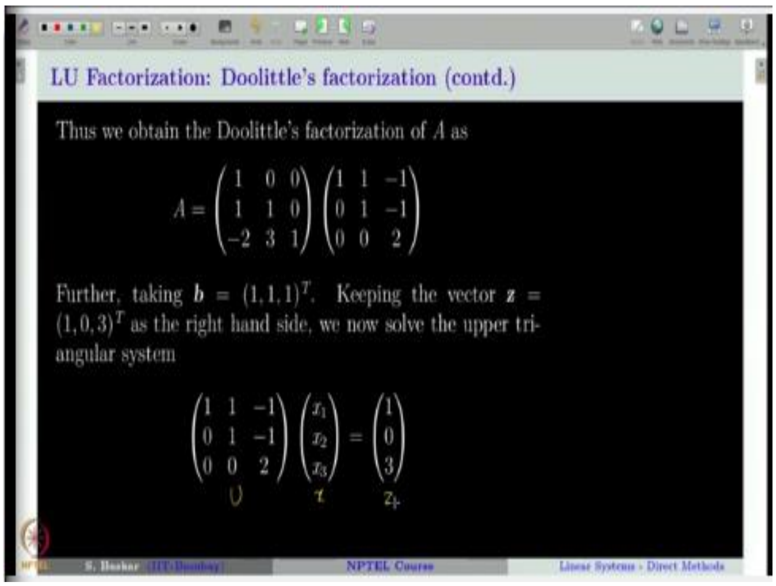

Once you have the z vector, remember z is nothing but Ux therefore you have to solve the system $Ux = z$. That again can be done by the backward substitution because now you have the coefficient matrix as the upper triangular matrix.

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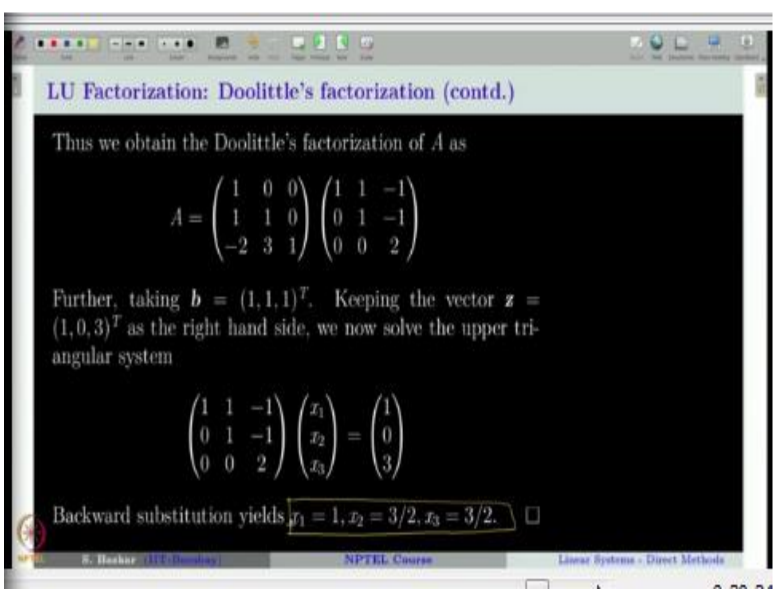

And that leads to the required solution given like this. You can notice that there is no approximation involved in this process and that is why it is a direct method. Why there is no approximation? Because we have not done any rounding process in this, all the calculations are done with infinite precision. Basically we have kept them in the fractional form and did all the calculations.

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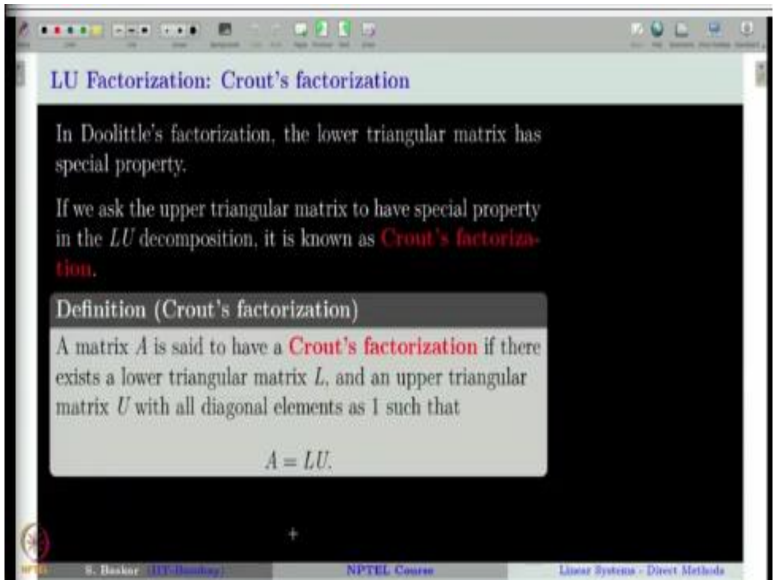

Once you understand Doolittle factorization, the Crout factorization is not very difficult for you to understand. Because the Crout factorization is the one where the upper triangular matrix has all its diagonal elements as 1 whereas the lower triangular matrix has no restrictions. Of course other than being a lower triangular matrix, that is all. Otherwise there is no restriction on these elements whereas the upper triangular matrix should have all it is diagonal elements as 1.

Now I hope you can construct the Crout factorization because the idea behind constructing crude factorization is very much similar to the way we have constructed the Doolittle factorization. Only thing is you have to write *U* with all it is diagonal elements as 1 whereas *L* is written without that condition. And with this we have finished the construction of Doolittle and Crout factorizations; we are left out with the Cholesky's factorization which we will study in the next lecture. Thank you for your attention.